Self-Consistent Cosmological Perturbations from Thermal Field Theory

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Finite-temperature field theory is used for a self-consistent and manifestly gauge-invariant derivation of the effective field equations governing the evolution of adiabatic perturbations of a homogeneous, isotropic cosmological model dominated by an ultrarelativistic plasma of weakly interacting particles in thermal equilibrium. Solutions are obtained in the form of rapidly converging power-series expansions.

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The analysis of linear perturbations of homogeneous, isotropic cosmological models as initiated by Lifshitz [1] is one of the most important ingredients in the theory of the formation of large-scale structure in the Universe [2]. There one considers the interplay between metric perturbations and perturbations in the energy-momentum tensor. Prescribing the possible response of the energy-momentum tensor, for example, by assuming a perfect-fluid behavior, allows one to determine the evolution of cosmological perturbations. For less simple forms of matter one of course needs a more fundamental description. A refinement of this type of analysis which is based on classical kinetic theory [3] has been developed by Peebles and Yu [4].

In this Letter we shall employ thermal field theory [5] to obtain a closed set of equations for cosmological perturbations. Thermal field theory, which is well developed for the thermal equilibrium case that is usually studied, provides a fully quantum theoretical framework that is particularly useful when dealing with gauge field theories. In the present application, apart from possible gauge fields which might provide constituents for the primordial plasma, the metric field is a gauge field, and the field-theoretic approach will prove useful in a proper treatment of the gauge degrees of freedom.

Thermal field theory has also just recently proved its power in the case of high-temperature quantum chromodynamics, where higher-loop effects in the plasmon dispersion relations have been evaluated [6], outrivalling kinetic theory, and potential problems concerning gauge dependence could be settled [7].

In Ref. [8] one of the present authors has recently obtained the leading temperature corrections to the graviton self-energy at finite temperature, which is the relevant quantity describing the response of thermal matter to metric perturbations. However, the results of Ref. [8] are restricted to the local tangent space, whereas the expected collective phenomena are sensitive to the large-scale structure of a given cosmological model, and, moreover, the temperature was unrealistically kept fixed. Here we shall extend these results to a specific cosmological model and derive self-consistent and manifestly gauge-invariant

effective field equations governing the evolution of adiabatic perturbations.

As in most of the literature on cosmological perturbations, we shall restrict our attention to a spatially flat Friedmann-Robertson-Walker model with line element

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2 = \sigma(\tau) [d\tau^2 - d\mathbf{x}^2], \quad (1)$$

where the conformal time $\tau = \int a^{-1}(t) dt = R_H$ measures the size of the horizon in comoving coordinates. In the radiation-dominated regime which we shall consider, $\tau \propto t^{1/2}$. All conventional cosmological models are of this form at some early stage of their evolution.

The basic equations for linear perturbations are derived from the Einstein equations

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu}, \quad (2)$$

with $G^{\mu\nu} = \hat{G}^{\mu\nu} + \delta G^{\mu\nu}$, $T^{\mu\nu} = \hat{T}^{\mu\nu} + \delta T^{\mu\nu}$, and $\hat{T}^{\mu\nu} = \text{diag}[\rho, -p, -p, -p]$, $p = \rho/3$. A corresponding metric perturbation can be decomposed into three independent modes, scalar, vector, and tensor,

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}^{(S)} + \delta g_{\mu\nu}^{(V)} + \delta g_{\mu\nu}^{(T)}, \quad (3)$$

of which the scalar part is physically the most interesting one in view of gravitational instability, and, surprisingly, also the most intricate one with respect to gauge modes [9,10].

Parametrizing $\delta g_{\mu\nu}^{(S)}$ in terms of scalar potentials,

$$\delta g_{\mu\nu}^{(S)} = \sigma(\tau) \begin{pmatrix} C & D_{,i} \\ D_{,j} & A\delta_{ij} + B_{,ij} \end{pmatrix}, \quad (4)$$

any two of them can be gauged away by $\delta g_{\mu\nu} = \delta \xi_{\mu;\nu} + \delta \xi_{\nu;\mu}$ and appropriate functions $\delta \xi_0 = f$ and $\delta \xi_i = g_{,i}$. This leaves two independent, manifestly gauge-invariant amplitudes,

$$\Phi = A + \frac{\dot{\sigma}}{\sigma} \left(D - \frac{1}{2} \dot{B} \right), \quad (5)$$

$$\Pi = \frac{1}{2} \left(\ddot{B} + \frac{\dot{\sigma}}{\sigma} \dot{B} + C - A \right) - \dot{D} - \frac{\dot{\sigma}}{\sigma} D,$$

where the overdot denotes differentiation with respect to τ . Physically, these two gauge-invariant potentials correspond to perturbations in energy density and anisotropic stress, respectively. For a given spatial Fourier mode with wave vector $\mathbf{k}$,

$$\delta = \frac{1}{3} x^2 \Phi, \quad \pi_{\text{anis}} = \frac{1}{3} x^2 \Pi, \quad (6)$$

where we have introduced the scale-invariant variable $x \equiv k\tau = 2\pi R_H/\lambda$ which gives the ratio of the size of the Hubble horizon to the comoving wavelength of the perturbation. While the anisotropic pressure π_{anis} is an unambiguous quantity, $\delta = \delta\rho/\rho$ is the energy-density contrast when measured relative to a spacelike hypersurface representing everywhere the local rest frame of $T^{\mu\nu}$ [9].

