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**ON MANIFEST COVARIANCE CONDITION
IN THE LAGRANGIAN FORMALISM**

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ABSTRACT

The manifest covariance condition for a Poincaré covariant theory is analysed in the framework of the Lagrangian formalism in a systematic way. The no-interaction theorem is established without the hypothesis of non-degeneracy of the Lagrangian. Then the manifest covariance condition is analysed in a Galilei invariant theory, obtaining the most general type of interaction allowed by this condition.

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1. Introduction

In this paper we clarify some points connected to the so-called “no-interaction” theorem in the framework of the Lagrangian formalism [1,2]. Essentially, we will show that a technical assumption concerning the non-degeneracy of the Lagrangian is not necessary. Also, we hope to present the whole problem in a more systematic way.

First, it is better to state the problem, namely to give a reasonable definition to an N-particle relativistic system.

1.The first point is purely kinematic and consists of choosing the configuration space Q . For an N-particle system it is natural to choose $Q \equiv R^3 \times \dots \times R^3$ (N factors), or perhaps one must subtract from this set the so-called collision set

$$\Delta \equiv \{(\mathbf{q}_1, \dots, \mathbf{q}_N) | \mathbf{q}_a = \mathbf{q}_b, \text{ for some } a \neq b\}$$

In any case, we can identify

$$T(Q) \equiv Q \times R^3 \times \dots \times R^3 (N \text{ factors})$$

In the configuration space Q we have a natural action of the special Euclidean group $SE(3)$:

$$\varphi_{R,\mathbf{a}}(\mathbf{q}_1, \dots, \mathbf{q}_N) \equiv (R\mathbf{q}_1 + \mathbf{a}, \dots, R\mathbf{q}_N + \mathbf{a}). \quad (1.1)$$

This action lifts to an action of $SE(3)$ on $T(Q)$:

$$\dot{\varphi}_{R,\mathbf{a}}(\mathbf{q}_1, \dots, \mathbf{q}_N, \mathbf{v}_1, \dots, \mathbf{v}_N) \equiv (R\mathbf{q}_1 + \mathbf{a}, \dots, R\mathbf{q}_N + \mathbf{a}, R\mathbf{v}_1, \dots, R\mathbf{v}_N). \quad (1.2)$$

2.Because we want to have a relativistic covariant theory, we suppose that we have an action Φ of the proper orthochronous Poincaré group $P_+^\uparrow$ on $T(Q)$, which extends $\dot{\varphi} : \Phi|_{SE(3)} = \dot{\varphi}$. Let us denote by X_ξ the vector field on $T(Q)$ which generates the infinitesimal transformation corresponding to the element ξ in the Lie algebra $Lie(P_+^\uparrow)$ of $P_+^\uparrow$. The generators of the time translation, space translations, rotations and boosts in $Lie P_+^\uparrow$ will be denoted by H, P_i, J_i, K_i respectively ($i = 1, 2, 3$).

3.The Euler-Lagrange equations (i.e. the dynamics) are now some algebraic conditions on X_H . Explicitly, one proceeds as follows. Let $L : T(Q) \rightarrow R$ be the Lagrangian of the system. We define first the 1-form θ_L by:

$$\theta_L \equiv \sum_{a=1}^N \frac{\partial L}{\partial v_{ai}} dq_{ai}. \quad (1.3)$$

(with the summation convention for $i = 1, 2, 3$), the 2-form ω_L by:

$$\omega_L \equiv d\theta_L = \sum_{a=1}^N \sum_{b=1}^N \left(\frac{\partial^2 L}{\partial v_{ai} \partial v_{bj}} dv_{bj} \wedge dq_{ai} + \frac{\partial^2 L}{\partial v_{ai} \partial q_{bj}} dq_{bj} \wedge dq_{ai} \right). \quad (1.4)$$

and the “energy” function $E_L : T(Q) \rightarrow R$ by:

$$E_L \equiv i_{X_H} \theta_L - L. \quad (1.5)$$

The Euler-Lagrange equations are now:

$$i_{X_H} \omega_L = -dE_L. \quad (1.6)$$

4. We now want the Poincaré transformations Φ_g to be symmetries for this Lagrangean theory, i.e. Φ_g must map solutions of the Euler-Lagrange equations into solutions of the same equations. It is well known that a sufficient condition is:

$$(\Phi_g)^* \omega_L = \omega_L. \quad (1.7)$$

for any $g \in P_+^\uparrow$.

5. Now, we impose the so-called “manifest covariance” (or “world line”) condition. This gives, essentially, the generic form of the vector fields X_{K_i} . The idea is to find a natural way to transform the elements of $T(Q)$ with respect to the boosts. One proceeds as follows [3,4]. We consider an evolution $t \mapsto (\mathbf{x}_1(t), \dots, \mathbf{x}_N(t))$ with $\mathbf{x}_a(0) = \mathbf{q}_a$ and $\dot{\mathbf{x}}_a(0) = \mathbf{v}_a$ for any $a = 1, \dots, N$. To obtain the evolution from the point of view of a moving observer, one applies the corresponding boost to the events $(t, \mathbf{x}_a(t))$ and obtains the evolutions in the moving frame, in a parametrised form: $t' = t'(t)$, $\mathbf{x}'_a = \mathbf{x}'_a(t)$. We must eliminate t , and compute $\mathbf{q}'_a \equiv \mathbf{x}'_a(t' = 0)$, $\mathbf{v}'_a \equiv \frac{d\mathbf{x}'_a}{dt'}(t' = 0)$. The desired transformation is then $(q, v) \mapsto (q', v')$, where we have used the shorthand notations: $q \equiv (\mathbf{q}_1, \dots, \mathbf{q}_N)$ and $v \equiv (\mathbf{v}_1, \dots, \mathbf{v}_N)$. Taking an infinitesimal boost, one finds that X_{K_i} must be of the following form:

$$X_{K_i} = \sum_{a=1}^N \left[q_{ai} v_{aj} \frac{\partial}{\partial q_{aj}} + (v_{ai} v_{aj} - \delta_{ij} + q_{ai} A_{aj}) \frac{\partial}{\partial v_{aj}} \right]. \quad (1.8)$$

where $\mathbf{A}_a$ are the accelerations. We will impose also the condition that $\mathbf{A}_a$ are functions of (q, v) . This amounts to supposing that the equations of motions are second order. This is the “manifest covariance” condition in the Lagrangian formalism.

