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**Solution of the Star-Triangle Equation
in the S_3 Spin Model**

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In the 2-dimensional spin model with spins taking values in the symmetric group S_3 a solution of the star-triangle equation is found associated to a genus-zero rapidity curve. The solution intersects the physical region of couplings at the point $(\beta_2^*, \beta_3^*) = (\frac{1}{8}\ln 2 + \frac{1}{6}\ln(1 + \frac{3}{\sqrt{2}}), \frac{1}{3}\ln(1 + \frac{3}{\sqrt{2}}))$. This point is conjectured to be a tricritical point where a non-Abelian orbifold symmetry is realized, namely the symmetry described by the Hopf algebra $\mathcal{D}(S_3)$, the double of S_3 .

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For any finite group G we consider G -spin models defined on the 2-dimensional square lattice. At each site $\mathbf{x} \in \mathbf{Z}^2$ we have a spin $g_{\mathbf{x}} \in G$. The interaction is a nearest neighbour interaction: to each bond $\langle \mathbf{x}, \mathbf{y} \rangle$ one associates a Boltzmann weight $W(g_{\mathbf{x}}^{-1} g_{\mathbf{y}})$ where W is a class function. The most general action of such a model can be expanded in terms of the irreducible characters χ_r of G :

$$-S(g) = \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} \sum_r \beta_r \chi_r(g_{\mathbf{x}}^{-1} g_{\mathbf{y}}) . \quad (1)$$

In the special case $G = \mathbf{Z}(N)$ this model is known to be integrable on the Potts model submanifold of the β -space in the sense that an infinite family of commuting transfer matrices can be found [1-3]. The members of these families do not live in general in the space of actions (1). To obtain them one has to introduce different weights W on horizontal and on vertical links. These Boltzmann weights lie on the algebraic curve, the genus of which is typically zero for the self-dual critical models. The critical points of the $N = 2, 3, 4$ Potts models were identified with the first members of the minimal series of conformal field theories [4]. At the self-dual points the $\mathbf{Z}(N) \otimes \mathbf{Z}(N)$ symmetry of these theories presents itself in the structure of sectors.

In this paper we make a step towards generalizing the above picture to non-Abelian finite groups. For $G = S_3$ we find a one-parameter family of commuting transfer matrices by solving the star-triangle equation. The solution lies on an algebraic curve of genus zero. The position of this solution on the (β_2, β_3) phase diagram compared to the Monte-Carlo simulation [5] and the vanishing of the genus suggests that we are at a critical point. Although a dynamical understanding of order-disorder duality in non-Abelian spin models is still missing (see however [6]), we know what is the generalization of the symmetry $G \otimes G^*$ (G^* is the Pontryagin-dual of the Abelian group G) to non-Abelian G : it is the quasi-triangular Hopf algebra $\mathcal{D}(G)$, the double of the group algebra $\mathbf{C}G$ [7,8]. It is worth noting that this quantum symmetry is the same as the one of a G -orbifold conformal field theory [8,9]. Thus G -spin systems at their 'self-dual' points have G -orbifold superselection structure.

Let G be arbitrary for a while and consider three systems with respective horizontal and vertical Boltzmann weight functions $(x, \bar{x})$, $(y, \bar{y})$ and $(z, \bar{z})$. The star-triangle equation is the following relation between them: there exists a constant C such that

$$\sum_{h \in G} \bar{z}(g_1^{-1} h) x(h^{-1} g_2) \bar{y}(h^{-1} g_3) = C z(g_3^{-1} g_2) \bar{x}(g_1^{-1} g_3) y(g_1^{-1} g_2) \quad (2)$$

holds for arbitrary $g_1, g_2, g_3 \in G$. As a linear system of equations for $z, \bar{z}$, (2) can be solved using the following trick (cf. [3]). Multiply (2) by $D_r^{ij}(g_1)$ where D_r is the r^{th} irreducible representation matrix and sum over g_1 . After the substitution $g_1 \rightarrow h g_1$ on the left-hand side one obtains the equation

$$\frac{|G|}{d_r} \bar{z}_r V_{rg}^{ij}(x, \bar{y}) = C z(g) V_{rg}^{ij}(y, \bar{x}) \quad (3)$$

