ANOMALOUS DIMENSIONS IN MULTIPARTICLE COLLISIONS AND THE EMPTY BIN EFFECT

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Abstract

Using the random cascade model, we systematically analyze various conditions where realistic multiparticle distributions from cascading processes are affected by systematic and statistical biases at high resolution. We show, both analytically and numerically, that the effect of such conditions ("empty bin effect") is to produce, in some cases, a modification of the power-law of factorial moments as a function of the bin size that is, of the anomalous dimensions governing the dynamical fluctuations. We examine how these may influence the intermittency analysis of multiparticle data. Simulations based on the α -model parametrization of random cascading are suggested to take into account the empty bin effects in the intermittency analysis. A systematic comparison of the fluctuation effects of known distributions, including the log-normal, negative binomial and Lévy-stable laws is performed in terms of the anomalous dimensions, in view of their use as useful approximations.

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1 Introduction and Motivations

The phenomenon of intermittency observed in the spectra of particles produced in high-energy reactions has recently been related to various cascading processes [1]. This phenomenon is characterized [2, 3] by non-statistical fluctuations in particle multiplicities, and can be described by a power-law dependence of the factorial moments F_q of order q ,

$$F_q = \frac{\langle K_m(K_m - 1) \dots (K_m - q + 1) \rangle}{\langle K_m \rangle^q} \quad (1)$$

on the experimental resolution δy (i.e., the typical bin-size)

$$F_q(\delta y) = F_q(\Delta y) \left(\frac{\Delta y}{\delta y} \right)^{(q-1)d_q} \quad (2)$$

when $\delta y \rightarrow 0$. In these expressions, Δy denotes an initial scale (e.g. in rapidity), K_m is the multiplicity in the m -th bin, and $\langle \dots \rangle$ denotes the average over all events. The (positive) quantities d_q can be interpreted (for reviews on this, see [4, 5]) as the anomalous fractal dimensions of the particle fluctuation pattern and, together with the F_q 's, they are accessible to experimental measurements.

The relation (2) seems to describe reasonably well a large amount of experimental observations ranging from heavy-ion collisions to e^+e^- multiparticle distributions in a range of the variable δy (up to 0.1 if δy is the rapidity bin size). The apparent universality of the power-law (2) strongly suggests that intermittency is a rather general phenomenon occurring in complex systems like the ensemble of partons and/or hadrons created as a result of high-energy collisions.

Recently however, the improvements in detection techniques have made it possible to obtain data with resolution of the order of $\delta y = 0.01$ in rapidity. Parallel to this increase in resolution, various other quantities, such as the azimuthal angle, can be used together with the rapidity, to parametrize the multiplicity distributions. This leads to data analysis in higher dimensions, and some experimental groups have recently started to produce and analyze this type of data [1], showing a stronger and clearer signal of intermittency.

However, it is natural to ask whether the data will keep following a power-law like in (2) at high resolutions, or if significant deviations with respect to (2) are to be expected, in the framework of the cascading models.

There are strong indications, both experimental and theoretical, for suspecting that such deviations will occur. On the experimental side such deviations were observed at very high resolutions, leading some authors to the conclusion that intermittency ceased to be a relevant concept in the limit $\delta y \rightarrow 0$. However, it has been shown [6, 7, 8] that the projection of the data collected in three dimensions into one dimension has the effect of weakening the intermittency signal, that is, reducing the anomalous dimensions with respect to their values in the full dimensional case. As we will discuss, those deviations are actually also expected for very different statistical reasons

(even for unprojected intermittent models), to be observed using a model of multiparticle production based on self-similar cascades [2, 3].

The reason for this apparent contradiction lies in the fact that the increase in the experimental resolution has the consequence of reducing the statistics per bin.

The most dramatic effect of this reduction is an inflection of the factorial moments F_q towards zero, in the region of the highest resolutions attainable. This is easy to understand given the definition (1) of the F_q 's, and has already been observed [1], as we remarked above.

From formula (1), it is clear that the factorial moments are biased observables, since there always exists a rapidity window δy small enough so that the maximum number of particles K_{max} in that window will always be smaller than q , the rank of the moment. This obvious fact is of unexpected importance, since it will affect the moments of high rank, which are very sensitive to the dynamical origin of the fluctuations. For instance, the existence or not of a transient phase transition can be related to the high moments [9, 10]. So, the "empty-bin" effect involves a non-trivial intertwining of statistical and dynamical aspects, which deserves to be studied.

By "empty-bin effect" we will refer to these various effects, more or less related to limited statistics per bin, and which show up as deviations of the factorial moments' behaviour with respect to (2), even for a purely intermittent model.

For reasons of method, and after presenting the random cascading model in section 2, we will disentangle some of these effects and consider the action of each one of them separately, in section 3. A synthesis of their separate studies and the general conclusions on the use of anomalous dimensions of fluctuations are finally presented in section 4.

2 Random Cascading Models

This study will be particularly transparent if done in the framework of the cascading model. We will proceed in this direction, starting by reviewing it briefly.

