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The Effective Action in $2 + 1$ Dimensional Gravity and Generalized BF Topological Field Theory

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Abstract

The one-loop off-shell effective action is studied for the case of generalized BF theories in three-dimensions, including, for example, $2 + 1$ dimensional gravity with a cosmological constant. The phase contribution to the effective action, originating from the η -function of a particular first order operator, is calculated using a momentum space technique. It is found that the η -function is proportional to the classical action.

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1 Introduction

In the study of three-dimensional topological field theory, a primary role is played by Chern-Simons theory [1, 2]. In particular, it gives a three-dimensional viewpoint for studying knot invariants and conformal field theories [2]. Quite distinct from this is the observation that 2+1 dimensional gravity can also be viewed as a Chern-Simons theory, when the gauge group is taken to be the Poincaré group $ISO(2,1)$ [3]. However, Chern-Simons theory with this gauge group is in fact a particular example of a so-called BF theory [4, 5, 6] with gauge group $SO(2,1)$; such a theory is described by a classical action $\int tr BF_A$.

In order to study 2+1 dimensional gravity with a cosmological constant λ , one is required to deal with a generalized BF theory, in which the action is of the form $\int tr(BF_A + \frac{\lambda}{3}B^3)$, and the gauge group is again $SO(2,1)$ [3].

Various studies have been made of the effective action in these different models [7, 8]. Our main concern here is with the generalized BF theory, with gauge group G . In particular, it was claimed in [7] that a Chern-Simons type term was induced at the 1-loop level. We provide an independent evaluation of the 1-loop off-shell effective action for this theory; the calculation amounts to an evaluation of the η -function of a certain first order operator which appears in a background field expansion. We find that the η -function is proportional to the original classical action $\int tr(BF_A + \frac{\lambda}{3}B^3)$, in disagreement with the results of [7].

In the following section we briefly review the calculational method, and then present the results. We conclude with a discussion and give general arguments for why the effective action should have the structure presented here.

2 Background

The main object of interest is the path integral

$$Z = \int d\phi e^{i \int \phi H \phi} , \quad (1)$$

where ϕ is a generic field and H is a first order elliptic Hermitian operator (which in our case will depend on the background fields). To derive a regulated expression for (1) we can assume that the fields are defined on some compact manifold so that the spectrum of H is discrete. A standard approach to defining Z_{reg} is to decompose the fields ϕ into eigenfunctions of H , as follows [9]:

$$\phi(x) = \sum_n a_n \phi_n(x) \quad , \quad H \phi_n(x) = \lambda_n \phi_n(x) \quad . \quad (2)$$

The measure $d\phi$ is then taken to be $\prod_n \frac{da_n}{\sqrt{\pi}}$, and, with the eigenfunctions appropriately normalized $(\phi_n, \phi_m) = \delta_{n,m}$, Z becomes

$$Z = \prod_n \int_{-\infty}^{\infty} \frac{da_n}{\sqrt{\pi}} e^{ia_n^2 \lambda_n} = \prod_n \frac{1}{|\sqrt{\lambda_n}|} e^{\frac{i\pi}{4} \text{sign} \lambda_n} \quad . \quad (3)$$

To regularize this expression we define the ζ and η functions of the operator H :

$$\begin{aligned} \zeta_H(s) &= \sum_n |\lambda_n|^{-s} \quad , \\ \eta_H(s) &= \sum_n (\text{sign} \lambda_n) |\lambda_n|^{-s} \quad . \end{aligned} \quad (4)$$

It then follows that

$$Z_{reg} = e^{\frac{1}{2}\zeta'(0) + \frac{i\pi}{4}\eta(0)} \quad . \quad (5)$$

It is the η -function evaluated at $s = 0$ which is of primary concern here, as it represents the possibility of a non-zero phase contribution to the partition function. To examine this η -function in more detail, it is most convenient to begin with the following integral representation [10]

$$\eta_H(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^{\infty} dt \, t^{(s-1)/2} \text{Tr}[H e^{-H^2 t}] \quad . \quad (6)$$

Although $\eta_H(s)$ is, in general, difficult to calculate, it is often possible to evaluate $\eta_H(0)$ which is, in any case, the object of interest. The important point to be stressed here is that $\eta_H(0)$ depends on the sign (but not the scale) of H .

