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BOUND STATES AND SCATTERING SOLUTIONS FOR 3D GRAVITY

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Abstract

We investigate the behaviour of test particles in the presence of the massive source in three-dimensional gravity. Bound states and scattering solutions are obtained both at the classical level and as solutions of the Schrödinger equation. The results depend crucially on whether the mass of the source is larger or smaller than the Planck mass and, in the quantum case, on the total angular momentum of the system.

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1. Introduction

Three-dimensional (3D) gravity has recently attracted much attention ([1] and [2]) for two reasons - it is related to a topological theory ([3], [4], [5] and [6]) and it is probably finite at the quantum level ([3], [7]).

In this paper we want to investigate three-dimensional gravitational systems - for simplicity we consider the case of only one massive source (which is assumed to be static) and test particles with a small mass (sufficiently small to neglect its impact on the source). Part of the results overlap with [8] and [9] but they are rederived in isotropic coordinates which allow for simpler interpretation in case of large masses (larger than the Planck mass). The case of colliding strings is considered in [10].

This paper is divided into two parts. In the first part we investigate the classical trajectories of test particles in the presence of a massive source in 2+1 dimensions. In the second part, starting from the Schrödinger equation we find the bound states in the presence of a massive source. The results, as can be expected, depend crucially on the mass of the source. The case of a low mass (in our notation $GM < 1$ or $M < M_P$) is fairly well understood, while for bigger masses there are some problems with the interpretation. For the latter case we obtain interesting constraints on the existence of bound states.

It should be emphasized that in the absence of time-dependent multiparticle solutions of Einstein equations, we are using in this work the static solution. Hence the classical trajectories and the solutions of the Schrödinger equation described below are only first approximations to the full scattering and bound state solutions in 2+1-dimensional gravity.

2. Classical trajectories for a test particle

In this Section we want to use the stationary solutions to the Einstein equations in the presence of pointlike sources as obtained in [11] to investigate the classical behaviour of a test particle in the presence of such sources. We will consider the case of one static source. The metric in so-called isotropic coordinates reads

$$ds^2 = dt^2 - \left| \frac{\mathbf{r} - \mathbf{R}}{b} \right|^{-2GM} (dr^2 + r^2 d\theta^2) \quad (1)$$

where M is the mass and $\mathbf{R}$ is the position of the source.

In order to derive the Lagrangian from the metric in (1) we use the following form of the action:

$$S = -m \int ds \quad (2)$$

where ds^2 is given by (1). The Lagrangian is hence given by

$$L = -m \left(1 - v^2 \left| \frac{\mathbf{r} - \mathbf{R}}{b} \right|^{-2GM} \right)^{1/2} \quad (3)$$

Using (3) we can derive momenta and the Hamiltonian

$$\mathbf{p} = \frac{m\mathbf{v} \left| \frac{\mathbf{r} - \mathbf{R}}{b} \right|^{-2GM}}{\sqrt{1 - v^2 \left| \frac{\mathbf{r} - \mathbf{R}}{b} \right|^{-2GM}}} \quad (4)$$

$$H = \left(m^2 + p^2 \left| \frac{\mathbf{r} - \mathbf{R}}{b} \right|^{2GM} \right)^{1/2} \quad (5)$$

The interaction described by (3) or (5) is not of the potential form.

We assume $\mathbf{R} = 0$ from now on - we rewrite (3):

$$L = -m \left(1 - v^2 \left| \frac{\mathbf{r}}{b} \right|^{-2GM} \right)^{1/2}. \quad (6)$$

Because (6) is obviously independent of translations in time and of rotations, we can write down the conserved quantities: energy and total angular momentum.