With the assumption that both the unperturbed and the perturbed energy-momentum tensor have perfect-fluid form, i.e., $\Pi=0$, and that the perturbation is adiabatic, $\delta\rho/\rho = \delta p/p$, Eq. (2) already determines the evolution of a scalar cosmological perturbation through an ordinary differential equation for Φ [9],

$$\Phi'' + (4/x)\Phi' + \frac{1}{3}\Phi = 0, \quad (7)$$

where a prime denotes differentiation with respect to x . This equation is solved in terms of the spherical Bessel function j_1 by

$$\Phi = Cx^{-1}j_1(x/\sqrt{3}). \quad (8)$$

For $x \ll 1$, i.e., at a time when the wavelength of the perturbation is still larger than the growing horizon, there is a growing density perturbation, $\delta \sim x^2$, whereas for $x \gg 1$, a perturbation propagates with constant amplitude and the speed of sound $c_s^2 = \frac{1}{3}$.

Such a purely hydrodynamical analysis is relevant in the case of a highly collisional primordial plasma, when the mean free path of particles $l \ll \lambda$. Deviations become important when $l > \lambda$ for some constituents of the plas-

ma, e.g., weakly interacting neutrinos [11].

The ultrarelativistic regime $k \gg l^{-1}$, where collisionless processes dominate, is more complicated than the hydrodynamical limit, because the wave vector $\mathbf{k}$ of a perturbation breaks isotropy and therefore the plasma can support anisotropic pressure perturbations which change the coupling to the metric perturbations.

It is this ultrarelativistic limit where a finite-temperature field-theoretic treatment is most appropriate as this limit turns out to correspond technically to the high-temperature one. In the field-theory approach the form of $\delta T^{\mu\nu}$ can be *derived* from an effective action. The energy-momentum tensor is defined as the one-point function

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta\Gamma}{\delta g^{\mu\nu}}, \quad (9)$$

where Γ contains all contributions to the effective action besides the classical Einstein-Hilbert action. Self-consistency requires that

$$\delta T_{\mu\nu} = (\delta T_{\mu\nu}/\delta g^{\alpha\beta})\delta g^{\alpha\beta}, \quad (10)$$

hence $\delta T_{\mu\nu}$ is determined by the graviton self-energy

$$\Pi_{\mu\nu\alpha\beta} \equiv \frac{\delta^2\Gamma}{\delta g^{\mu\nu}\delta g^{\alpha\beta}} = \frac{1}{2} \frac{\delta(\sqrt{-g} T_{\mu\nu})}{\delta g^{\alpha\beta}}. \quad (11)$$

The leading temperature (one-loop) contributions to (11) with $T \gg k, k_0, m$ have been obtained in Ref. [8] in a local momentum-space representation, where it has been shown that their form is generally covariant and universal; i.e., apart from an overall factor it is the same for any (effectively massless) matter, including gauge fields and even gravitons, if thermalized. It can furthermore be shown that the leading temperature contributions to the effective action are conformally invariant, so in the present case of a conformally flat curved space (1), the global form of the graviton self-energy is given by

$$\Pi_{\mu\nu\alpha\beta}(x, y)|_{g=\sigma\eta} = \sigma(x) \int d^4k e^{-ik(x-y)} \tilde{\Pi}_{\mu\nu\alpha\beta}(k_0, \mathbf{k})|_{g=\eta} \sigma(y), \quad (12)$$

in terms of the momentum-space results of Ref. [8]. Because at finite temperature there is a preferred Lorentz frame, $\tilde{\Pi}_{\mu\nu\alpha\beta}$ can be decomposed into fourteen independent tensors and as many structure functions, which turn out to involve polynomials and logarithmic functions of $k_0/|\mathbf{k}|$ [8].

With (10), (11), and (12), self-consistent perturbations of the Einstein equations (2) on a conformally flat background are determined by

$$\left\{ \frac{1}{2} (g^{\alpha\beta} g^{\mu\nu} - g^{\alpha(\mu} g^{\nu)\beta}) \nabla^2 - \frac{1}{2} (g^{\alpha\beta} \nabla^\mu \nabla^\nu + g^{\mu\nu} \nabla^\alpha \nabla^\beta) + g^{\alpha(\mu} \nabla^{\nu)} \nabla^\beta + \frac{1}{12} g^{\alpha\beta} g^{\mu\nu} R - \frac{1}{3} g^{\alpha(\mu} g^{\nu)\beta} R \right. \\ \left. - 16\pi G \sigma^{-1} \tilde{\Pi}^{\alpha\beta\mu\nu}(i\partial_0, i\partial) |_{g=\eta} \sigma^{-1} \right\} \delta g_{\alpha\beta} = 0, \quad (13)$$

where brackets around indices denote symmetrization. Because the effective action Γ is a nonlocal object, the last term in (13) is, in general, a differential operator of infinite order.