Following Gilkey [10], one can introduce a 1-parameter family of operators $H(\lambda)$ such that $H(1) = H$ and $H(0)$ has a kernel, i.e. $\text{Tr}[He^{-H^2 t}]$, which can easily be evaluated. By differentiating with respect to λ , we find

$$\frac{d\eta_{H(\lambda)}(s)}{d\lambda} = \frac{-s}{\Gamma(\frac{s+1}{2})} \int_0^\infty dt t^{(s-1)/2} \text{Tr}[\frac{dH(\lambda)}{d\lambda} e^{-H^2(\lambda)t}] . \quad (7)$$

This shows that the $\lim_{s \rightarrow 0}$ is given by integrating $(\int_0^1 d\lambda)$ the residue of the pole at $s = 0$ in the above t integral. By writing

$$M \equiv H^2(\lambda) = M_0 + M_1 , \quad (8)$$

one can then make use of a Schwinger expansion [11] of e^{-Mt} :

$$e^{-Mt} = \sum_n K_n(t) , \quad (9)$$

where

$$\begin{aligned} K_0(t) &= e^{-M_0 t} , \\ K_1(t) &= -t \int_0^1 du e^{-M_0(1-u)t} M_1 e^{-M_0 u t} , \\ K_2(t) &= t^2 \int_0^1 u du \int_0^1 dv e^{-M_0(1-u)t} M_1 e^{-M_0(1-v)ut} M_1 e^{-M_0 v t} , \\ K_{n+1}(t) &= -t \int_0^t dt' e^{-M_0(t-t')} M_1 K_n(t') , \end{aligned} \quad (10)$$

Writing

$$H(\lambda) = H_0 + \lambda H_1 , M_0 = H_0^2 , M_1 = \lambda \{H_0, H_1\} + \lambda^2 H_1^2 , \quad (11)$$

we establish that the η -function at $s = 0$ is given by

$$\eta_H(0) = \eta_{H(0)}(0) + \lim_{s \rightarrow 0} \frac{-s}{\Gamma(\frac{s+1}{2})} \int_0^\infty dt t^{(s-1)/2} [\text{Tr}_0 + \text{Tr}_1 + \text{Tr}_2 + \dots] , \quad (12)$$

where

$$\begin{aligned} \text{Tr}_0 &= \text{Tr}[H_1 e^{-H_0^2 t}] , \\ \text{Tr}_1 &= -t \int_0^1 du \text{Tr}[e^{-H_0^2 u t} H_1 e^{-H_0^2(1-u)t} (\frac{1}{2}\{H_0, H_1\} + \frac{1}{3}H_1^3)] , \\ \text{Tr}_2 &= \frac{t^2}{3} \int_0^1 u du \int_0^1 dv \text{Tr}[e^{-H_0^2 u v t} H_1 e^{-H_0^2(1-u)t} \{H_0, H_1\} e^{-H_0^2 u(1-v)t} \{H_0, H_1\}] , \end{aligned} \quad (13)$$

It is most convenient to evaluate the functional traces in momentum space with the definition

$$\begin{aligned} \text{Tr}[\mathcal{O}] &= \int \frac{d^n p}{(2\pi)^n} \langle p | \text{Tr}' \mathcal{O} | p \rangle \\ &= \int \frac{d^n p}{(2\pi)^n} d^n x d^n y \langle p | x \rangle \langle x | \text{Tr}' \mathcal{O} | y \rangle \langle y | p \rangle . \end{aligned} \quad (14)$$

Here, the prime indicates a trace over any other indices carried by the operator $\mathcal{O}$. Our conventions are such that $\langle p | x \rangle = e^{-ipx}$ and $A(p) = \int d^n x e^{-ipx} A(x)$.