According to this model, the final particle density ρ , measured with a resolution δy , relates to an initial density $\rho(\Delta y)$ by an N -step multiplicative random process

$$\rho(\delta y) = w(1) \dots w(N) \rho(\Delta y) \quad (3)$$

where $w(1), \dots, w(N)$ are independent (and positive) random quantities describing the effect of stepwise scale changes, from the k -th to the $(k+1)$ -th step ($k = 1, \dots, N-1$).

As is well known [2, 3], the normalized moments of the density $\rho(\delta y)$ are equal, in average, to the factorial moments (1) and lead exactly to the relation (2). One has:

For a given distribution $r(w)$ one can vary the number N of cascading steps in formula (3). Physically, this will correspond to different assumptions on the dynamics underlying the intermittency, namely a limited (N small) or extended (N large) range for the scales at which intermittency shows up.

3 Analysis of the Empty-Bin Effects

Now we proceed by studying each “empty-bin” effect separately, taking as a case example, the random cascading models themselves:

i) Truncation :

The first consequence of the finite statistics can be easily understood in terms of the particle density ρ (say, along the rapidity axis) or better, in terms of its logarithm, for a given resolution δy , assuming it is everywhere positive:

$$\xi \equiv \ln \rho(\delta y) \quad (10)$$

The (normalized) distribution of ξ , $P_{thcor}(\xi)$, can be computed using the cascade model [12, 13], and a typical result is represented by the full line in fig. 1. One can tell the quality of $P_{thcor}(\xi)$ by comparing it with experimental distributions. This problem will be discussed later (see paragraph (iii)). Here we are interested in the consequences of the following fact: the comparison between the theoretical curve $P_{thcor}(\xi)$ and the experimental distribution $P_{real}(\xi)$ can only be made inside a finite interval

$$I_{CUT} \equiv (\xi_{min}, \xi_{max}) \quad (11)$$

of size

$$\Delta \xi = \xi_{max} - \xi_{min} \quad (12)$$

which is the range of statistics actually accessible to the experiment. We will denote this by “truncation”. It should be pointed out that the quantity $\exp(\Delta \xi)$ is approximately related to the maximal density of particles, $(K_m)_{max}$, observed in windows of bin-size δy . In fact, we have approximately

$$(K_m)_{max} \simeq \langle K_m \rangle e^{\xi_{max}} \quad (13)$$

and, since the distributions are generally nearly symmetric close to the maximum, this reduces to

$$\frac{(K_m)_{max}}{\langle K_m \rangle} \simeq e^{\Delta \xi / 2} \quad (14)$$

The definition of ξ_{min} and ξ_{max} is dictated by the “events” (i.e., occurrences of a certain value of ξ) with the minimal probability. In general this means one “event” out of a total of N_{events} . Hence ξ_{min} and ξ_{max} are those values of ξ where the distribution $P(\xi)$ reaches the level of $(N_{events})^{-1}$. For

$$\frac{(\rho(\delta y)^q)}{(\rho(\delta y))^q} \equiv F_q(\delta y) \equiv \frac{(\rho(\Delta y)^q)}{(\rho(\Delta y))^q} \left(\frac{\Delta y}{\delta y} \right)^{(q-1)d_q} \quad (4)$$

The actual bin multiplicities K_m , which are integers, can be determined from the bin densities ρ through the action of an additional noise. One of the advantages of the factorial moments is that they filter out this noise and the densities become directly accessible [2, 3]. Typically this noise is modelled by the Poisson distribution

$$\mathcal{P}(K_m) = \frac{(\Lambda \rho_m)^{K_m}}{K_m!} e^{-\Lambda \rho_m} \quad (5)$$

which we will use later on when performing some numerical simulations of the cascade. Here $\Lambda \rho_m$ is the average number of particles per bin, $\langle K_m \rangle$. Other distributions are to be used as alternatives to the Poissonian when there are constraints e.g. on the multiplicity. In particular the Bernoulli distribution seems to be a realistic model of the noise.

The self-similarity of the cascade means that the N random variables $w(i)$ have the same (normalized) distribution, $r(w)$. In order to avoid the trivial blow-up of the product of w 's in $\rho(\delta)$ due to the large value of N , and to focus the attention on the fluctuations around a given average, one imposes the condition

$$(w)_r = \int dw \, r(w) \, w = 1 \quad (6)$$

The anomalous fractal dimensions are then given by

$$d_q = \frac{1}{q-1} \frac{\ln \langle w^q \rangle_r}{\ln \lambda} \quad (7)$$

The average $\langle \dots \rangle_r$ is calculated with respect to the distribution $r(w)$ and λ measures the stepwise scale changes and is defined as

$$\lambda = \left(\frac{\Delta y}{\delta y} \right)^{1/N} \quad (8)$$

The analytical study of random multiplicative processes becomes particularly simple if we discretize the random variable $w(i)$ [11]. In this case we assume that $w(i)$ can have only two values, namely $w = \beta_+$ with probability u , and $w = \beta_-$ with probability $v = 1 - u$ (these are called α -models). From (7) one immediately gets

$$d_q = \frac{\ln \{u(\beta_+)^q + v(\beta_-)^q\}}{(q-1) \ln \lambda} \quad (9)$$

Although this result is exact and independent of N , its numerical calculation (by simulating the α -model) still demands a sufficient number of steps (N) in order to attain statistical significance. Indeed the comparison between the analytical values (9) and those obtained through realistic simulations will be done in the next section, together with the study of the truncation effects.

a distribution with a single maximum (as the ones we will be dealing with) this occurs only at the points $\xi_{\min}$ and $\xi_{\max}$.

