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**MANIFEST COVARIANCE CONDITION
IN THE HAMILTONIAN THEORY OF RELATIVISTIC STRINGS**

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ABSTRACT

The manifest covariance condition of Currie, Jordan and Sudarshan is generalised in a natural manner to the theory of one relativistic string. Then, all "local" solutions of a certain type (suggested by the existing literature) are classified. Two solutions, which seem to be new in the literature, are found. The analysis is performed in the framework of the Hamiltonian formalism.

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1. Introduction

The purpose of this paper is to use the so-called “world line” condition (or “manifest covariance” condition) for the theory of a relativistic string.

First, let us show the use of this condition in the Hamiltonian theory of one relativistic particle.

1. For a point particle evolving in R^3 , we take as the configuration space $Q = R^3$. According to the usual prescription of the Hamiltonian formalism [1] the phase space is $M = T^*(Q) \cong R^3 \times R^3$, with the Poisson bracket :

$$\{f, g\} \equiv \frac{\partial f}{\partial q_i} \cdot \frac{\partial g}{\partial p_i} - (f \leftrightarrow g), \quad (1.1)$$

for any f, g smooth functions on M . We use the summation convention over the repeated indices.

2. In the configuration space Q we have a natural action of the special Euclidean group $SE(3)$:

$$\varphi_{R, \mathbf{a}}(\mathbf{q}) \equiv R\mathbf{q} + \mathbf{a}. \quad (1.2)$$

This action lifts to a symplectic action of $SE(3)$ on M :

$$\dot{\varphi}_{R, \mathbf{a}}(\mathbf{q}, \mathbf{p}) \equiv (R\mathbf{q} + \mathbf{a}, R\mathbf{p}). \quad (1.3)$$

3. We search now for a symplectic action Φ of the special orthochronous Poincaré group $P_+^\uparrow$ on M which extends $\dot{\varphi} : \Phi|_{SE(3)} = \dot{\varphi}$.

To investigate if this problem has solutions, one proceeds as follows. First, one uses the fact that any symplectic action of the Poincaré group admits an equivariant momentum map [2], $J : M \rightarrow (Lie P_+^\uparrow)^*$. (This is related to well-known cohomology properties of the Poincaré group.) In other words, one has a set of ten smooth functions P_i, J_i, K_i, H on M (which are the infinitesimal generators of space translations, rotations, boosts and time translations) and which fulfil the following relations:

$$\{J_i, J_j\} = \epsilon_{ijk} J_k. \quad (1.4)$$

$$\{P_i, P_j\} = 0. \quad (1.5)$$

$$\{J_i, P_j\} = \epsilon_{ijk} P_k. \quad (1.6)$$

$$\{P_i, H\} = 0. \quad (1.7)$$

$$\{J_i, H\} = 0. \quad (1.8)$$

$$\{K_i, P_j\} = \delta_{ij} H. \quad (1.9)$$

$$\{J_i, K_j\} = \epsilon_{ijl} K_l. \quad (1.10)$$

$$\{K_i, H\} = P_i. \quad (1.11)$$

$$\{K_i, K_j\} = -\epsilon_{ijk} J_k. \quad (1.12)$$

Using (1.3), one shows easily that we have:

$$P_i(\mathbf{q}, \mathbf{p}) = p_i. \quad (1.13)$$

$$J_i(\mathbf{q}, \mathbf{p}) = \epsilon_{ijk} q_j p_k. \quad (1.14)$$

and as a consequence (1.4)-(1.6) are fulfilled identically.