3 Application

The theory of interest is defined by the partition function [3, 7]

$$Z = \int dA_\alpha dB_\alpha \exp \frac{ik}{\sqrt{\lambda}} S , \quad (15)$$

with classical action given by

$$\begin{aligned} S &= \frac{1}{2\pi} \int \text{tr} [B \wedge F_A + \frac{1}{3} B \wedge B \wedge B] \\ &= \frac{1}{2\pi} \int d^3 x \epsilon^{\alpha\beta\gamma} \text{tr} [\frac{1}{2} B_\alpha F_{\beta\gamma} + \frac{1}{3} B_\alpha B_\beta B_\gamma] . \end{aligned} \quad (16)$$

Here, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$ is the curvature of the gauge connection A_α , $B = B_\alpha^a T^a dx^\alpha$ is a 1-form, and we are taking the theory to be defined on $\mathbf{R}^3$, where the momentum space procedure described above is valid. We are using the conventions where the structure constants are real and completely antisymmetric, with $[T^a, T^b] = f^{abc} T^c$. For the fundamental representation of $SU(n)$, the matrices T^a are skew-hermitian and we take $\text{tr} T^a T^b = -\frac{1}{2} \delta^{ab}$, while for the quadratic Casimir we have $f^{acd} f^{bcd} = c_v \delta^{ab}$. For the case of $SO(2, 1)$, we simply replace δ^{ab} with η^{ab} .

It is straightforward to establish the local symmetries of the above action, namely

$$\begin{aligned} \delta A_\alpha &= D_\alpha \omega + [B_\alpha, \theta] , \\ \delta B_\alpha &= D_\alpha \theta + [B_\alpha, \omega] . \end{aligned} \quad (17)$$

Now, for the purposes of a one-loop background field calculation, we shall decompose the fields into a background plus a quantum part as follows:

$$A \rightarrow A + A^q \equiv \hat{A} , B \rightarrow B + B^q \equiv \hat{B} , \quad (18)$$

where we are taking arbitrary background configurations which are not necessarily on-shell. The classical symmetry then takes the form

$$\begin{aligned} \delta A_\alpha &= D_\alpha \omega , \\ \delta A_\alpha^q &= [A_\alpha^q, \omega] + [B_\alpha^q, \theta] , \\ \delta B_\alpha &= D_\alpha \theta + [B_\alpha, \omega] , \\ \delta B_\alpha^q &= [B_\alpha^q, \omega] + [A_\alpha^q, \theta] , \end{aligned} \quad (19)$$

where it is important to note that the covariant derivative here is with respect to the background field. One should also note that the quantum fields transform linearly as vectors. Our aim now is to quantize the theory, and we wish to find a set of gauge fixing conditions which are covariant (i.e. they transform as vectors) with respect to (19). One such set is given by

$$\begin{aligned} G_1 &\equiv D \cdot A^q + [B_\alpha, B^{q\alpha}] , \\ G_2 &\equiv D \cdot B^q + [B_\alpha, A^{q\alpha}] , \end{aligned} \quad (20)$$

and indeed it can be checked that they transform as

$$\begin{aligned} \delta G_1 &= [G_1, \omega] + [G_2, \theta] , \\ \delta G_2 &= [G_2, \omega] + [G_1, \theta] . \end{aligned} \quad (21)$$

The quantum action is then given by

$$\begin{aligned} S_q &= S(\hat{A}, \hat{B}) + \frac{1}{2\pi\epsilon} \int d^3x \text{tr} \delta[\bar{c}(D_\alpha(\hat{A}^\alpha - A^\alpha) + [B_\alpha, \hat{B}^\alpha - B^\alpha]) \\ &\quad + \bar{c}'(D_\alpha(\hat{B}^\alpha - B^\alpha) + [B_\alpha, \hat{A}^\alpha - A^\alpha])] , \end{aligned} \quad (22)$$

with the off-shell nilpotent BRST transformations defined by

$$\begin{aligned} \delta A_\alpha^q &= \epsilon(\hat{D}_\alpha c + [\hat{B}_\alpha, c']) , \\ \delta B_\alpha^q &= \epsilon(\hat{D}_\alpha c' + [\hat{B}_\alpha, c]) , \end{aligned}$$