Generally the interval measured by $\Delta\xi$ will be centred around the maximum of $P_{\text{theor}}(\xi)$, unless $P_{\text{theor}}(\xi)$ is very asymmetric, in which case the determination of $\xi_{\min}$ and $\xi_{\max}$ will be done by functional inversion of $P_{\text{theor}}(\xi)$.

Compared with the theoretical curve — which is normalized over its entire range — the experimental distribution is normalized over the interval I_{cut} . Any other normalization of the experimental distribution beyond this interval would involve some kind of additional knowledge (or bias) about the shape of the theoretical distribution, not supplied by the experimental data.

We will now extract some consequences arising from the existence of the finite cuts $\xi_{\min}$ and $\xi_{\max}$.

The quantity $\mathcal{N}_{\text{events}}$ depends in general on the bin (m) considered, and is measured by the total number of particles falling into that bin m , summed over all the events (i.e., high energy collisions). Often the dependence of $\mathcal{N}_{\text{events}}$ on the bin m can be very weak, in which case $\mathcal{N}_{\text{events}}$ is simply given by the product

$$\mathcal{N}_{\text{events}} = (\text{Number of Collisions}) * (\text{Number of Bins}) \quad (15)$$

This simplification occurs in the random cascade model which we now use. Thus, $\xi_{\min}$ and $\xi_{\max}$ measure the minimal and the maximal average particle density fluctuations per bin.

The effect of $\Delta\xi$, and therefore of the limited statistics, on the determination of the anomalous dimensions d_q using the formula (7) can be rather pronounced, and we estimate it by using a cascade with $N = 5$ steps and values for β_4 and u typical [2, 3] of one-dimensional studies of intermittency ($\beta_4 = 1.3$, $u = 0.2$). We considered cases with $\Delta\xi = 0.64$, 1.0 and 1.60 (or equivalently, $\mathcal{N}_{\text{events}} \simeq 5$, 10 and 10⁴), which correspond to maximal bin densities given by $(K_m)_{\max}/(K_m) \sim 1.3$, 1.6 and 2.2. We conclude that, with q fixed, the anomalous dimension d_q tends to increase with increasing $\Delta\xi$. The results are indicated in fig. 2 a) where, for each $\Delta\xi$, we used two expressions for $P_{\text{theor}}(\xi)$ — to be discussed in paragraph (iii) — and the theoretical curve (9), obtained as the limit $N \rightarrow \infty$.

We may further notice that the case $\Delta\xi = 0.64$ represents a rather severe truncation, whereas the interval I_{cut} with $\Delta\xi = 1.60$ includes, in this case, most of the statistics and thus it represents a weak truncation (see paragraph (ii)).

With longer cascades the qualitative behaviour of d_q is essentially the same as the one above. In fig. 2 b) we show the case with $N = 30$ steps and cuts at $\Delta\xi = 2.0$, 4.0 and 10.0, (or equivalently, $\mathcal{N}_{\text{events}} \simeq 5$, 10³ and $> 10^{10}$), corresponding roughly to $(K_m)_{\max}/(K_m) \simeq 3$, 7 and 150. In this case $\Delta\xi = 2.0$ represents a severe truncation over the dynamical fluctuations, whereas $\Delta\xi = 10.0$ is a weak one.

To be more precise, the definition (7) of the Rényi dimension itself, using the truncated distribution, becomes questionable for a severe truncation. Indeed, the power-law behaviour (2) is affected in such a case (see e.g. ref. [14, 15]).

The truncation effect can thus be studied directly on the factorial moments F_q , using a numerical simulation along the lines proposed in [14, 15]. The results are represented in fig. 3 a) and show the typical bending down of the F_q 's as the resolution increases ($\delta y \rightarrow 0$).

In the same figure we plot the results of a similar simulation, done however with no truncation, this meaning in practice that we did not impose any constraints on the average particle density per bin. We observe first of all no bending-down of the F_q 's at small δy . Instead, we have a large intermediate region where the $\ln F_q$'s grow almost linearly as expected, and for the smallest values of δy , the values of $\ln F_q$ again start increasing.

Figure 3 b) depicts the anomalous dimensions d_q versus $-\ln \delta y$, obtained from the definition (2), where we chose $F_q(\Delta y)$ as the value of F_q generated at an intermediate resolution (not represented) of figure 3 a).