where $\bar{z}_r$ is the r^{th} coefficient in the character expansion of $\bar{z}(g)$, d_r is the dimension of the representation D_r and

$$V_{rg}^{ij}(x, \bar{y}) = \sum_{h \in G} D_r^{ij}(h^{-1}) x(hg) \bar{y}(h) .$$

In order for (3) to be solvable for $z, \bar{z}$ the coefficients $V(x, \bar{y})$ and $V(y, \bar{x})$ have to satisfy certain consistency conditions that we shall write up for the S_3 case. The strategy is to find functions F_i , $i = 1, 2, \dots$ such that the consistency conditions will have the form $F_i(x, \bar{x}) = F_i(y, \bar{y})$.

Focusing on the S_3 case, let us introduce the polynomials $p_1(x, \bar{y}) = x_1 \bar{y}_1 + x_2 \bar{y}_2 + x_3 \bar{y}_3$, $p_2(x, \bar{y}) = x_1 \bar{y}_1 - x_2 \bar{y}_2$, $p_3(x, \bar{y}) = 2x_1 \bar{y}_1 + 2x_2 \bar{y}_2 - x_3 \bar{y}_3$ and the mappings $R : (x_1, x_2, x_3) \rightarrow (x_2, x_1, x_3)$, $S : (x_1, x_2, x_3) \rightarrow (x_3, x_3, x_1 + x_2 + x_3)$. In all expressions the indices $r = 1, 2, 3$ refer to the trivial, the 1-dimensional and the 2-dimensional representations, respectively. Now the consistency conditions can be written as follows.

$$\frac{p_2(x, \bar{y}) p_3(x, R\bar{y})}{p_3(x, \bar{y}) p_2(x, R\bar{y})} = \frac{p_2(y, \bar{x}) p_3(y, R\bar{x})}{p_3(y, \bar{x}) p_2(y, R\bar{x})} \quad (4)$$

$$\frac{x_1 - x_2}{\bar{x}_3} \cdot \frac{p_1(x, \bar{y})}{p_2(x, \bar{y}) p_1(x, S\bar{y})} = \frac{y_1 - y_2}{\bar{y}_3} \cdot \frac{p_1(y, \bar{x})}{p_2(y, \bar{x}) p_1(y, S\bar{x})} \quad (5)$$

$$\frac{x_1 + x_2 - x_3}{x_1 - x_2} \cdot \frac{p_1(x, \bar{y})}{p_2(x, \bar{y})} = \frac{y_1 + y_2 - y_3}{y_1 - y_2} \cdot \frac{p_1(y, \bar{x})}{p_2(y, \bar{x})} \quad (6)$$

$$\frac{x_1 - x_2}{x_3} \cdot \frac{\bar{x}_1 - \bar{x}_2}{\bar{x}_3} = \frac{y_1 - y_2}{y_3} \cdot \frac{\bar{y}_1 - \bar{y}_2}{\bar{y}_3} \quad (7)$$

$$\frac{x_1 + x_2 - x_3}{x_3} \cdot \frac{\bar{x}_1 + \bar{x}_2 - \bar{x}_3}{\bar{x}_3} = \frac{y_1 + y_2 - y_3}{y_3} \cdot \frac{\bar{y}_1 + \bar{y}_2 - \bar{y}_3}{\bar{y}_3} \quad (8)$$