$$\begin{aligned}
\delta c &= -\frac{\epsilon}{2}(\{c, c\} + \{c', c'\}) , \\
\delta c' &= -\epsilon\{c, c'\} , \\
\delta \bar{c} &= \epsilon\phi , \delta\phi = 0 , \\
\delta \bar{c}' &= \epsilon\phi' , \delta\phi' = 0 , \\
\delta A_\alpha &= \delta B_\alpha = 0 .
\end{aligned} \tag{23}$$

In the above, c and c' are the ghosts corresponding to the ω and θ symmetries, while $\bar{c}, \phi$ and $\bar{c}', \phi'$ are the antighost-multiplier pairs. Here, $\hat{D}$ denotes the covariant derivative with respect to the full $\hat{A}$ field. The part of the action which is quadratic in the quantum fields (recalling that the ghosts and multipliers are purely quantum) is given by

$$\begin{aligned}
S_q^{(2)} &= \frac{1}{2\pi} \int d^3x \operatorname{tr} \epsilon^{\alpha\beta\gamma} (B_\alpha^q D_\beta A_\gamma^q + \frac{1}{2} B_\alpha [A_\beta^q, A_\gamma^q] + \frac{1}{2} B_\alpha [B_\beta^q, A_\gamma^q]) \\
&+ \phi(D \cdot A^q + [B_\alpha, B_\alpha^q]) + \phi'(D \cdot B^q + [B_\alpha, A_\alpha^q]) + \text{ghosts} .
\end{aligned} \tag{24}$$

We are concerned here with the first order operator $H_{ab}^{\alpha\beta}$ connecting the gauge fields and multipliers; specifically this is given by

$$S_q^{(2)} = \frac{1}{8\pi} \int d^3x \begin{pmatrix} B_\alpha^q & A_\alpha^q & \phi & \pi \end{pmatrix}^a H_{\alpha\beta}^{ab} \begin{pmatrix} B_\beta^q \\ A_\beta^q \\ \phi \\ \pi \end{pmatrix}^b , \tag{25}$$

where

$$H_{\alpha\beta}^{ab} = \begin{pmatrix} -\epsilon^{\alpha\gamma\beta} f^{acb} B_\gamma^c & -\epsilon^{\alpha\gamma\beta} D_\gamma^{ab} & f^{acb} B_\alpha^c & D_\alpha^{ab} \\ -\epsilon^{\alpha\gamma\beta} D_\gamma^{ab} & -\epsilon^{\alpha\gamma\beta} f^{acb} B_\gamma^c & D_\alpha^{ab} & f^{acb} B_\alpha^c \\ -f^{acb} B_\beta^c & -D_\beta^{ab} & 0 & 0 \\ -D_\beta^{ab} & -f^{acb} B_\beta^c & 0 & 0 \end{pmatrix} . \tag{26}$$

We can now proceed with the calculation. We find that the Tr_0 and Tr_2 terms in (13) do not contribute to $\eta_H(0)$, while the remaining terms give

$$\begin{aligned}
&Tr [e^{-H_0^2 u t} H_1 e^{-H_0^2 (1-u)t} \{H_0, H_1\}] \\
&= - \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} e^{-p^2 t - k^2 u(1-u)t} (16i c_v \epsilon^{\alpha\beta\gamma} B_\alpha^a(-k) k_\beta A_\gamma^a(k)) ,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
& Tr [e^{-H_0^2 u t} H_1 e^{-H_0^2 (1-u) t} H_1^2] \\
&= - \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 r}{(2\pi)^3} e^{-p^2 t - k^2 u (1-u) t} 4c_v f^{abc} \epsilon^{\alpha\beta\gamma} \times \\
&\quad (B_\alpha^a(k) B_\beta^b(-k-r) B_\gamma^c(r) + A_\alpha^a(k) B_\beta^b(-k-r) A_\gamma^c(r) \\
&\quad + A_\alpha^a(k) A_\beta^b(-k-r) B_\gamma^c(r) + B_\alpha^a(k) A_\beta^b(-k-r) A_\gamma^c(r)) . \quad (28)
\end{aligned}$$