The “no-interaction” theorem asserts that if the conditions 1-5 above are satisfied and also the $3N \times 3N$ matrix of entries $\frac{\partial^2 L}{\partial v_{a i} \partial v_{b j}}$ is non-degenerate, then there is no interaction between the N particles, i.e. the Lagrangian is a sum of one-particle Lagrangians.

We will prove that this technical condition is redundant. The proof will be divided in two parts. In Section 2 we will show that the assumptions 2 and 4 above imply that $\mathbf{A}_a = \mathbf{0}$ for any $a = 1, \dots, N$, so, on physical grounds, it is clear that there is no interaction. Next, in Section 3 we will show that the Lagrangian L is a sum of one-particle Lagrangians. In this part of the proof, we will see that in fact, we must generalise somehow the Lagrangian formalism. Indeed, if we insist that L is defined all over $T(Q)$, then we get only a trivial solution. But it is easy to see from the explicit formulae appearing in the proof that if we define L only on some open subset of $T(Q)$, which is invariant with respect to Φ , then we have well-defined solutions for L which are sums of one-particle Lagrangians.

In Section 4 we will analyse the same problem for a Galilei invariant theory and we will find that non trivial interactions are possible. In fact, after finding the expression for X_{K_i} dictated by the “manifest covariance” condition, but with Poincaré boosts replaced by Galilei boosts, we will obtain the most general form for the Lagrangian L .

2. The determination of the accelerations

1. The idea is to exploit completely the action property of Φ in the infinitesimal form:

$$[X_\xi, X_\eta] = X_{[\xi, \eta]}. \quad (2.1)$$

for any $\xi, \eta \in Lie(P_+^\uparrow)$. In an explicit form these relations are:

$$[X_{P_i}, X_{P_j}] = 0. \quad (2.2)$$

$$[X_{J_i}, X_{J_j}] = \epsilon_{ijk} X_{J_k}. \quad (2.3)$$

$$[X_{J_i}, X_{P_j}] = \epsilon_{ijk} X_{P_k}. \quad (2.4)$$

$$[X_{K_i}, X_{P_j}] = \delta_{ij} X_H. \quad (2.5)$$

$$[X_{K_i}, X_H] = X_{P_i}. \quad (2.6)$$

$$[X_{J_i}, X_H] = 0. \quad (2.7)$$

$$[X_{K_i}, X_{K_j}] = -\epsilon_{ijk} X_{J_k}. \quad (2.8)$$

$$[X_{P_i}, X_H] = 0. \quad (2.9)$$

$$[X_{J_i}, X_{K_j}] = \epsilon_{ijl} X_{K_l}. \quad (2.10)$$

- We notice first that the hypothesis 2 and the formula (1.2) allow us to compute X_{P_i} and X_{J_i} :

$$X_{P_i} = -\sum_{a=1}^N \frac{\partial}{\partial q_{ai}}. \quad (2.11)$$

$$X_{J_i} = -\epsilon_{ijk} \sum_{a=1}^N (q_{aj} \frac{\partial}{\partial q_{ak}} + v_{aj} \frac{\partial}{\partial v_{ak}}). \quad (2.12)$$

Not surprisingly, (2.2)-(2.4) are fulfilled identically.

- Next, we use (2.5) to determine X_H . Computing the left-hand side of this relation with the help of (2.11) and (1.8), we get easily that X_H must be of the form:

$$X_H = \sum_{a=1}^N (v_{ai} \frac{\partial}{\partial q_{ai}} + A_{ai} \frac{\partial}{\partial v_{ai}}). \quad (2.13)$$

and also that the accelerations must satisfy:

$$\sum_{a=1}^N \frac{\partial A_{ak}}{\partial q_{bj}} = 0. \quad (2.14)$$

This expresses the translational invariance of the functions $\mathbf{A}_a$ in an infinitesimal form.

- Using (1.8), (2.13) and (2.11) we show rather easily that (2.6) is equivalent to:

$$\sum_{b=1}^N \{v_{bj}(q_{bi} - q_{ai}) \frac{\partial A_{ak}}{\partial q_{bj}} + [v_{bi}v_{bj} - \delta_{ij} + (q_{bi} - q_{ai})A_{bj}] \frac{\partial A_{ak}}{\partial v_{bj}}\} = 2v_{ai}A_{ak} + v_{ak}A_{ai}. \quad (2.15)$$

- Using (2.12) and (2.13) we show that:

$$\epsilon_{ijl} \sum_{b=1}^N (q_{bj} \frac{\partial A_{ak}}{\partial q_{bl}} + v_{bj} \frac{\partial A_{ak}}{\partial v_{bl}}) = \epsilon_{ijk} A_{aj}. \quad (2.16)$$

which expresses the rotational covariance of the functions $\mathbf{A}_a$ in an infinitesimal form. The relations (2.14)-(2.16) are valid for any $a = 1, \dots, N$, and any $i, k = 1, 2, 3$.

- Finally, it is easy to prove that the remaining relations (2.8)-(2.10) are identically satisfied. So, the action property of Φ is encoded in the relations (2.14)-(2.16). These are the so-called Currie-Hill equations [3-5].