The fact that d_q generally depends on the resolution δy is evident from the figures and should be considered as another realistic aspect of the model, since similar results have been reported by experimental groups [14], independently of the projection effect. In this sense the values of d_q determined theoretically (fig. 2) should be understood as some type of averages of δy -dependent parameters which themselves can be identified with true anomalous dimensions only in the limits $\delta y \rightarrow 0$ and infinite statistics.

The comparisons, such as in fig. 2, are thus typical of how truncation affects, in the average, the determination of the anomalous dimension when one assumes an exactly (i.e. unprojected) intermittent distribution. Note however, that simulations are necessary to analyze the resolution-dependent truncation effects, as is demonstrated by the discrepancies shown in fig. 3 b), between the theoretical determinations of d_q and those obtained from simulations. One possible interpretation is that a residual effect of the statistical fluctuations remains present in the simulation analysis.

ii) Saturation :

When the truncation is sufficiently mild (i.e., when $\Delta\xi$ is sufficiently large), further increases of $\Delta\xi$ will not increase the statistics significantly. Consequently, the anomalous dimensions d_q are expected to become relatively insensitive to $\Delta\xi$ when $\Delta\xi \geq \Delta\xi_{\text{sat}}$, where the saturation values $\Delta\xi_{\text{sat}}$ depend on q as well as on the remaining parameters of the system ($N, \beta_4, \beta_-, \dots$). Thus, increasing the statistics (number of events) will not help in making progress on the determination of the fluctuation patterns. This effect of saturation is easily measured in the random cascade model, and is represented in the figures 4 a) and 4 b) for $N = 5$ and $N = 30$, respectively. It appears as a flattening of the curves of d_q versus $\Delta\xi$, at fixed q and for sufficiently large $\Delta\xi$.

Notice that at the other extreme (i.e., small $\Delta\xi$), the d_q 's may even become negative. This results from the strong effect of bending-down in the F_q 's when the truncation is severe, as we mentioned earlier. In such cases, the average slope of the F_q 's versus $-\ln \delta y$ can easily become negative, leading

to negative values of the d_q 's, and thus becomes meaningless.

Comparing the figures 4 a) and 4 b), we can conclude that, although the saturation sets in at larger values of $\Delta\xi$ when N is large ($N = 30$), the values of $\Delta\xi_{sat}$ are smaller, in relative terms, for larger N .

Another conclusion from the figures 4 is that the saturation of d_q is reached at smaller $\Delta\xi_{sat}$ for smaller values of q . This makes sense since d_q 's with higher q 's, are derived from moments F_q 's which probe deeper into the tails of the distribution ($P_{theor}(\xi)$ or $P_{real}(\xi)$) and thus demand higher values of $\Delta\xi$ in order to reach significant statistical levels. Besides, the simulation performed for fig. 3 b) shows that real saturation could require higher statistics than for the theoretical "averaged" values, in particular for high rank moments, which are sensitive to exceptional fluctuations, including those of the statistical noise.

iii) Approximations :

Here we will compare the predictions for the factorial moments and for the anomalous dimensions, made using different analytical approximations, P_{theor} , to the real distribution, taken here to be P_{test} :

$$P_{test} \simeq P_{theor} = \begin{cases} P_{normal} \\ P_{NBD} \\ P_{Levy} \end{cases} \quad (16)$$

to be defined in the following. Note that P_{Levy} does not possess an explicit analytical expression.

In the random cascade model that we are considering, ξ is a sum of a large number (N) of random variables

$$\xi_i = \ln w(i) \quad (i = 1, \dots, N) \quad (17)$$

with common mean-value

$$\xi_0 \equiv \langle \ln w \rangle_r \quad (18)$$

and common variance

$$\sigma^2 = \langle (\ln w)^2 \rangle_r - \langle \ln w \rangle_r^2 \quad (19)$$

Then, by the central limit theorem, ξ is distributed according to the normal law

$$P_{normal}(\xi)d\xi = \frac{d\xi}{\sqrt{2\pi N\sigma^2}} \exp\left(-\frac{(\xi - N\xi_0)^2}{2N\sigma^2}\right) \quad (20)$$

This is one of the mostly used approximations to real multiplicity distributions [3]. When expressed in terms of the particle density ρ (see (10)) it is known as log-normal distribution.

By using the statistical independence of the random variables $w(i)$, the expression of d_q in terms of the moments $\langle w^q \rangle_r$ can be rewritten in terms of the variable ξ

$$\langle w^q \rangle_r \sim \langle \rho^q \rangle_r^{1/N} = (e^{q\xi_0})^{1/N} P_{theor} \quad (21)$$

The effect of these moments is to emphasize the importance of the tails of the distribution (20) by effectively shifting the position of its maximum in the positive direction. Such deformations are already present in the case of additive random processes, but they become exponentially stronger with multiplicative processes. It is thus crucial to control the shape of the tails that is, the regions of low statistics, when studying the higher moments.