$$E = m \left(1 - v^2 \left(\frac{r}{b} \right)^{-2GM} \right)^{-1/2}. \quad (7)$$

$$\mathbf{J} = \mathbf{r} \times \mathbf{p}. \quad (8)$$

where $r = |\mathbf{r}|$. In polar coordinates

$$J = Er^2 \dot{\theta} \left(\frac{r}{b} \right)^{-2GM}. \quad (9)$$

$$v^2 = \dot{r}^2 + r^2 \dot{\theta}^2. \quad (10)$$

Using (7), (9) and (10) we can write the differential equation for r :

$$\dot{r}^2 = \left(1 - \frac{m^2}{E^2} \right) \left(\frac{r}{b} \right)^{2GM} - \frac{J^2}{E^2 r^2} \left(\frac{r}{b} \right)^{4GM} \quad (11)$$

This equation can be solved for arbitrary GM :

$$\left(\frac{r}{b}\right) = \left[\frac{(1-GM)^2}{b^2 E^2} (E^2 - m^2)(t - t_0)^2 + \frac{J^2}{b^2(E^2 - m^2)} \right]^{\frac{1}{2(1-GM)}} \quad (12)$$

Plugging this solution into (9) and integrating we get

$$\theta - \theta_0 = \frac{1}{1-GM} \arctan \left[\frac{(1-GM)(E^2 - m^2)}{J E} (t - t_0) \right] \quad (13)$$

Combining (12) and (13) we get the trajectory

$$\left(\frac{r}{b}\right) = \left[\frac{J^2}{b^2(E^2 - m^2) \cos^2((1-GM)(\theta - \theta_0))} \right]^{\frac{1}{2(1-GM)}} \quad (14)$$

The case $GM = 1$ should be considered independently - the result reads:

$$r = C \exp \left(-\frac{1}{b^2 E} \sqrt{b^2(E^2 - m^2) - J^2} (t - t_0) \right) \quad (15)$$

$$\theta = \frac{J}{E b^2} (t - t_0) \quad (16)$$

$$r = C \exp \left(-\frac{1}{J} \sqrt{b^2(E^2 - m^2) - J^2} (\theta - \theta_0) \right) \quad (17)$$

We can see from (14) that the trajectories are open (the scattering case) for $GM \leq 1$ and closed (the bound state case) for $GM > 1$. The deflection angle for the scattering case is equal to

$$\theta_{defl} = \frac{\pi}{1-GM} - \pi = \frac{\pi GM}{1-GM} \quad (18)$$

When $GM \rightarrow 1$ the deflection angle grows to infinity, i.e. the particle winds around the source more and more times. For $GM > 1$ all the trajectories start and end in the source; it takes, however, an infinite time to reach the source. This is not unexpected since the two-dimensional invariant distance to the source is in this case infinite.

It is worth noting that combining (6) and (7) we obtain the connection between the invariant distance ds along the trajectory and the isotropic time dt :

$$ds = \frac{m}{E} dt \quad (19)$$

Hence the particle's clock and the isotropic clock are simply scaled versions of one another.

3.Scattering of waves in the Schrödinger equation

In this Section we want to investigate the solutions of the Schrödinger equation for a test particle in the presence of a source. For $GM < 1$ we find a solution which resembles the scattering of plane waves in the usual potential problems. We assume that the wave packet is described by the wave function ψ and is the eigenfunction of the Hamiltonian with the eigenvalue E .

We rewrite (5) in the following form:

$$(E^2 - m^2) \left(\frac{r}{b}\right)^{-2GM} \psi = p^2 \psi \quad (20)$$

With the usual substitution $p^2 \rightarrow -\nabla^2$ we have the differential equation - it can be solved exactly by means of Bessel functions. We try the following form of ψ :

$$\psi = f(r)e^{i\alpha\theta} \quad (21)$$

When substituted into (20) it gives the ordinary differential equation for $f(r)$:

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{\alpha^2}{r^2} f + (E^2 - m^2) \left(\frac{r}{b}\right)^{-2GM} f = 0 \quad (22)$$

The general solution to this equation can be expressed in terms of Bessel functions

$$f(r) = J_{\pm \frac{\alpha}{1-GM}} \left(\frac{1}{1-GM} \sqrt{E^2 - m^2} b^{GM} r^{1-GM} \right) \quad (23)$$