2. The main point of this paper is to prove that in fact (2.14)-(2.16) imply that $\mathbf{A}_a = \mathbf{0}$ for any $a = 1, \dots, N$.

- First, we use (2.14). The idea is to use, for a fixed a , the relative coordinates:

$$\xi_1 \equiv \mathbf{q}_1 - \mathbf{q}_a, \dots, \xi_{a-1} \equiv \mathbf{q}_{a-1} - \mathbf{q}_a, \xi_{a+1} \equiv \mathbf{q}_{a+1} - \mathbf{q}_a, \dots, \xi_N \equiv \mathbf{q}_N - \mathbf{q}_a$$

and $\mathbf{q}_a$. Then, (2.14) tells us that $\mathbf{A}_a$ is a function only of $\xi^a \equiv (\xi_1, \dots, \xi_{a-1}, \xi_{a+1}, \dots, \xi_N)$ and v .

- Next, we write (2.15) and (2.16) in these new coordinates. We get immediately:

$$\sum_{b \neq a} \xi_{bi} (v_{bj} \frac{\partial A_{ak}}{\partial \xi_{bj}} + A_{bj} \frac{\partial A_{ak}}{\partial v_{bj}}) + \sum_b (v_{bi} v_{bj} - \delta_{ij}) \frac{\partial A_{ak}}{\partial v_{bj}} = 2v_{ai} A_{ak} + v_{ak} A_{ai}. \quad (2.17)$$

$$\epsilon_{ijl} (\sum_{b \neq a} \xi_{bj} \frac{\partial A_{ak}}{\partial \xi_{bl}} + \sum_b v_{bj} \frac{\partial A_{ak}}{\partial v_{bl}}) = \epsilon_{ijk} A_{aj}. \quad (2.18)$$

for any $i, k = 1, 2, 3$. Remark that in the system (2.17)+(2.18), the variable

$$A^a \equiv (\mathbf{A}_1, \dots, \mathbf{A}_{a-1}, \mathbf{A}_{a+1}, \dots, \mathbf{A}_N)$$

is a parameter.

- To analyse this system, we proceed as follows. We define the functions F_{ai} by:

$$F_{ai}(\xi^a, v, \mathbf{a}) \equiv A_{ai}(\xi^a, v) - a_i. \quad (2.19)$$

where $\mathbf{A}_a$ is a solution of (2.17)+(2.18). (For notational simplicity we did not write explicitly the dependence on the parameters A^a). We define also the differential operators:

$$K_i \equiv \sum_{b \neq a} \xi_{bi} (v_{bj} \frac{\partial}{\partial \xi_{bj}} + A_{bj} \frac{\partial}{\partial v_{bj}}) + \sum_b (v_{bi} v_{bj} - \delta_{ij}) \frac{\partial}{\partial v_{bj}} + (2v_{ai} a_k + v_{ak} a_i) \frac{\partial}{\partial a_k}. \quad (2.20)$$

$$J_i \equiv \epsilon_{ijk} (\sum_{b \neq a} \xi_{bj} \frac{\partial}{\partial \xi_{bk}} + \sum_b v_{bj} \frac{\partial}{\partial v_{bk}} + a_j \frac{\partial}{\partial a_k}). \quad (2.21)$$

If a function F of variables (ξ^a, v, a) is a linear combination of the functions F_{ai} with coefficient functions only of (ξ^a, v) , then we write $F \approx 0$. Now, it is easy to establish that (2.17) and (2.18) imply that for any F verifying $F \approx 0$, we also have:

$$K_i F \approx 0. \quad (2.22)$$

$$J_i F \approx 0. \quad (2.23)$$

- By a straightforward computation one establishes that:

$$[J_i, K_j] = -\epsilon_{ijl} K_l - \epsilon_{ikl} \sum_{b \neq a} \xi_{bj} A_{bk} \frac{\partial}{\partial v_{bl}}. \quad (2.24)$$

So, (2.22) and (2.23) imply also that:

$$X_{ij} F \approx 0. \quad (2.25)$$

for any F verifying $F \approx 0$. Here:

$$X_{ij} \equiv \epsilon_{ikl} \sum_{b \neq a} \xi_{bj} A_{bk} \frac{\partial}{\partial v_{bk}}. \quad (2.26)$$

- Now, we extract from (2.22) and (2.25) the following consequences:

$$[K_{i_1}, [K_{i_2}, \dots, [K_{i_n}, X_{ij}], \dots], F \approx 0. \quad (2.27)$$

for any F such that $F \approx 0$.

- We will prove that, for different n 's, the multiple commutators are linear independent differential operators. The proof is based on induction. We show first by recurrence that we have:

$$[K_{i_1}, \dots, [K_{i_n}, X_{ij}], \dots] = \sum_{b \neq a} [P_{i_1 \dots i_n; ijk}(\xi_b, \mathbf{v}_b, \mathbf{A}_b) \frac{\partial}{\partial \xi_{bk}} + Q_{i_1 \dots i_n; ijk}(\dots) \frac{\partial}{\partial v_{bk}}]. \quad (2.28)$$

where $P_{\dots}$ and $Q_{\dots}$ are polynomials. Next, we use the recurrence relation for $P_{\dots}$ and $Q_{\dots}$ to prove that these polynomials have the following structure:

$$P_{i_1 \dots i_n; ijk}(\xi, \mathbf{v}, \mathbf{A}) = \sum_{p=0}^{n-1} v_{l_1} \dots v_{l_p} P_{i_1 \dots i_n; ijk}^{l_1 \dots l_p}(\xi, \mathbf{A})$$

$$Q_{i_1 \dots i_n; ijk}(\xi, \mathbf{v}, \mathbf{A}) = \sum_{p=0}^n v_{l_1} \dots v_{l_p} Q_{i_1 \dots i_n; ijk}^{l_1 \dots l_p}(\xi, \mathbf{A})$$