To do this, we will systematically compare the statistical properties of two closely related distributions, namely the **Gaussian distribution** (20) in ξ , and the distribution resulting from an (nearly) exact summation of all the corrections to the Gaussian form of the central limit theorem [13]:

$$P_{test}(\xi)d\xi \sim d\xi e^{-\Phi(y)} \quad (22)$$

where

$$y = \frac{\xi - N\xi_0}{Nuv \ln(\beta_+/\beta_-)} \quad (23)$$

and

$$\Phi(y) = Nuv \left\{ \left(\frac{1}{u} - y \right) \ln(1 - uy) + \left(\frac{1}{v} + y \right) \ln(1 + vy) \right\} \quad (24)$$

It is easy to show that this second distribution (which we will call "test-distribution") reproduces the normal one above for values of ξ sufficiently close to ξ_0 . In general this corresponds to a fairly large interval around the average value ξ_0 . Furthermore (22) has the additional property of being analytic only over the finite interval (we assume $\beta_+ > \beta_-$)

$$I_{test} = (N\xi_0 - Nu \ln(\beta_+/\beta_-), N\xi_0 + Nv \ln(\beta_+/\beta_-)) \quad (25)$$

Thus the experimental cut I_{cut} must be contained inside I_{test} ¹. This means that the comparison between the statistical properties of (20) and

¹At this point we should make two remarks. The first one is that, at the limits of I_{test} , the distribution (22) becomes singular but remains finite for N finite. In fact, at ξ_{min} and ξ_{max} , P_{theor} becomes equal to v^N and u^N , respectively. Hence, from the statistical point of view, reaching one of these limits would not require infinite statistics (as would be the case if (22) vanished there). However outside I_{test} the distribution $P_{test}(\xi)$ acquires a non-zero imaginary part and its real part starts oscillating and can become negative.

The second remark is that (22) is not the exact sum of all corrections to the central limit theorem but still involves a (mild) approximation. Very near the limits of I_{test} this approximation becomes poorer and there (22) does not represent the (truly) exact distribution. However it deviates only slightly from the exact value and can be considered on its own in that region, after being normalized over the interval I_{test} .

of (22) has to be made inside I_{test} , even before the cuts (ξ_{min}, ξ_{max}) are considered. This creates an additional effect on the quantities derived from the log-normal distribution, which would be absent if its range were taken as usual as the interval $I_\infty \equiv (-\infty, +\infty)$. This effect comes mainly from the restriction of the shape of P_{normal} to I_{test} and its normalization over this interval ("Gaussian+Cut"). The comparison is shown in figs. 2 and summarized in fig. 5. From these figures one sees that the dependence of d_q on q , obtained from P_{test} , converges to the theoretical limit $d_q\{N = \infty\}$, given by eq. (9), faster than the d_q 's calculated using P_{normal} . Hence we suggest, for further studies, the use of P_{test} and the moments derived from it, to parametrize the data.

Another well-known approximation to particle multiplicities is given by the negative binomial distribution (NBD):

$$P_{NBD}(n) = \left\{ \begin{matrix} n+k-1 \\ n \end{matrix} \right\} \frac{(\bar{n}/k)^n}{(1+\bar{n}/k)^{n+k}} \quad (26)$$

where $\{\dots\}$ denote the usual binomial coefficients. Here n counts the multiplicity per bin, $\bar{n}$ is its average, and k is an additional parameter which roughly controls the width of the distribution P_{NBD} .

The parameter $\bar{n}$ depends on the overall normalization and can easily be eliminated from the problem by using the well-known property of the factorial moments computed with the NBD (see e.g., reference [16]):

$$\left(\frac{F_{q+1}}{F_q} \right)_{NBD} = 1 + \frac{q}{k} \quad (27)$$

Then, by comparing the moments (rather than the distributions)

$$(F_q)_{NBD} \simeq (F_q)_{test} \quad (28)$$

and using the relation

$$(F_q)_{test} = \langle \rho^q \rangle_{test} = \lambda^{N(q-1)d_q} \quad (29)$$

which can be deduced from the previous equations, we immediately get the predicted value of the parameter k , as a function of the anomalous dimensions

$$N(qd_{q+1} - (q-1)d_q) \ln \lambda \simeq \ln(1 + \frac{q}{k}) \quad (30)$$

The independence of k on the rank q was checked for different parameters of the α -model, and the results are indicated in fig. 6, where λ was chosen at $\lambda = 2$. If the NBD provided a good fit to the data (which is here represented by the test distribution), we would expect basically no dependence of k on the rank q . However, as the figure indicates, that is generally not the case, except in the case of small cascades ($N \simeq 5$). The figure also suggests that the truncation may introduce stability in the values of $k(q)$. Hence the NBD may still provide reasonable fits to strongly truncated data; however, they would mask the difference from the "true" distribution.