The last two equations are already of the required form $F(x, \bar{x}) = F(y, \bar{y})$. (4) and (6) can also be cast into this form if we use (7) and (8). In this way we find 4 functions on the projective space $\mathbf{CP}^2 \times \mathbf{CP}^2$ such that all of them should take the same value on the rays $(Cx, C\bar{x})$ and $(Cy, C\bar{y})$ in order for the corresponding transfer matrices to commute. Introducing the homogeneous coordinates $X_+ = (x_1 + x_2 - x_3)/(x_1 - x_2)$, $X_0 = (x_1 + x_2 + 2x_3)/(x_1 - x_2)$ and $\bar{X}_+$, $\bar{X}_0$ analogously the four functions are the following: $F_1(x, \bar{x}) = X_+ \bar{X}_+$, $F_2(x, \bar{x}) = (X_0 - X_+)(\bar{X}_0 - \bar{X}_+)$, $F_3(x, \bar{x}) = (X_0 + 2X_+ - X_0 \bar{X}_+^2)/\bar{X}_+$ and $F_4(x, \bar{x}) = (\bar{X}_0 + 2\bar{X}_+ - \bar{X}_0 X_+^2)/X_+$. For generic values $a, b, c, \bar{c}$ of $F_1, \dots, F_4$ respectively the niveau surfaces of these functions meet only at a point. If we want, let us say, X_+ to be a free complex parameter on the common niveau surface the following relations must hold: $b = 3(a + c)$, $c = \bar{c} = c_{\pm}$, where $c_{\pm} = -a + a^{\pm 1}$. The case $c = c_+$ never leads to positive Boltzmann weight so we restrict ourselves to the c_- case. This solution would be a two-parameter solution (depending on a and X_+) if there were not the equation (5). This restricts the value of a to be $\sqrt{2}$ or $-\sqrt{2}$. The latter case is again unphysical and we end up with the solution $a = \sqrt{2}$ depending on the complex parameter $u = X_+$.

Summarizing the result: we have found three polynomials

$$\begin{aligned}
f_1(u) &= -2u + 4u^3 \\
f_2(u) &= 4 - 2u^2 \\
f_3(u) &= 4u - 2u^3
\end{aligned} \tag{9}$$

such that a solution of (2) is obtained by setting $x(g) = f_s(u)$, $\bar{x}(g) = f_s(\bar{u})$, $y(g) = f_s(v)$, $\bar{y}(g) = f_s(\bar{v})$, $z(g) = f_s(w)$ and $\bar{z}(g) = f_s(\bar{w})$ where s is the order of g , $w = u/v$ and the bar denotes the mapping $u \rightarrow \bar{u} \equiv \sqrt{2}/u$.

The polynomials (9) define an algebraic curve of degree 3 embedding the complex sphere to $\mathbf{CP}^2$ with one self-intersection point. Therefore the spectral parameter u lives on a Riemann surface of genus 0.

In the interval $1/\sqrt{2} < u < \sqrt{2}$ the Boltzmann weights are exponentials of an action (1) with real β -s. For $1 < u < \sqrt{2}$ the β -s are positive. The point where the vertical and horizontal interactions coincide is the point $u = 2^{1/4}$. The corresponding point in the (β_2, β_3) -plane was given in the abstract and does not lie on the 6-state Potts model curve $\beta_3 = 2\beta_2$.

Next we turn to the quantum interpretation of this result. Consider for each $h \in S_3$ a copy $\mathcal{H}(h)$ of the Hilbert space of functions of N variables $g_1, \dots, g_N \in S_3$, to be imagined to lie on the diagonal of the square lattice. $\mathcal{H}(h)$ will be called the sector twisted by h for the following reason. Let the operators $Q(g)$ and $P(h)$ be defined by $Q(g)\psi_h(g_1, \dots, g_N) = \psi_{ghg^{-1}}(g^{-1}g_1, \dots, g^{-1}g_N)$ and $P(g)\psi_h = \delta_{g,h}\psi_h$ with ψ_h denoting any element of $\mathcal{H}(h)$. These operators satisfy the defining relations of the Hopf algebra $\mathcal{D}(S_3)$, namely: $Q(g_1)Q(g_2) = Q(g_1g_2)$, $P(h_1)P(h_2) = \delta_{h_1,h_2}P(h_1)$ and $Q(g)P(h) = P(ghg^{-1})Q(g)$. Furthermore we can define transfer matrices $T(u, h) : \mathcal{H}(h) \rightarrow \mathcal{H}(h)$ depending on the spectral parameter u such that the full transfer matrix $T(u) = \oplus_h T(u, h)$ commutes with $\mathcal{D}(S_3)$.