The coefficients $P_{\dots}$ and $Q_{\dots}$ appearing on the right-hand side of the preceding relation are polynomials in ξ and $\mathbf{A}$. Moreover, it is easy to establish a recurrence relation for the highest order coefficients $Q_{i_1 \dots i_n; ijk}^{l_1 \dots l_n}$. Using this recurrence relation, one establishes next that we have:

$$Q_{i_1 \dots i_n; ijk}^{l_1 \dots l_n}(\xi, \mathbf{A}) = Q_{i_1 \dots i_n; ijk l m}^{l_1 \dots l_n} \xi_l A_m. \quad (2.30)$$

with $Q_{i_1 \dots i_n; ijk l m}^{l_1 \dots l_n}$ constants subjected to some recurrence relations which we do not write in detail. These recurrence relations can be used to show that the above constant coefficients are non-zero. With this information it is easy to prove the desired linear independence for the multiple commutators appearing in (2.28) for different n 's. This means that by taking suitable linear combinations of these multiple commutators, we can generate from (2.27):

$$\frac{\partial F_{ai}}{\partial \xi_{bj}} \approx 0, \quad \frac{\partial F_{ai}}{\partial v_{bj}} \approx 0. \quad (2.31)$$

for any $b \neq a$ and any $i, j = 1, 2, 3$. But using the definition of (2.19) F_{ai} it is easy to see that in fact we must have:

$$\frac{\partial A_{ai}}{\partial \xi_{bj}} = 0, \quad \frac{\partial A_{ai}}{\partial v_{bj}} = 0. \quad (2.32)$$

for any $b \neq a$ and any $i, j = 1, 2, 3$.

- So, $\mathbf{A}_a$ depends only on $\mathbf{v}_a$. This implies that in (2.22) and (2.23) we can take K_i and J_i to be:

$$K_i \equiv (v_{ai} v_{aj} - \delta_{ij}) \frac{\partial}{\partial v_{aj}} + (2v_{ai} a_j + v_{aj} a_i) \frac{\partial}{\partial a_j}. \quad (2.33)$$

$$J_i \equiv \epsilon_{ijk} (v_{aj} \frac{\partial}{\partial v_{ak}} + a_j \frac{\partial}{\partial a_k}). \quad (2.33)$$

Because K_i and J_i are linear independent, by taking suitable linear combinations of (2.22) and (2.23) we can arrive at:

$$\frac{\partial F_{ai}}{\partial v_{aj}} \approx 0. \quad (2.35)$$

Taking into account the definition (2.19) of F_{ai} , we have from here:

$$\frac{\partial A_{ai}}{\partial v_{aj}} \approx 0. \quad (2.36)$$

for any $i, j = 1, 2, 3$, i.e. $\mathbf{A}_a$ is a constant vector.

- Finally, using (2.18) we get:

$$\mathbf{A}_a = \mathbf{0}. \quad (2.37)$$

So, the relations (1.8) and (2.13) become:

$$X_{K_i} = \sum_{a=1}^N [q_{ai} v_{aj} \frac{\partial}{\partial q_{aj}} + (v_{ai} v_{aj} - \delta_{ij}) \frac{\partial}{\partial v_{aj}}]. \quad (2.38)$$

and respectively:

$$X_H = \sum_{a=1}^N v_{ai} \frac{\partial}{\partial q_{ai}}. \quad (2.39)$$

3.The determination of the Lagrangian

The determination of the Lagrangian proceeds in two steps. First, one proves that L is a sum of one-particle Lagrangians, and then one is reduced to a one-particle problem.

1. For the first step, we follow essentially [2]. First, we impose the condition (1.7). It is convenient to use it in the infinitesimal form:

$$L_{X_\xi} \omega_L = 0. \quad (3.1)$$

for any $\xi \in Lie(P_+^\uparrow)$. In particular we have:

$$L_{X_{K_i}} \omega_L = 0. \quad (3.2)$$

for any $i = 1, 2, 3$. Using (1.4) and (2.38) it is elementary to prove that the relation(3.2) is equivalent to the following set of relations:

$$(q_{ai} - q_{bi}) \frac{\partial^2 L}{\partial v_{aj} \partial v_{bk}} = 0. \quad (3.3)$$

$$\begin{aligned} X_{K_i} \frac{\partial^2 L}{\partial v_{ai} \partial v_{bk}} + \delta_{ij} v_{al} \frac{\partial^2 L}{\partial v_{al} \partial v_{bk}} + \delta_{ik} v_{bl} \frac{\partial^2 L}{\partial v_{bl} \partial v_{aj}} + v_{bj} \frac{\partial^2 L}{\partial v_{bk} \partial v_{aj}} + \\ q_{bi} \left(\frac{\partial^2 L}{\partial v_{bk} \partial q_{aj}} - \frac{\partial^2 L}{\partial v_{aj} \partial q_{bk}} \right) = 0. \end{aligned} \quad (3.4)$$

$$X_{K_i} \left(\frac{\partial^2 L}{\partial v_{aj} \partial q_{bk}} - \frac{\partial^2 L}{\partial v_{bk} \partial q_{aj}} \right) - \delta_{ij} v_{al} \left(\frac{\partial^2 L}{\partial v_{al} \partial q_{bk}} - \frac{\partial^2 L}{\partial v_{bk} \partial q_{al}} \right) + \delta_{ik} v_{bl} \left(\frac{\partial^2 L}{\partial v_{bl} \partial q_{aj}} - \frac{\partial^2 L}{\partial v_{aj} \partial q_{bl}} \right) = 0. \quad (3.5)$$