Using the equality

$$e^{\pm iz \cos \phi} = \sum_{n=-\infty}^{\infty} (\pm i)^n J_n(z) \cos n\phi \quad (24)$$

we obtain the following solution to (20) :

$$\psi_{\pm}(r, \theta) = A \exp \left[\frac{\pm i}{1-GM} \sqrt{E^2 - m^2} b^{GM} r^{1-GM} \cos((1-GM)(\theta - \theta_0)) \right] \quad (25)$$

It is an analogue of $\exp(\pm ikz)$ but the outgoing wave is deflected (without any change in intensity) with respect to the incoming wave. So (in some sense) the S matrix is equal to 1 when we properly interpret the 1. Such an interpretation is valid only for $GM < 1$ - it is not clear if for $GM > 1$ one can have a solution which can be interpreted as a scattering solution.

4. Bound states for pointlike source

In this section we want to investigate the existence of bound states i.e. normalizable stationary solutions of the Schrödinger equation with $E < m$ in the presence of a pointlike source of mass M . We assume that the solutions are periodic in the polar angle ϕ .

The analogue of (23) for $E < m$ are Bessel functions of imaginary argument: $K_{\frac{|n|}{1-GM}}$ and $I_{\frac{|n|}{1-GM}}$. The first vanish exponentially at infinity and blow out at the origin, while the second vanish (or are finite) at the origin and blow out exponentially at infinity. To check the normalizability of states we have to remember that

$$\langle \psi | \psi \rangle = \int d^2r \sqrt{g} |\psi|^2 \sim \int dr r^{1-2GM} |\psi|^2 \quad (26)$$

a) $GM < 1$

Normalizability for $r \rightarrow \infty$ requires the solution to be of the form:

$$\psi(r, \phi) = A K_{\frac{|n|}{1-GM}} \left(\frac{1}{1-GM} \sqrt{m^2 - E^2} b^{GM} r^{1-GM} \right) e^{in\phi}. \quad (27)$$

For $r \rightarrow 0$ this function behaves like $r^{-|n|}$ so the solution is normalizable only for $n = 0$. The energy spectrum of this bound state is continuous.

b) $GM > 1$

Normalizability for $r \rightarrow 0$ requires the solution to be of the form:

$$\psi(r, \phi) = A K_{\frac{|n|}{GM-1}} \left(\frac{1}{GM-1} \sqrt{m^2 - E^2} b^{GM} \left(\frac{1}{r} \right)^{GM-1} \right) e^{in\phi}. \quad (28)$$

For $r \rightarrow \infty$ this function behaves like $r^{|n|}$ so it is normalizable when

$$1 - 2GM + 2|n| < -1. \quad (29)$$

Hence the bound states exist only for

$$|n| < GM - 1. \quad (30)$$

The energy spectrum of these bound states is also continuous. It is amusing to note that in two dimensions the angular momentum of a state should be less than the mass squared while here it should not exceed the mass of a source - similar constraints may also exist in higher dimensions.

5. Bound states for a massive disk

In the previous section we have obtained the constraints that have to be satisfied for the bound states ($n = 0$ for $GM > 1$ and (30) for $GM > 1$). The existence of these constraints may be connected to our assumption that the source is pointlike. In this section we want to investigate the existence of bound states for a "smeared" source - it turns out that the constraints are exactly the same.

In order to derive the metric for an arbitrary continuous source we use the generalization of the metric given in [11] :

$$ds^2 = dt^2 - \exp \left(-2G \int d^2x \rho(\mathbf{x}) \ln \left| \frac{\mathbf{r} - \mathbf{x}}{b} \right| \right) (dr^2 + r^2 d\theta^2) \quad (31)$$

To simplify the equations as much as possible we require the metric inside the disc to be quasi-flat. To this end we assume the density inside the disc to be of the form:

$$\rho(r) = \frac{M \ln \left(\frac{b}{R} \right)}{2\pi r^2 \left(\ln \left(\frac{b}{r} \right) \right)^2} \quad (32)$$