Finally, a different generalization of the central limit theorem was proposed [17], leading to Lévy-stable laws (P_{Levy}). In this case the anomalous dimensions d_q in random cascade models, can be written as

$$d_q = d_2 \frac{q^\mu - q}{(2^\mu - 2)(q - 1)} \quad (31)$$

which reduces to the Gaussian case (obtained from P_{normal}) when $\mu = 2$. Values of μ closer to zero were shown to describe well the q -dependence of d_q observed in heavy-ion collisions [17]. The explicit form of P_{Levy} is not known in general and has to be numerically determined. Nevertheless, as we show in Fig. 5, for some value(s) of the μ parameter, (31) provides values of d_q which coincide almost exactly with the theoretical ones given by (9). As Fig. 5 also suggests, for N finite (e.g. $N = 30$), an adequate choice of μ may well lead to d_q 's which are closer to $d_q\{test\}$ and $d_q\{N = \infty\}$ than those obtained from the Gaussian distribution P_{normal} . Based on this, we conjecture that the Lévy-law result (31), when adequately tuned for μ , improves over the Gaussian estimates of d_q , even for finite statistics (i.e., with truncation). Hence, and similarly to P_{test} , we also suggest, for further studies, the use of approximation schemes using P_{Levy} implemented numerically. The relations (if any) between P_{test} and P_{Levy} remain to be clarified.

iv) Dimension/Projection :

In higher dimensions (i.e., with more than one variable to parametrize the multiplicity distributions), the data can be substantially diluted and consequently the statistics per bin is further reduced. This can have consequences for the F_q 's and the d_q 's which should be qualitatively analogous to some of the effects considered in the previous paragraphs.

Besides this effect of dilution, another one takes place, which specifically concerns the dependency of intermittency on the number of dimensions. It was suggested [7, 8] that systems exhibiting strong intermittency in several variables may behave like weak intermittent systems once that number of variables is reduced, and even break the effective intermittent behaviour at high resolution.

Qualitatively, the effect is due to the averaging associated with the integration of the distributions over some of the variables. This implies that, under identical conditions, the anomalous dimensions d_q at higher dimensionality will tend to be larger than those derived from distributions containing a smaller number of parameters. This is illustrated in the figures 7 a) and 7 b) where we computed d_q for various values of q and for various truncations $\Delta\xi$. We took values for β_4 and u typical of two-dimensional intermittency [7, 8] ($\beta_4 = 1.74$ and $u = 0.32$).

We observe the same qualitative behaviour as for one-dimensional intermittency. Quantitatively however, the values of d_q tend to be larger than in one dimension (stronger intermittency), as expected.

4 Synthesis

The study of the anomalous dimensions of fluctuations in multiparticle collisions has revealed various aspects of interest in the search for appropriate parametrizations of the intermittency phenomenon. Let us make an attempt to reach a synthetic view on the studies performed in the previous sections, and point out the main lessons for further phenomenological investigations.

Parametrizations of intermittency patterns:

The comparison of anomalous dimensions has been performed for three main type of distributions, namely, the log-normal ($P_{\text{log-normal}}$), the negative binomial (P_{NBD}) and the Lévy-stable ($P_{\text{Lévy}}$) laws. A fourth distribution (P_{test}) was used to produce reference values for these anomalous dimensions. Concerning their ability to reproducing the features of random cascading models, considered as the prototype of physical intermittency, the following conclusions have been reached:

- i) The log-normal and the negative binomial approximations are better for small number of cascading steps (N), while the Lévy-law approximation is better for long cascades (N large).
- ii) The anomalous dimensions allow one to confront two different noticeable aspects of multiparticle production, namely the relevance of a negative binomial parametrization of multiplicity distributions [18] and the intermittency phenomena. The question is whether a negative binomial distribution can parametrize a given pattern of intermittent fluctuations and, if so, for which value of the NBD parameter k . The result of our analysis is independent of the other NBD parameter (the average multiplicity) and is positive only for small cascades and rather weak intermittent behaviour.

Application to the empty-bin effect:

The analysis of fluctuations in experimental data on multiplicity distributions has proven not to be very easy. The damping of statistical fluctuations has been made possible by the use of factorial observables and revealed novel aspects of multiparticle dynamics, in particular the intermittency phenomenon. However, limitations are due both to statistical and systematic problems, among which the empty-bin effect plays an important role. In particular, this drawback is found when looking for fluctuations in very small phase-space windows and/or moderate overall multiplicity. Using the tools provided by the anomalous dimensions, we reached the following conclusions:

- i) In regions where the empty-bin effect is stronger, the normalization of the anomalous dimensions is lost. However, the dependence of the anomalous dimensions on the rank can be preserved. In other words, the strength of the intermittency singularity [8] is masked, but the fractal or multi-fractal behaviour of the fluctuations remains accessible to analysis. This point is indeed important in order to decide on whether a phase transition or a genuine

cascading process is the origin of intermittency [17, 19].