$(\forall a, b = 1, \dots, N; \forall i, j, k = 1, 2, 3).$

It is clear from (3.3) that:

$$\frac{\partial^2 L}{\partial v_{aj} \partial v_{bk}} = 0$$

$(\forall a \neq b)$, so L is of the form:

$$L(q, v) = \sum_{a=1}^N l_a(q, \mathbf{v}_a). \quad (3.6)$$

If we take now $a \neq b$ in (3.4), we get:

$$\frac{\partial^2 l_a}{\partial v_{aj} \partial q_{bk}} = \frac{\partial^2 l_b}{\partial v_{bk} \partial q_{aj}}$$

Because the left-hand side of this relation does not depend on $\mathbf{v}_b$ and the right-hand side does not depend on $\mathbf{v}_a$, both sides are functions only of q . Now it is easy to prove that l_a must be of the form:

$$l_a(q, \mathbf{v}_a) = L_a(\mathbf{q}_a, \mathbf{v}_a) + F_{ai}(q)v_{ai} - V_a(q)$$

Inserting back in (3.5) we get:

$$\frac{\partial F_{ai}}{\partial q_{bj}} = \frac{\partial F_{bj}}{\partial q_{ai}}$$

So, using the Frobenius theorem it follows that F_{ai} is of the form $F_{ai} = \frac{\partial F}{\partial q_{ai}}$ for some function F depending only on q . Hence,

$$L(q, v) + \sum_{a=1}^N L_a(\mathbf{q}_a, \mathbf{v}_a) + \sum_{a=1}^N \frac{\partial F}{\partial q_{ai}}(q)v_{ai} - V(q)$$

The second term is a total derivative, and so can be neglected i.e. L can be taken of the form:

$$L(q, v) = \sum_{a=1}^N L_a(\mathbf{q}_a, \mathbf{v}_a) - V(q). \quad (3.7)$$

Finally, we use the relation (1.6). From (1.6) and (2.39) we have:

$$E_L = \sum_{a=1}^N \frac{\partial L}{\partial v_{ai}} v_{ai} - L. \quad (3.8)$$

so, (1.6) can be shown to be equivalent with:

$$\sum_{b=1}^N \frac{\partial^2 L_a}{\partial v_{ai} \partial q_{bj}} v_{bj} = \frac{\partial L}{\partial q_{ai}}$$

($\forall a = 1, \dots, N; \forall i = 1, 2, 3$). Inserting (3.7) in this relation we get that $\frac{\partial V}{\partial q_{ai}}$ is a function only of $\mathbf{q}_a$. So, V decomposes also into the form:

$$V(q) = \sum_{a=1}^N V_a(\mathbf{q}_a)$$

Combining this with (3.7), we get that L can be taken in a completely decomposed form:

$$L(q, v) = \sum_{a=1}^N L_a(\mathbf{q}_a, \mathbf{v}_a). \quad (3.9)$$

2. Now we determine the one-particle Lagrangians L_a . First, we note that (1.6) becomes:

$$\frac{\partial^2 L_a}{\partial v_{ai} \partial q_{aj}} v_{aj} = \frac{\partial L_a}{\partial q_{ai}}. \quad (3.10)$$

for any $a = 1, \dots, N$. We differentiate this relation with respect to v_{ai} and get:

$$\frac{\partial^2 L_a}{\partial v_{ai} \partial q_{ak}} + \frac{\partial^3 L_a}{\partial v_{ai} \partial v_{ak} \partial q_{aj}} v_{aj} = \frac{\partial^2 L_a}{\partial q_{ai} \partial v_{ak}}$$

Now, we separate the symmetric and the antisymmetric parts in the indices i and k ; we get:

$$\frac{\partial^2 L_a}{\partial v_{ai} \partial q_{ak}} = \frac{\partial^2 L_a}{\partial q_{ai} \partial v_{ak}}. \quad (3.11)$$

and respectively:

$$\frac{\partial^3 L_a}{\partial v_{ai} \partial v_{ak} \partial q_{aj}} v_{aj} = 0. \quad (3.12)$$

The relation (3.11) implies that (3.5) is automatically fulfilled. Also (3.4) is non-trivial only for $a = b$ and considering also (3.11) we get for any a :

$$X_{K_i} \frac{\partial^2 L_a}{\partial v_{aj} \partial v_{ak}} + \delta_{ij} v_{al} \frac{\partial^2 L_a}{\partial v_{al} \partial v_{ak}} + \delta_{ik} v_{al} \frac{\partial^2 L_a}{\partial v_{al} \partial v_{aj}} + v_{ai} \frac{\partial^2 L_a}{\partial v_{aj} \partial v_{ak}} = 0$$

If we substitute the expression (2.34) of X_{K_i} , we get:

$$\begin{aligned} q_{ai} v_{al} \frac{\partial^3 L_a}{\partial q_{al} \partial v_{aj} \partial v_{ak}} + (v_{ai} v_{al} - \delta_{il}) \frac{\partial^3 L_a}{\partial v_{al} \partial v_{aj} \partial v_{ak}} + \delta_{ij} v_{al} \frac{\partial^2 L_a}{\partial v_{al} \partial v_{ak}} + \\ \delta_{ik} v_{al} \frac{\partial^2 L_a}{\partial v_{al} \partial v_{aj}} + v_{ai} \frac{\partial^2 L_a}{\partial v_{aj} \partial v_{ak}} = 0 \end{aligned}$$