Let D_{reg} denote the left regular representation of S_3 acting on the group algebra $\mathbf{CS}_3$. $\mathbf{CS}_3$ plays the role of the auxiliary space in the QISM language. We define L -matrices $L_i(u)_{gg'} : \mathcal{H}(h) \rightarrow \mathcal{H}(h)$ via the formula

$$L_i(u)_{gg'}\psi_h(g_1, \dots, g_i, \dots, g_N) = \sum_k R(u|g_i, g; g', k)\psi_h(g_1, \dots, k, \dots, g_N),$$

where $R(u|g_i, g; g', k) = x(g^{-1}g_i)\bar{x}(g^{-1}k)x(k^{-1}g')\bar{x}(g_i^{-1}g')$ are the coefficients of the R -matrix $R(u) : \mathbf{CS}_3 \otimes \mathbf{CS}_3 \rightarrow \mathbf{CS}_3 \otimes \mathbf{CS}_3$ and $x, \bar{x}$ denote the Boltzmann weights solving equation (2) taken at the spectral parameter value u . By repeated application of the star-triangle relation one can show that the R -matrices obey the QYBE

$$R_{23}(u/v)R_{12}(u/w)R_{23}(v/w) = R_{12}(v/w)R_{23}(u/w)R_{12}(u/v) \tag{10}$$

and an inversion formula

$$R(1/u)R(u) = \left[\frac{12\sqrt{2}}{u^2}(2u^2 - 1)(2 - u^2) \right]^4 \text{id} \otimes \text{id}. \tag{11}$$

The transfer matrix $T(u, h)$ on the twisted sector $\mathcal{H}(h)$ is defined as the trace in auxiliary space

$$T(u, h) = \text{tr} (T(u) D_{\text{reg}}(h)) ,$$

where $T(u) = L_1(u)L_2(u)\dots L_N(u)$. The components of $T(u)$ are subjected to the commutation relations

$$R(u/v) (T(u) \otimes T(v)) = (T(v) \otimes T(u)) R(u/v) . \quad (12)$$

Since the R -matrix $R(u/v)$ commutes with $D_{\text{reg}}(h) \otimes D_{\text{reg}}(h)$, (12) implies that for arbitrary complex numbers u, v and $h \in S_3$

$$[T(u, h), T(v, h)] = 0 . \quad (13)$$

This means integrability in the sector twisted by h .

The symmetry operators and $T(u, h)$ obey the relations: $[P(g), T(u, h)] = 0$ and $Q(g)T(u, h)Q(g^{-1}) = T(u, ghg^{-1})$. Therefore the full transfer matrix $T(u)$, as well as the untwisted one $T(u, e)$, belong to the algebra of $\mathcal{D}(S_3)$ -invariant operators (i.e. are ‘observables’). Although the invariance of an operator under the action (the adjoint action [10]) of a Hopf algebra a priori means that it transforms under the trivial representation (the co-unit), for semisimple Hopf algebras like $\mathcal{D}(G)$ for any finite group G this notion is equivalent to saying that the operator commutes with the Hopf algebra. It follows then that the spectrum of $T(u)$ and $T(u, e)$ can be classified according to the irreducible representations of $\mathcal{D}(S_3)$.

Since this was a purely kinematical reasoning, it is valid for arbitrary coupling constants. The real question is what representations of $\mathcal{D}(S_3)$ occur as superselection sectors with respect to the observable algebra at a particular point of the phase diagram. The answer to this question lies in the behaviour of various correlation functions. For example in the $\mathbf{Z}(N)$ spin system the point where both $\mathbf{Z}(N)$ and dual- $\mathbf{Z}(N)$ symmetry are realized is the self-dual point ($N < 5$), where both the order and the disorder correlation functions go to zero. Although in the S_3 spin model we were up to now unable to calculate correlation functions, it is perhaps worth examining what order and disorder parameters look like in a non-Abelian spin model and how a correlation inequality can be derived which limits the possible phases.

There are two order parameter fields $\chi_2(g_{\mathbf{x}})$ and $\chi_3(g_{\mathbf{x}})$. $\chi_2(g_{\mathbf{x}})$ is the Ising variable obtained by factoring out the normal subgroup $\mathbf{Z}(3)$. So the vanishing of $\langle \chi_2(g_{\mathbf{x}}^{-1}) \chi_2(g_{\mathbf{y}}) \rangle$ at large separation is the signal of the existence of a $\mathbf{Z}(2)$ charge. $\chi_3(g_{\mathbf{x}})$ is a true S_3 -field but there is a correlation inequality (valid above the Potts model curve) that relates the two:

$$\langle \chi_3(g_{\mathbf{x}}^{-1} g_{\mathbf{y}}) \rangle \leq 2 \langle \chi_2(g_{\mathbf{x}}^{-1} g_{\mathbf{y}}) \rangle, \quad \text{if } 2\beta_2 \geq \beta_3 . \quad (14)$$