Specializing to the background model (1) and to scalar metric perturbations (4), one finds that the latter appear in (13) only in gauge-invariant combinations as in Eq. (5), and that only two independent equations arise,

$$\Phi'' + \frac{4}{x}\Phi' + \frac{1}{3}\Phi = \frac{2}{3}\Pi - \frac{2}{x}\Pi', \quad (14)$$

which is a generalization of Eq. (7), to which it reduces if $\Pi \equiv 0$, and

$$(x^2 + 3)\Phi + 3x\Phi' = 6 \left[1 - 2 \frac{d}{dx} \operatorname{arccot} \left(\frac{d}{dx} \right) \right] (\Phi + \Pi), \quad (15)$$

where the formal infinite-order differential operator is defined by

$$\left[\frac{d}{dx} \operatorname{arccot} \left(\frac{d}{dx} \right) \right] (\Phi + \Pi) = \frac{1}{2\pi} \int d\omega e^{-i\omega x} \frac{\omega}{2} \ln \frac{\omega+1}{\omega-1} \int dx' e^{i\omega x'} (\Phi + \Pi)(x'). \quad (16)$$

Choosing the integration contour over ω such that analyticity holds in the upper half plane, which corresponds to retarded boundary conditions, the right-hand side of Eq. (16) can be evaluated and Eq. (15) becomes

$$(x^2 - 3)\Phi + 3x\Phi' = 6\Pi - 12 \int_0^x dx' j_0(x - x') (\Phi + \Pi)'(x') + c j_0(x), \quad (17)$$

where c parametrizes the initial conditions at $x=0 = \tau$. Requiring regularity at $\tau=0$ fixes $c = -3[\Phi(0) + 2\Pi(0)]$, and the system of the ordinary differential equation (14) and the integro-differential equation (17) can be solved by a power-series ansatz. These power series turn out to be rapidly convergent with coefficients alternating in sign and decreasing faster than factorials, and the first few terms are given by

$$\Phi(x) = C \left(7 - \frac{43}{63} \frac{x^2}{2!} + \frac{14249}{80115} \frac{x^4}{4!} \mp \dots \right), \quad (18)$$

$$\Pi(x) = C \left(-1 + \frac{13}{63} \frac{x^2}{2!} - \frac{857}{11445} \frac{x^4}{4!} \pm \dots \right).$$

FIG. 1. The evolution of density perturbations (solid line) and anisotropic pressure (dashed line) as a function of $x/\pi \equiv k\tau/\pi = R_H/(\lambda/2)$. The amplitudes are normalized so that the peak value of $\delta(x)$ is 1. The classical hydrodynamical solution for δ ($\pi_{\text{anis}} \equiv 0$ in that case) is given by the dot-dashed line for comparison.

In Figs. 1 and 2, $\delta \propto x^2 \Phi$ and $\pi_{\text{anis}} \propto x^2 \Pi$ are plotted for x ranging up to 10π , and in Fig. 1 the hydrodynamical solution (8) has been added for comparison. For small values of x , where $\lambda > R_H$, the familiar growth of the perturbation can be seen. $x/\pi = 1$ corresponds to the time when half of a wavelength of the perturbation fits inside the horizon. For larger values of x , where the horizon has grown larger than the wavelength, an oscillatory behavior arises which, in contrast to the hydrodynamical solution, is strongly damped, and the propagation velocity is about the speed of light rather than speed of sound. This phenomenon is due to the fact that a perturbation made out of nearly collisionless, effectively massless particles propagates with the velocity of their constituents, which is the speed of light, and collisionless damping occurs in the form of directional dispersion [11,12]. Neglecting the effects of gravity one can derive [12] $\delta \sim \sin(x)/x$ for that case, which indeed describes well

FIG. 2. Same as Fig. 1, but with a magnified ordinate.

the asymptotic behavior of the full solutions.

Our exact results [13] are also in good agreement with the results obtained previously by numerical integration of the coupled Einstein-Boltzmann equations [11]. In fact, the latter approach yields integro-differential equations like Eq. (17) which have been derived in the synchronous gauge [11] [$C=D=0$ in (4)] and in the longitudinal gauge [14,15] ($B=D=0$), and we have checked that their asymptotic behavior $x \rightarrow 0$ is consistent with (18). Our equations, on the other hand, are manifestly gauge invariant, and, gratifying, somewhat simpler.

We thus believe we have demonstrated the usefulness of a thermal-field-theory approach to cosmological perturbations. Further applications suggest themselves: Besides the inclusion of vector and tensor perturbations, the analysis could be extended to other cosmological models as well as to more-component systems.

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