The first term is zero because of (3.12), so we get the following relation (for simplicity we drop the index a):

$$(\delta_{il} - v_i v_l) \frac{\partial^3 L}{\partial v_j \partial v_k \partial v_l} - \delta_{ij} v_l \frac{\partial^2 L}{\partial v_l \partial v_k} - \delta_{ik} v_l \frac{\partial^2 L}{\partial v_l \partial v_j} - v_i \frac{\partial^2 L}{\partial v_l \partial v_j} = 0$$

This relation integrates once and gives:

$$(\delta_{il} - v_i v_l) \frac{\partial^2 L}{\partial v_k \partial v_l} - \delta_{ik} (v_l \frac{\partial L}{\partial v_l} - L) = c_{ik}(\mathbf{q})$$

where $c_{ik}(\mathbf{q})$ are some unknown functions of $\mathbf{q}$. Using the function E (see (3.8)) we can write this relation as:

$$\frac{\partial^2 L}{\partial v_i \partial v_k} = v_i \frac{\partial E}{\partial v_k} + \delta_{ik} E + c_{ik}(\mathbf{q}). \quad (3.13)$$

The antisymmetric part of this relation is:

$$\epsilon_{ijk} v_j \frac{\partial E}{\partial v_k} = c_i(\mathbf{q}); (c_i \equiv \frac{1}{2} \epsilon_{ijk} c_{jk}). \quad (3.14)$$

This relation is the infinitesimal form of a relation of the form:

$$E(\mathbf{q}, R\mathbf{v}) - E(\mathbf{q}, \mathbf{v}) = \Delta(R, \mathbf{q}). \quad (3.15)$$

It follows easily that $R \mapsto \exp(\Delta(R, \mathbf{q}))$ is a one-dimensional representation of $SO(3)$. So, necessarily $\Delta = 0$. As a consequence:

$$E(\mathbf{q}, R\mathbf{v}) = E(\mathbf{q}, \mathbf{v}). \quad (3.16)$$

From (3.16) it follows that E is of the form:

$$E(\mathbf{q}, \mathbf{v}) = e(\mathbf{q}, \mathbf{v}^2). \quad (3.17)$$

If we multiply (3.13) by v_i and sum over i , we get the following equation for E :

$$(1 - \mathbf{v}^2) \frac{\partial E}{\partial v_k} - v_k E = c_{ik}(\mathbf{q}) v_i$$

or, taking into account (3.17):

$$v_k [2(1 - \mathbf{v}^2) \frac{de}{d\xi}(\mathbf{q}, \mathbf{v}^2) - e(\mathbf{q}, \mathbf{v}^2)] = c_{ik}(\mathbf{q}) v_i$$

If we make $\mathbf{v} \mapsto R\mathbf{v}$ (here R is an arbitrary element of $SO(3)$) we get easily that c_{ik} is a rotational invariant tensor. So, $c_{ik}(\mathbf{q}) = \delta_{ik}c(\mathbf{q})$, and the preceding relation becomes:

$$2(1 - \xi) \frac{de}{d\xi}(\xi) - e(\xi) = c. \quad (3.18)$$

(where the $\mathbf{q}$ dependence is ignored for the moment). A particular solution of (3.18) is $e = -c$. So, we must still find the general solution of the homogeneous equation:

$$2(1 - \xi) \frac{de_0}{d\xi}(\xi) - e_0(\xi) = 0. \quad (3.19)$$

It is easy to prove that (3.19) implies:

$$(1 - \xi)[e_0(\xi)]^2 = \text{const}. \quad (3.20)$$

Thus, if L (and *a posteriori* e_0) is defined for all values of $\mathbf{v}$, the only smooth solution of (3.20) is $e_0 = 0$ i.e. $e(\xi) = -c$. But then (3.13) becomes:

$$\frac{\partial^2 L}{\partial v_i \partial v_k} = 0$$

so L is of the form:

$$L(\mathbf{q}, \mathbf{v}) = \mathbf{a}(\mathbf{q}) \cdot \mathbf{v} - V(\mathbf{q})$$

In this case (3.10) gives $V = \text{const}$ and

$$\frac{\partial a_i}{\partial q_j} = \frac{\partial a_j}{\partial q_i}$$

So, using the Frobenius theorem L is of the form:

$$L(\mathbf{q}, \mathbf{v}) = \frac{\partial a}{\partial q_i}(\mathbf{q})v_i + \text{const}$$

i.e. a total derivative.

If we want non-trivial solutions, it is natural to look for Lagrangians which are defined only on some open subsets of $T(Q)$. It is clear from (3.20) that we have only two convenient cases:

a) $C_< \equiv \{(\mathbf{q}, \mathbf{v}) | \mathbf{v}^2 < 1\}.$

In this case the general solution of (3.18) is:

$$E(\mathbf{q}, \mathbf{v}) = m(\mathbf{q})(1 - \mathbf{v}^2)^{-1/2} - c(\mathbf{q})$$

Substituting in (3.13) we get that L is of the form:

$$L(\mathbf{q}, \mathbf{v}) = -m(\mathbf{q})(1 - \mathbf{v}^2)^{1/2} - V(\mathbf{q})$$

Finally, substituting this expression into (3.10) we get that m and V are constants. So, neglecting the constant V , we get the well-known expression:

$$L(\mathbf{q}, \mathbf{v}) = -m(1 - \mathbf{v}^2)^{1/2}. \quad (3.21)$$

b) $C_{>} \equiv \{(\mathbf{q}, \mathbf{v}) | \mathbf{v}^2 > 1\}$.

Proceeding in the same way, we get:

$$L(\mathbf{q}, \mathbf{v}) = m(\mathbf{v}^2 - 1). \quad (3.22)$$

which corresponds to the so-called *tachions*.

The Lagrangians (3.21) and (3.22) are non-trivial *iff* $m \in R^*$. Thus, the general solution of our problem is (3.9) where the one-particle Lagrangians are of the type (3.21) or (3.22) and L is defined on a suitable open subset of $T(Q)$.