So, now one tries to find smooth functions H and K_i , such that (1.7)-(1.12) are fulfilled. It is quite easy to find the most general solution of this problem. In fact, (1.7) tells us that H depends only on $\mathbf{p}$, then (1.9) shows that $\mathbf{K}$ is of the form:

$$\mathbf{K}(\mathbf{q}, \mathbf{p}) = \mathbf{q}H(\mathbf{q}, \mathbf{p}) + \mathbf{k}(\mathbf{p}). \quad (1.15)$$

for some smooth function $\mathbf{k}$ depending only on $\mathbf{p}$. Next, (1.11) gives us the equation:

$$H \frac{\partial H}{\partial p_i} = p_i \Leftrightarrow H^2 - \mathbf{p}^2 = \text{const.}$$

Because we want H to be a smooth function on M , we must have $\text{const} > 0$, i.e. H is of the form:

$$H(\mathbf{q}, \mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}. \quad (1.16)$$

for some $m > 0$. Then (1.8) is automatically fulfilled, and (1.10) is equivalent with:

$$k_i(R\mathbf{p}) = R_{ij} k_j(\mathbf{p}). \quad (1.17)$$

(for all R in $SO(3)$). This ensures also that (1.12) is true.

So, the solution of our problem is not unique: it involves the unknown function $\mathbf{k}$. If we want a unique solution, we can impose an additional assumption, namely the “manifest covariance” (or the “world line”) condition. Let us explain it in detail. If we know the symplectic action Φ of the Poincaré group, we know in principle the time evolution in phase space. If (L, a) is a generic element of the Poincaré group, with L a proper orthochronous Lorentz transformation and a a space-time translation, then the time evolution is $\Phi_{\mathbf{1}, -te_0}$: here $\mathbf{1}$ is the identity transformation in the Minkowski space M^4 and e_0, e_1, e_2, e_3 is a Lorentz basis in M^4 . So, the time evolution of the coordinates $\mathbf{q}$ is:

$$\mathbf{x}(t) \equiv \mathbf{q} \circ \Phi_{\mathbf{1}, -te_0}$$

($\mathbf{x}(t)$ is a function on the phase space M). Now, one computes this time evolution from the point of view of a moving observer in two ways. One can extract from Φ the action of the boosts $L_B(\beta, \chi)$ of velocity $v = t\hbar\chi$ in the direction β : $\Phi_{L_B(\beta, \chi), 0}$. On the other hand, one can apply the boost $L_B(\beta, \chi)$ to the quadrivector $(t, \mathbf{x}(t))$ and then extract the spatial part. The “manifest covariance” condition of Currie, Jordan and Sudarshan [3] requires that both operations produce the same result. In formulae:

$$\mathbf{x}(t) + \beta[(ch\chi - 1)\mathbf{x}(t) \cdot \beta + t(sh\chi)] = \mathbf{x}(t) \circ \Phi_{L_B(\beta, \chi)}^{-1}. \quad (1.18)$$

One proves that (1.18) implies infinitesimally:

$$\{q_i, K_j\} = q_j \{q_i, H\} \Leftrightarrow \frac{\partial K_j}{\partial p_i} = q_j \frac{\partial H}{\partial p_i}. \quad (1.19)$$

Using (1.15) in this relation, one gets that $\mathbf{k}$ is in fact constant. Then (1.17) gives $\mathbf{k} = \mathbf{0}$, and we get the unique solution:

$$\mathbf{K}(\mathbf{q}, \mathbf{p}) = \mathbf{q} \sqrt{\mathbf{p}^2 + m^2}. \quad (1.20)$$

Let us mention briefly that the “manifest covariance” condition is at the origin of a very interesting result in Hamiltonian mechanics, namely the so-called “no-interaction” theorem [3-5]. Namely, one generalises naturally the framework 1-3 for an N -particle system, taking for instance $Q = R^3 \times \dots \times R^3$ (N factors). Also the “manifest covariance” condition generalises to this case naturally: we require (1.18) for every particle. Then, imposing a technical assumption, namely that the $3N \times 3N$ Hessian matrix of the Hamiltonian with respect to the momenta is non-degenerate, one can prove that H and $\mathbf{K}$ are sums of expressions of the form (1.16) and (1.20) respectively for the individual particles i.e. the particles do not interact.