Combining these results (and noting that $\eta_{H(0)}(0) = 0$) we find that the complete expression for $\eta_H(0)$ is

$$\eta_H(0) = \frac{4c_v}{\pi^2} \int d^3 x \epsilon^{\alpha\beta\gamma} tr \left(\frac{1}{2} B_\alpha F_{\beta\gamma} + \frac{1}{3} B_\alpha B_\beta B_\gamma \right) . \quad (29)$$

Using (5) and (15), we see that the phase contribution to the 1-loop effective action is

$$\Gamma^{(1)} = \left(\frac{k}{\sqrt{\lambda}} + sign\left(\frac{k}{\sqrt{\lambda}}\right) 2c_v \right) S , \quad (30)$$

where it is important to note the dependence of the 1-loop phase on the sign of the coupling $\frac{k}{\sqrt{\lambda}}$.

4 Discussion

We can establish the validity of the above computation by making the following observations: In addition to (16), we can define an action

$$S' = \frac{1}{4\pi} \int tr [AdA + \frac{2}{3} A^3 + Bd_A B] , \quad (31)$$

which also has the symmetry (17). Defining

$$A^\pm = A \pm B , \quad (32)$$

together with the corresponding Chern-Simons actions

$$S^\pm = \frac{1}{4\pi} \int tr [A^\pm dA^\pm + \frac{2}{3} (A^\pm)^3] , \quad (33)$$

we find

$$S^\pm = S' \pm S . \quad (34)$$

Thus

$$\frac{k}{\sqrt{\lambda}}S = \frac{k}{2\sqrt{\lambda}}(S^+ - S^-) . \quad (35)$$

We can now obtain the 1-loop effective action for S by invoking the known results from Chern-Simons theory. Given a Chern-Simons action kS_{CS} , the η -function contribution to the effective action is [2, 12, 13]

$$kS_{CS} \rightarrow [k + \text{sign}(k)c_v]S_{CS} . \quad (36)$$

Using this in (35), we obtain agreement with our explicit calculation (30). Furthermore, we can also consider a coupled theory with action

$$\frac{k}{\sqrt{\lambda}}S + k'S' = \frac{1}{2}\left(\frac{k}{\sqrt{\lambda}} + k'\right)S^+ + \frac{1}{2}\left(-\frac{k}{\sqrt{\lambda}} + k'\right)S^- , \quad (37)$$

It is then clear that the η -function phase contribution to the effective action will depend on the relative signs of the couplings, and for example, setting $k = 0$ we find the S' action has an induced S' phase. Alternatively, one may wish to explicitly check this along the lines presented above.

At this point, one may raise the question as to the gauge dependence of the above results. It is well known that the off-shell effective action is, in general, dependent on the gauge fixing condition chosen. The solution to this problem has been provided by Vilkovisky-DeWitt [14, 15], whereby a meaning can be assigned to the notion of a unique off-shell effective action. Within the Vilkovisky-DeWitt procedure, a particular role is played by the so-called “no Vilkovisky-DeWitt correction” gauge; such a gauge provides the unique off-shell result, and a computation in any other gauge will require the addition of the Vilkovisky-DeWitt correction term. For the case of Chern-Simons theory, it has been shown that the “no correction” gauge is the Landau gauge (which in background field terms is $D \cdot A^q = 0$) [12, 13]. Thus, viewing the action (16) from the Chern-Simons point of view (equation (35)) we see that the unique off-shell effective action requires use of the Landau gauges: $D^+ \cdot A_q^+ = D^- \cdot A_q^- = 0$, where $D^\pm$ is the covariant derivative with respect to the background field $A^\pm$. In fact, one can now see that this is precisely the gauge we have chosen (namely (20)) for our computation. It would be interesting to examine other possible gauge choices and to compute explicitly the Vilkovisky-DeWitt correction terms.

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