Remark Apparently, we did not use completely (3.1), but only a consequence, namely (3.2). However, it is easy to prove that (1.6) implies $L_{X_H}\omega_L = 0$. Using now (2.7) and (2.9) it is clear that $L_{P_i}\omega_L = 0$ and $L_{J_i}\omega_L = 0$ are consequences of (3.2) and (1.6).

4. The manifest covariance condition in the Galilean relativity

1. If we work with the proper orthochronous Galilei group $G_+^\uparrow$ instead of $P_+^\uparrow$, it is easy to see that the argument in Section 1.5 leads to the following expressions for the boost generators:

$$X_{K_i} = - \sum_{a=1}^N \frac{\partial}{\partial v_{ai}}. \quad (4.1)$$

From now on it is convenient to separate the centre-of-mass motion, using the new coordinates:

$$\xi_1 \equiv \mathbf{q}_1 - \mathbf{q}_2, \dots, \xi_{N-1} \equiv \mathbf{q}_{N-1} - \mathbf{q}_N, \xi_N \equiv \frac{1}{N}(\mathbf{q}_1 + \dots + \mathbf{q}_N). \quad (4.2)$$

$$\rho_1 \equiv \mathbf{v}_1 - \mathbf{v}_2, \dots, \rho_{N-1} \equiv \mathbf{v}_{N-1} - \mathbf{v}_N, \rho_N \equiv \frac{1}{N}(\mathbf{v}_1 + \dots + \mathbf{v}_N). \quad (4.3)$$

Then (4.1) becomes:

$$X_{K_i} = -\frac{\partial}{\partial \rho_{ni}}. \quad (4.4)$$

Also (2.11) and (2.12) become:

$$X_{P_i} = -\frac{\partial}{\partial \xi_{ni}}. \quad (4.5)$$

$$X_{J_i} = -\epsilon_{ijk} \sum_{a=1}^N (\xi_{aj} \frac{\partial}{\partial \xi_{ak}} + \rho_{aj} \frac{\partial}{\partial \rho_{ak}}). \quad (4.6)$$

The relations (2.2)-(2.4),(2.6),(2.7),(2.9) and (2.10) are true for $G_+^\dagger$ also, but instead of (2.5) and (2.8) we have:

$$[X_{K_i}, X_{P_j}] = 0. \quad (4.7)$$

$$[X_{K_i}, X_{K_j}] = 0. \quad (4.8)$$

All the commutation relations not involving X_H are automatically satisfied by (4.4)-(4.6). It remains to determine X_H using (2.6),(2.7) and (2.9). We start with a generic form for X_H :

$$X_H = \sum_{a=1}^N (A_{ai} \frac{\partial}{\partial \xi_{ai}} + B_{ai} \frac{\partial}{\partial \rho_{ai}}). \quad (4.9)$$

with $\mathbf{A}_a$ and $\mathbf{B}_a$ smooth functions of $\xi \equiv (\xi_1, \dots, \xi_N)$ and $\rho \equiv (\rho_1, \dots, \rho_N)$.

- Imposing (2.9) we establish easily that $\mathbf{A}_a$ and $\mathbf{B}_a$ do not depend on ξ_N .
- Imposing (2.6) we get immediately that:

$$\frac{\partial A_{ai}}{\partial \rho_{Nj}} = \delta_{aN} \delta_{ij}, \quad \frac{\partial B_{ai}}{\partial \rho_{Nj}} = 0$$

for any $i, j = 1, 2, 3$. So, in fact X_H is of the form:

$$X_H = \sum_{a=1}^N (A_{ai} \frac{\partial}{\partial \xi_{ai}} + B_{ai} \frac{\partial}{\partial \rho_{ai}} + \rho_{Ni} \frac{\partial}{\partial \xi_{Ni}}). \quad (4.10)$$

where $\mathbf{A}_a$ and $\mathbf{B}_a$ are smooth functions of the variables: $\xi' \equiv (\xi_1, \dots, \xi_{N-1})$ and $\rho' \equiv (\rho_1, \dots, \rho_{N-1})$.

- Finally (2.7) is equivalent to the rotational covariance of the functions $\mathbf{A}_a$ and $\mathbf{B}_a$ appearing in (4.10).

2. We must now impose (1.6) and (1.7) (with $G_+^\dagger$ instead of $P_+^\dagger$). We start with (1.7) in the infinitesimal form:

$$L_{X_\xi} \omega_L = 0. \quad (4.11)$$

for any $\xi \in \text{Lie}(G_+^\dagger)$. It is easy to establish, using (4.4) that:

$$L_{X_{K_i}} \omega_L = 0$$

is equivalent to:

$$\frac{\partial}{\partial \rho_{Ni}} \left(\frac{\partial^2 L}{\partial \rho_{aj} \partial \rho_{bk}} \right) = 0. \quad (4.12)$$

and:

$$\frac{\partial}{\partial \rho_{Ni}} \left(\frac{\partial^2 L}{\partial \rho_{aj} \partial \xi_{bk}} - \frac{\partial^2 L}{\partial \rho_{bk} \partial \xi_{aj}} \right) = 0. \quad (4.13)$$

Also,

$$L_{X_{P_i}} \omega_L = 0$$

is equivalent to:

$$\frac{\partial}{\partial \xi_{Ni}} \left(\frac{\partial^2 L}{\partial \rho_{aj} \partial \rho_{bk}} \right) = 0. \quad (4.14)$$

and:

$$\frac{\partial}{\partial \xi_{Ni}} \left(\frac{\partial^2 L}{\partial \rho_{aj} \partial \xi_{bk}} - \frac{\partial^2 L}{\partial \rho_{bk} \partial \xi_{aj}} \right) = 0. \quad (4.15)$$

The relations (4.12)-(4.15) are valid for any $a, b = 1, \dots, N$ and any $i, j, k = 1, 2, 3$, and tell us that the expressions $\frac{\partial^2 L}{\partial \rho_{aj} \partial \rho_{bk}}$ and $\frac{\partial^2 L}{\partial \rho_{aj} \partial \xi_{bk}} - \frac{\partial^2 L}{\partial \rho_{bk} \partial \xi_{aj}}$ depend only on the variables ξ' and ρ' .