To prove (14) one parametrizes the spins as $g_{\mathbf{x}} = t^{n(\mathbf{x})} r^{m(\mathbf{x})}$, $n(\mathbf{x}) = 0, 1$; $m(\mathbf{x}) = 0, 1, 2$ where t and r are the second-, respectively third-order elements of S_3 . The Boltzmann weight $W(g_{\mathbf{x}}^{-1} g_{\mathbf{y}})$ then becomes a linear combination of 1, $\delta_{n(\mathbf{x}), n(\mathbf{y})}$ and $\delta_{n(\mathbf{x}), n(\mathbf{y})}$.

$\delta_{m(\mathbf{x}),m(\mathbf{y})}$. Related to the fact that S_3 is a solvable group, the action we obtained is that of a coupled pair of $\mathbf{Z}(2)$ and $\mathbf{Z}(3)$ Potts models. It is easy to see that the model is equivalent to a bond percolation problem with two kinds of bond sets, L and K subjected to the condition $K \subset L$. The order parameters turn out to be $\langle \chi_2(g_{\mathbf{x}}^{-1}g_{\mathbf{y}}) \rangle = \text{Prob}\{\mathbf{x} \sim_L \mathbf{y}\}$ and $\langle \chi_3(g_{\mathbf{x}}^{-1}g_{\mathbf{y}}) \rangle = 2\text{Prob}\{\mathbf{x} \sim_K \mathbf{y}\}$ where $\mathbf{x} \sim_L \mathbf{y}$ and $\mathbf{x} \sim_K \mathbf{y}$ denote the events that $\mathbf{x}$ and $\mathbf{y}$ belong to the same component of L respectively of K . If $2\beta_2 > \beta_3$ then all the terms are positive in the expression of the expectation values; therefore, (14) follows from $K \subset L$.

The disorder parameters or twist field correlations $\rho_i(p, q)$; $i = 2, 3$ where p and q are plaquettes, are defined as follows. Take a path from p to q on the dual lattice and multiply $W(g_{\mathbf{x}}^{-1}h_i g_{\mathbf{y}})/W(g_{\mathbf{x}}^{-1}g_{\mathbf{y}})$, where $h_2 = t$, $h_3 = r$, over the links $\langle \mathbf{x}, \mathbf{y} \rangle$ that are dual to the links of the path. Let L^* and K^* denote the set of links on the dual lattice that are duals of links not belonging to L and K , respectively. Then the disorder parameters can again be written as probabilities: $\langle \rho_2(p, q) \rangle = \text{Prob}\{p \sim_{L^*} q\}$, $\langle \rho_3(p, q) \rangle = \text{Prob}\{p \sim_{K^*} q\}$. It follows the correlation inequality

$$\langle \rho_2(p, q) \rangle \leq \langle \rho_3(p, q) \rangle, \quad 2\beta_2 \geq \beta_3. \quad (15)$$

These correlation inequalities suggest the following phase structure. At small β_2 and β_3 there is a phase with unbroken S_3 and broken dual- S_3 , that is $\langle \chi_i(g_{\mathbf{x}}^{-1}g_{\mathbf{y}}) \rangle$ for $i = 2, 3$ go to zero but $\langle \rho_i(p, q) \rangle$ for $i = 2, 3$ go to a positive constant. The latter means that twists have no effect at large distances; there are no twisted sectors. For large β_2 and small β_3 there is a phase with $\mathbf{Z}(3)$ symmetry where $\mathbf{Z}(2)$ -twists are possible: the χ_2 and ρ_3 correlation functions go to some positive value while the χ_3 and ρ_2 correlation functions go to zero. If β_3 is large S_3 is completely broken, but there are twisted sectors for all $h \in S_3$. The point(s) where the system shows full $\mathcal{D}(S_3)$ symmetry is (are) expected to lie on the borderline between the first and third phases. The point (β_2^*, β_3^*) where the system was found to be integrable has a good chance, according to the numerical results [5], of being the triple point where the three phase boundaries meet.

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