As we have mentioned at the beginning of this rather long introduction, we want to investigate the consequences of these ideas for the theory of relativistic string. Namely, after identifying the configuration space of a string, we construct a framework analogous to points 1-3 above, and we impose also the “manifest covariance” condition, which generalises naturally for strings. The emerging problem is very complicated, but assuming a special “local” form for H and $\mathbf{K}$ (which is suggested by the literature), we are able to solve it in this special case. We obtain two types of solutions which seem to be new in the literature.

Let us close this introduction by pointing out some related approaches. It is known that one way to circumvent the “no-interaction theorem” is to use the so-called “many-time formalism” [6]. So, it is interesting to analyse a relativistic string with this formalism. In this context, the analysis was performed recently on the open Nambu action [7]. On the other hand, it is clear from our analysis that a important rôle was played by the choice of the configuration space. Another way to exhibit the configuration space in a symplectic formalism is to look for a position operator in the sense of Newton-Wigner [8]. This idea was applied recently for relativistic systems with finite number of degrees of freedom in [9], and for the open Nambu string in [10].

2. The “manifest covariance” condition in string theory

1. For definiteness we consider only the case of a closed string. It is clear that the configuration space for a single closed string is [11]:

$$Q = C^\infty(S^1, R^3) / \text{Diff}(S^1). \quad (2.1)$$

Indeed, a parametrised closed string is described by a smooth map $\mathbf{q} : S^1 \rightarrow R^3$. But if $\phi \in \text{Diff}(S^1)$, then $\mathbf{q}$ and $\mathbf{q} \circ \phi$ describe evidently the same configuration, so we have (2.1) with the right action of $\text{Diff}(S^1)$:

$$(\phi, \mathbf{q}) \longmapsto \mathbf{q} \circ \phi. \quad (2.2)$$

Next, we can identify:

$$T(Q) = (C^\infty(S^1, R^3) \times C^\infty(S^1, R^3)) / \text{Diff}(S^1). \quad (2.3)$$

where the action of $\text{Diff}(S^1)$ is:

$$(\phi, (\mathbf{q}, \mathbf{v})) \longmapsto (\mathbf{q} \circ \phi, \mathbf{v} \circ \phi). \quad (2.4)$$

Finally, we identify:

$$M \equiv T^*(Q) = (C^\infty(S^1, R^3) \times C^\infty(S^1, R^3))/\text{Diff}(S^1). \quad (2.5)$$

with the action of $\text{Diff}(S^1)$:

$$(\phi, (\mathbf{q}, \mathbf{p})) \longmapsto (\mathbf{q} \circ \phi, (\mathbf{p} \circ \phi)\phi'). \quad (2.6)$$

In this way, the Poisson bracket on M is:

$$\{F, G\}([\mathbf{q}, \mathbf{p}]) = \int d\sigma \left(\frac{\delta F \circ \pi}{\delta q_i(\sigma)} \cdot \frac{\delta G \circ \pi}{\delta p_i(\sigma)} - (F \leftrightarrow G) \right). \quad (2.7)$$

Here, $[\mathbf{q}, \mathbf{p}]$ is the equivalence class of $(\mathbf{q}, \mathbf{p})$ with respect to (2.6), $d\sigma$ is the Lebesgue measure on the circle and π is the canonical projection which associates to $(\mathbf{q}, \mathbf{p})$ its equivalence class in M . One computes the right-hand side using Fréchet derivatives.

2. In Q we have a natural action of $\text{SE}(3)$, namely the factorisation of:

$$(\varphi_{R, \mathbf{a}}(\mathbf{q})(\cdot)) = R\mathbf{q}(\cdot) + \mathbf{a}. \quad (2.8)$$

This action lifts to M , and as in sect.1, we have:

$$(\dot{\varphi}_{R, \mathbf{a}}(\mathbf{q}(\cdot), \mathbf{p}(\cdot)) = (R\mathbf{q}(\cdot) + \mathbf{a}, R\mathbf{p}(\cdot)). \quad (2.9)$$

projected by π .