Repeated applications of the Frobenius theorem show that in this case L can be taken of the following form:

$$\begin{aligned} L(\xi, \rho) = & \frac{1}{2} M_{ij} \rho_{Ni} \rho_{Nj} + \sum_{a=1}^{N-1} \frac{\partial N_i}{\partial \xi_{aj}}(\xi') \rho_{Ni} \rho_{aj} + \frac{1}{2} C_{ij} \rho_{Ni} \xi_{Nj} + \\ & \sum_{a=1}^{N-1} \frac{\partial L_i}{\partial \xi_{aj}}(\xi') \rho_{aj} \xi_{Ni} - V(\xi', \rho') + l(\xi). \end{aligned} \quad (4.16)$$

Here M and C are constant matrices, N_i and L_i are smooth functions of ξ' , V is a smooth function of ξ' and ρ' , and l is a smooth function of ξ . Also, M is a symmetric matrix and C is an antisymmetric matrix. The computations are long but elementary, so we do not give them here.

Using $L_{J_i} \omega_L = 0$ we can establish that M_{ij} and C_{ij} are rotational invariant tensors, so we must have: $C = 0$ and $M_{ij} = M \delta_{ij}$. Also N_i and L_i are rotational covariant functions, and one can choose V and l to be rotational invariant.

3. We use now (1.6). Using (4.10), elementary computations show that (1.6) is equivalent to the following set of relations:

$$\sum_{b=1}^N \left(\frac{\partial L}{\partial \rho_{bj}} \frac{\partial A_{bj}}{\partial \xi_{ai}} + \frac{\partial^2 L}{\partial \rho_{ai} \partial \xi_{bj}} A_{bj} + \frac{\partial^2 L}{\partial \rho_{ai} \partial \rho_{bj}} B_{bj} \right) = \frac{\partial L}{\partial \xi_{Ni}}. \quad (4.17)$$

$$\sum_{b=1}^N \left(\frac{\partial^2 L}{\partial \rho_{Ni} \partial \xi_{bj}} A_{bj} + \frac{\partial^2 L}{\partial \rho_{Ni} \partial \rho_{bj}} B_{bj} \right) = \frac{\partial L}{\partial \xi_{Ni}}. \quad (4.18)$$

$$\sum_{b=1}^N \frac{\partial L}{\partial \rho_{bj}} \frac{\partial A_{bj}}{\partial \rho_{ai}} = \frac{\partial L}{\partial \rho_{ai}}. \quad (4.19)$$

for any $a = 1, \dots, N-1$ and any $i = 1, 2, 3$. Inserting (4.16) into (4.18) we get easily that in fact l does not depend on ξ_N so it can be absorbed in V . Finally, adding to L a total derivative we get the expression:

$$L(\xi, \rho) = \frac{1}{2} M \rho_N^2 + \sum_{a=1}^{N-1} \left[\frac{\partial N_i}{\partial \xi_{aj}}(\xi') \rho_{aj} - L_i(\xi') \right] \rho_{Ni} - V(\xi', \rho'). \quad (4.20)$$

Inserting (4.20) into (4.17)-(4.19) we get a system of partial differential equations for the functions $\mathbf{A}_a$ and $\mathbf{B}_a$. From (4.20) it is clear that the $3N \times 3N$ matrix of entries $\frac{\partial^2 L}{\partial \rho_{ai} \partial \rho_{bj}}$ has the determinant M . So, if $M \neq 0$, this matrix is invertible. Then we can determine the functions $\mathbf{B}_a$ from (4.17) and (4.18) in terms of L and $\mathbf{A}_1, \dots, \mathbf{A}_N$ (which remain constrained only by rotational covariance and (4.19)). The case $M = 0$, when the Lagrangian is degenerate, seems to be more difficult to analyse.

5. Conclusions

We have succeeded in showing that the no-interaction theorem can be proved without a technical condition of non-degeneracy for the Lagrangean. This seems to indicate that this must also be true in the Hamiltonian analysis of the N-particle system [6-8], i.e. the non-degeneracy condition is not necessary.

For the Galilean case, the analysis seems to be new in the literature. It remains however to analyse completely the case $M = 0$. Also, even in the case $M \neq 0$, the fact that the functions $\mathbf{A}_1, \dots, \mathbf{A}_N$ remain quite arbitrary is rather bothersome. It would be desirable to eliminate this arbitrariness by imposing, perhaps, some additional physical conditions.

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References

1. A.P.Balachandran, G.Marmo, and A.Stern, *Nuovo Cimento* 69A(1982)175
2. G.Marmo, N.Mukunda, and E.C.G.Sudarshan, *Phys.Rev.D*30(1984)2110
3. D.G.Currie, *Phys.Rev.*142(1966)817
4. R.N.Hill, *Journ.Math.Phys.*8(1967)201
5. L.Bell, *Ann.Inst.H.Poincaré* 12(1970)307
6. D.G.Currie, T.F.Jordan, and E.C.G.Sudarshan, *Rev.Mod.Phys.*35(1963)350
7. J.T.Cannon, and T.F.Jordan, *Journ.Math.Phys.*5(1964)299
8. H.Leutwyler, *Nuovo Cim.*37(1965)556