- ii) As previously mentioned, $P_{\text{log-normal}}$ and P_{NBD} are suitable approximations for small cascades, and $P_{\text{Lévy}}$ for long ones. However, this last type of approximation seems more promising in applications to studies of real data. In fact, we used only a restrictive version of the Lévy-law distribution without realistic truncation. It is thus expected that a numerical determination of $P_{\text{Lévy}}$ will provide an interesting tool. Moreover, P_{test} and $P_{\text{Lévy}}$ are the only approximations considered which are valid for large number of steps², which are present only when high level fluctuations are important. This is also crucial in higher-dimensional studies of intermittency, where the strength of the fluctuations is larger.

- iii) As is often the case in problems of intermittency, simulations are required for checking the validity of the analytical results. In our analysis, the simulations have shown the approximate validity of averaged anomalous dimensions, independent of the resolution scale, to describe the truncation effects. However, in the case of the empty-bin effect, it is clear that the analytically determined averaged anomalous dimensions, do not reflect all the truncation effects, as we could see e.g. in fig. 3 b). Indeed, our study suggests a better assumption on the statistical noise than the truncated Poisson probability distribution currently used [14, 15], namely the Bernoulli distribution with a maximal multiplicity per bin given by the experimental limitations.

The analysis with Bernoulli noise remains to be done. It should be noted that in the numerical study of 2-D phase transitions, a similar type of assumption [19] leads to remarkable results [20] which, in this case, can be confronted with precise predictions [21]

²Compared with the log-normal distribution, P_{test} provides a better approximation even for small number of steps (see text).

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Figure Captions

Fig. 1: The effect of truncation resulting from finite statistics.

Shape of a theoretical distribution $P_{\text{theor}}(\xi)$ and the limits ($\xi_{\min}$ and $\xi_{\max}$) within which it can be compared to statistical distributions obtained from a finite number of events (N_{events}). For illustration we chose $N_{\text{events}} = 1000$ events ($\equiv$ No. Collisions * No. Bins).

Fig. 2: Dependence of the anomalous dimensions on the rank q and on the statistics, for short and longer cascades (1-D intermittency).

a) Variation of the anomalous dimension d_q as a function of q for three different statistical cuts: $\Delta\xi = 0.64, 1.00, 1.60$. The full lines represent the values ($d_q\{\text{test}\}$) obtained using the test-distribution (22) and the dashed lines ($d_q\{\text{normal}\}$) were obtained from the normal distribution (20). In both cases (test and normal) we used the definition (7) of d_q in terms of the moments (w^q), which, in turn, were related to the moments of ξ using the relation (21).

For small $\Delta\xi$'s the two lines almost coincide, reflecting the fact that small $\Delta\xi$'s are associated with intervals in which the test-distribution is indistinguishable from the normal one. For larger $\Delta\xi$'s on the other hand, the curves differ.

The curves were obtained with $N = 5$, $\beta_+ = 1.3$ and $\beta_- = 0.925$ (corresponding to forward and backward probabilities equal to $u = 0.2$ and $v = 0.8$ respectively).

b) The same as fig. 2 a), but with cuts at $\Delta\xi = 2.0, 4.0, 10.0$, and a longer cascade with $N = 30$ steps.

Figs. 3: Numerical simulation of the random cascade model. Factorial moments and anomalous dimensions.

a) Dependence of the factorial moments F_q ($q = 2, 3, 4, 5$) on the resolution δy , for a typical one-dimensional intermittent system ($\beta_+ = 1.30$ and $\beta_- = 0.925$) with $N = 5$ steps along the cascade. The full lines represent a case with essentially no truncation in $\Delta\xi$ (maximal and minimal multiplicities per bin chosen at 161 and 1, respectively). The dashed lines represent a truncated case with maximal and minimal multiplicities per bin equal to 5 and 1, respectively, corresponding to a truncation of $\Delta\xi \simeq 1.8$. The effect of the truncation is clear in the bending-down of the F_q 's at the highest resolutions. In both cases, the number of simulations is 1600 and the smallest

bin size is equal to $\delta y_{\min} = 0.04$.

b) Variation of the anomalous dimensions d_q ($q = 2, 3, 4, 5$) on the resolution δy . The conditions chosen are those of a). Dashed (full) lines represent the case with (no) truncation. The horizontal lines indicate in each case the values obtained using the distribution P_{theor} , which were already plotted in figure 2. We see that the discrepancy between the numerical values and those of figure 2 is smallest at intermediate resolutions and, for each resolution, it increases with the rank q . Finally, we chose the arbitrary scale Δy at intermediate resolutions, where the measurement of d_q can lead to values closer to their theoretical definition (2). The vertical bars represent numerical fluctuations and not the real estimated errors.

Fig. 4: Saturation of the d_q 's with increasing statistics.

a) Saturation of the anomalous dimensions d_q 's appears as a flattening of the d_q 's versus $\Delta\xi$ near the largest values of $\Delta\xi$. For simplicity, we only consider the cases with $q = 2$ and 6. The full lines represent the values ($d_q\{\text{test}\}$) obtained using the test-distribution (22) and the dashed lines ($d_q\{\text{normal}\}$) were obtained from the normal distribution (20).