With this choice of the density the metric reads:

$$\begin{cases} ds^2 = dt^2 - \left(\frac{R}{b} \right)^{-2GM} (dr^2 + r^2 d\theta^2) & r < R \\ ds^2 = dt^2 - \left(\frac{r}{b} \right)^{-2GM} (dr^2 + r^2 d\theta^2) & r > R \end{cases} \quad (33)$$

and the Schrödinger equation

$$\begin{cases} \nabla^2 \psi - (m^2 - E^2) \left(\frac{R}{b} \right)^{-2GM} \psi = 0 & r < R \\ \nabla^2 \psi - (m^2 - E^2) \left(\frac{r}{b} \right)^{-2GM} \psi = 0 & r > R \end{cases} \quad (34)$$

The bound state solutions (i.e. with $E < m$) are as follows:

a) $GM < 1$

$$\psi(r, \phi) = \begin{cases} A I_{|n|} \left(\sqrt{m^2 - E^2} \left(\frac{R}{b} \right)^{-GM} r \right) e^{\pm i n \phi} & r < R \\ B K_{\frac{|n|}{1-GM}} \left(\frac{1}{1-GM} \sqrt{m^2 - E^2} b^{GM} r^{1-GM} \right) e^{\pm i n \phi} & r > R \end{cases} \quad (35)$$

For $r < R$ also K_0 is admissible. It is easy to check that the solution and its first derivative do not have a jump at $r = R$ only for $n = 0$ so we recover the previous condition.

b) $GM > 1$

For $|n| < GM - 1$ we have the following solution:

$$\psi(r, \phi) = \begin{cases} A I_{|n|} \left(\sqrt{m^2 - E^2} \left(\frac{R}{b} \right)^{-GM} r \right) e^{\pm i n \phi} & r < R \\ B_{\pm} I_{\frac{\pm n}{1-GM}} \left(\frac{1}{1-GM} \sqrt{m^2 - E^2} b^{GM} \left(\frac{1}{r} \right)^{GM-1} \right) e^{\pm i n \phi} & r > R \end{cases} \quad (36)$$

while for $|n| > GM - 1$ we have

$$\psi(r, \phi) = \begin{cases} A I_{|n|} \left(\sqrt{m^2 - E^2} \left(\frac{R}{b} \right)^{-GM} r \right) e^{\pm i n \phi} & r < R \\ B I_{\frac{|n|}{1-GM}} \left(\frac{1}{1-GM} \sqrt{m^2 - E^2} b^{GM} \left(\frac{1}{r} \right)^{GM-1} \right) e^{\pm i n \phi} & r > R \end{cases} \quad (37)$$

For $r < R$ also K_0 is admissible. It is easy to check that the solution and its first derivative do not have a jump at $r = R$ for

$$|n| < GM - 1. \quad (38)$$

and we again recover the previous condition.

6. Discussion

In this paper we have investigated bound state and scattering solutions at the classical level and as solutions of the Schrödinger equation. The classical trajectories of a test particle are open for a small mass of the source ($GM < 1$) and closed for bigger masses. When GM approaches 1 from below, the "winding number" of a trajectory grows to infinity. For the quantum case the scattering solution ($E > m$) describes a wave which comes from infinity, deflects, and goes to infinity without any change in shape or intensity (the deflection angle is the same as for a classical trajectory). The bound state solutions ($E < m$) exist both for GM less and bigger than 1. In the former case there exists only one state with no angular momentum, while for the latter there exist states with the angular momentum less than $GM - 1$. These constraints are the same for pointlike and smeared sources. In the case of $GM > 1$ the test particle is infinitely far away (in the sense of the invariant distance) from the source - the fact that both at the classical and quantum levels the massive source even at an infinite distance acts like a black hole confirms that

the three-dimensional gravitational interaction has a different nature than in any other dimension, where at least asymptotically flat regions can guide our intuition. This is also the reason why the solution to the full scattering problem is so difficult to find in spite of the apparent simplicity of three-dimensional gravity.

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