3. We assume that in M we have a symplectic action Φ of the proper orthochronous Poincaré group which extends $\dot{\varphi}$. Then, we have ten functionals P_i, J_i, K_i and H fulfilling (1.4)-(1.12). Instead of (1.13) and (1.14) we have here:

$$P_i([\mathbf{q}, \mathbf{p}]) = \int d\sigma p_i(\sigma). \quad (2.10)$$

$$J_i([\mathbf{q}, \mathbf{p}]) = \epsilon_{ijk} \int d\sigma q_j(\sigma) p_k(\sigma). \quad (2.11)$$

Note that the right-hand sides of these two relations are well defined, i.e. do not depend on the representative $(\mathbf{q}, \mathbf{p}) \in [\mathbf{q}, \mathbf{p}]$.

4. Next, we generalise straightforwardly (1.18) to this case, and we get instead of (1.19):

$$\frac{\delta K_j}{\delta p_i(\sigma)} = q_j(\sigma) \frac{\delta H}{\delta p_i(\sigma)}. \quad (2.12)$$

The problem at which we have arrived is rather difficult to solve. But the expressions usually found in the literature (see [12], formula (2.6)) suggest that we look for H and $\mathbf{K}$ of the particular form:

$$H([\mathbf{q}, \mathbf{p}]) = \int d\sigma H(\mathbf{q}(\sigma), \mathbf{q}'(\sigma), \mathbf{p}(\sigma)). \quad (2.13)$$

$$K_i([\mathbf{q}, \mathbf{p}]) = \int d\sigma K_i(\mathbf{q}(\sigma), \mathbf{q}'(\sigma), \mathbf{p}(\sigma)). \quad (2.14)$$

where H and K_i must satisfy some identities in order that the right-hand sides of (2.13) and (2.14) be well defined.

The rest of this section is devoted to the determination of the most general solution of the form (2.13),(2.14) for the equations (1.7)-(1.12)+(2.2). The proof consists of elementary computations and repeated use of the Frobenius condition of integrability (*FCI*). For the sake of clarity, we fragment this proof in several steps.

a) First, (1.7) is equivalent to:

$$\int \frac{\delta H}{\delta q_i} d\sigma = 0$$

Using the lemma in the Appendix and computing the functional derivative, one infers that there exist some functions f_i depending only on $\mathbf{q}$, $\mathbf{q}'$ and $\mathbf{p}$, such that:

$$\frac{\partial H}{\partial q_i} - \frac{\partial^2 H}{\partial q'_i \partial q_j} q'_j = \frac{\partial f_i}{\partial q_j} q'_j. \quad (2.15)$$

$$\frac{\partial^2 H}{\partial q'_i \partial q'_j} = -\frac{\partial f_i}{\partial q'_j}. \quad (2.16)$$

$$\frac{\partial^2 H}{\partial q'_i \partial p_j} = -\frac{\partial f_i}{\partial p_j}. \quad (2.17)$$

From (2.16) we get $\frac{\partial f_i}{\partial q'_j} = (i \leftrightarrow j)$, so with *FCI* we have a function f depending on $\mathbf{q}, \mathbf{q}'$ and $\mathbf{p}$, such that $f_i = \frac{\partial f}{\partial q'_i}$. Then (2.16) and (2.17) show that $H + f$ is of the form:

$$H + f = g_i(\mathbf{q}) q'_i + g(\mathbf{q}, \mathbf{p})$$

for some functions g_i of $\mathbf{q}$ and g of $\mathbf{q}$ and $\mathbf{p}$. This can be taken as the definition of f , and substituting into (2.15) we end up with:

$$\frac{\partial H}{\partial q_i} = \frac{\partial g_i}{\partial q_j} q'_j. \quad (2.18)$$