For small $\Delta\xi$'s the two lines coincide, reflecting the fact that small $\Delta\xi$'s are associated with intervals in which the exact distribution is indistinguishable from the normal one. For larger $\Delta\xi$'s on the other hand, the curves diverge.

The curves were obtained with $N = 5$, $\beta_+ = 1.3$ and $\beta_- = 0.925$ (corresponding to forward and backward probabilities equal to $u = 0.2$ and $v = 0.8$ respectively). In the upper horizontal scale we translated the values of $\Delta\xi$ into statistical levels N_{events} (see text and Fig. 1), computed from the test-distribution (22) normalized over the interval I_{test} .

b) The same as Fig. 4 a), but with $N = 30$.

Fig. 5: Analytical approximations of d_q , for N finite.

Comparing different analytical approximations to d_q at finite N , with the exact curve given by the α -model (9). The parameters are $\beta_+ = 1.3$ and $\beta_- = 0.925$ (1-D intermittency).

i) $N = \infty$ theoretical curve (9) (dotted line).

ii) Test distribution with N finite (full line). It is evident from this figure that this converges to the theoretical curve when $N \rightarrow \infty$. The normalization of P_{test} was done over the interval I_{cut} of size $\Delta\xi$, and $\Delta\xi$ itself was chosen to be saturating in each case, that is, $\Delta\xi \simeq 1.6$ for $N = 5$ and $\Delta\xi \simeq 10$ for $N = 30$.

iii) Gaussian approximation (dashed line) normalized in such a way that the approximated value of d_2 coincides with the one obtained from the test distribution (d_2 -normalization). Contrary to the case ii), its shape and slope do not converge to the theoretical limit as N increases from $N = 5$ to $N = 30$.

iv) Lévy-law approximation (dashed-dotted lines). As discussed in the text, the shape of this approximation can be tuned by means of a parameter μ . On the other hand no *CUT* is implemented on it (see text). We show this approximation when $\mu = 0.659$ (squares), $\mu = 1.801$ (circles) and $\mu = 2.176$ (stars). This last value of μ makes the Lévy approximation practically exact, as we can see from the figure, where the stars lie on top of $N = \infty$ theoretical line.

Fig. 6: Comparing the negative binomial and the test distributions.

The anomalous dimensions can be used to compare the predictions of the negative binomial distribution (*NBD*) with data or with predictions from other distributions. In this figure we compare them with those obtained from the test-distribution (see text). We used the parameter k of the *NBD* and plot its variation as a function of the rank q of the factorial moments F_q . A good description of the moments by the *NBD* should yield values of k almost insensitive to q . We observe that this approximately happens only for small cascades. We consider cases with and without truncation.

Notice that the truncation (curves 3 and 3') generally improves the quality of the fit with the *NBD*.

The parameters were chosen with the following values:

Line 1 : $\beta_+ = 1.3$, $\beta_- = 0.925$, $N = 5$, $\Delta\xi = 1.6$ (saturation).

Line 2 : $\beta_+ = 1.3$, $\beta_- = 0.925$, $N = 5$, $\Delta\xi = 0.85$ (truncation).

Line 3 : $\beta_+ = 1.74$, $\beta_- = 0.65$, $N = 5$, $\Delta\xi = 4.93$ (saturation).

Line 1' : $\beta_+ = 1.3$, $\beta_- = 0.925$, $N = 30$, $\Delta\xi = 10.0$ (saturation).

Line 2' : $\beta_+ = 1.3$, $\beta_- = 0.925$, $N = 30$, $\Delta\xi = 4.0$ (truncation).

Line 3' : $\beta_+ = 1.74$, $\beta_- = 0.65$, $N = 30$, $\Delta\xi = 29.5$ (saturation).

Lines 3 and 3' correspond to 2-D intermittency.

Fig. 7: Anomalous dimensions versus rank q for intermittency in two dimensions (to be compared with figure 2).

a) Variation of the anomalous dimension d_q as a function of q for two different statistical cuts: $\Delta\xi = 4.0$, 4.9 . The full lines represent the values ($d_q\{\text{test}\}$) obtained using the test-distribution (22) and the dashed lines ($d_q\{\text{normal}\}$) were obtained from the normal distribution (20). In both cases (test and normal) we used the definition (7) of d_q in terms of the moments

$(w^q)_r$ which, in turn, were related to the moments of ξ using the relation (21).

The curves were obtained with $N = 5$, $\beta_+ = 1.74$ and $\beta_- = 0.65$ (corresponding to forward and backward probabilities equal to $u = 0.32$ and $v = 0.68$ respectively). The curves are not normalized with respect to d_2 .

b) The same as fig. 7 a), but with cuts at $\Delta\xi = 20.0$, 29.5 , and a longer cascade with $N = 30$ steps.

FIG. 2

FIG. 1

FIG. 3 a)

FIG. 3 b)

FIG. 4

FIG. 5

FIG. 7

FIG. 6