Applying *FCI* to this system we get that:

$$\frac{\partial g_i}{\partial q_j} - (i \leftrightarrow j) = C_{ij}, \quad (2.19)$$

for some constants C_{ij} verifying $C_{ij} = -C_{ji}$. If we define now:

$$G_i(\mathbf{q}) \equiv g_i(\mathbf{q}) - \frac{1}{2} C_{ij} q_j$$

then (2.19) gives:

$$\frac{\partial G_i}{\partial q_j} = (i \leftrightarrow j)$$

So, with *FCI* we get a function g of $\mathbf{q}$ such that $G_i = \frac{\partial g}{\partial q_i}$. It follows that the most general solution of (2.19) is:

$$g_i(\mathbf{q}) = \frac{\partial g}{\partial q_i}(\mathbf{q}) + \frac{1}{2} C_{ij} q_j$$

Substituting into (2.18) we get:

$$H(\mathbf{q}, \mathbf{q}', \mathbf{p}) = \frac{\partial g}{\partial q_i}(\mathbf{q})q'_i + \frac{1}{2}C_{ij}q_iq'_j + h(\mathbf{q}', \mathbf{p})$$

for some functions h of $\mathbf{q}'$ and $\mathbf{p}$. The first term gives a null contribution in (2.13), so in fact we can take:

$$H(\mathbf{q}, \mathbf{q}', \mathbf{p}) = \frac{1}{2}C_{ij}q_iq'_j + h(\mathbf{q}', \mathbf{p}). \quad (2.20)$$

In this case (2.18) becomes the definition of g_i .

b) We impose now (1.8), which is the infinitesimal form of:

$$H([R\mathbf{q}, R\mathbf{p}]) = H([\mathbf{q}, \mathbf{p}]). \quad (2.21)$$

(for any R in $SO(3)$). Using now (2.20) and the lemma in the Appendix, we get a function g_R of $\mathbf{q}$ such that:

$$\frac{1}{2}(C_{ij}^R - C_{ji})q_iq'_j + h(R\mathbf{q}', R\mathbf{p}) - h(\mathbf{q}', \mathbf{p}) = \frac{\partial g_R}{\partial q_j}(\mathbf{q})q'_j. \quad (2.22)$$

Here $C_{ij}^R \equiv R_{ki}R_{lj}C_{kl}$. Differentiating this relation with respect to q_i yields:

$$\frac{1}{2}(C_{ij}^R - C_{ji}) = \frac{\partial^2 g_R}{\partial q_i \partial q_j}$$

Because the left-hand side is antisymmetric in i and j and the right-hand side is symmetric, they are both zero. So, we have in particular $C^R = C$, i.e. C is a second degree rotational invariant tensor. Because C is antisymmetric, we must have $C = 0$. Also, g_R must be of the form: $g_R = \Delta(R)_i q_i + \delta(R)$. So, (2.22) reduces to:

$$h(R\mathbf{q}', R\mathbf{p}) - h(\mathbf{q}', \mathbf{p}) = \Delta(R)_i q'_i$$

From this relation it is easy to prove that the map $SO(3) \ni R \mapsto \Delta(R) \in R^3$ is an R^3 -valued one-cocycle for $SO(3)$, where on R^3 one considers the usual action of $SO(3)$. But it is well known that $H^1(G, V) = 0$ for any compact group, any finite dimensional vector space V and any action of G on V . So, in fact $R \mapsto \Delta(R)$ must be a coboundary, i.e. of the form: $\Delta(R) = R\mathbf{v} - \mathbf{v}$ for some $\mathbf{v} \in R^3$. It follows that by redefining $h \mapsto h - \mathbf{v} \cdot \mathbf{q}'$, we can take h to be rotational invariant, without changing H . Summing up, it follows that in (2.13) we can take H to be $\mathbf{q}$ -independent and verifying:

$$H(R\mathbf{q}', R\mathbf{p}) = H(\mathbf{q}', \mathbf{p}). \quad (2.23)$$

(for any R in $SO(3)$).

c) Now we use (1.9), and proceeding exactly as in step a), we can show that K_i is of the form:

$$K_i(\mathbf{q}, \mathbf{q}', \mathbf{p}) = \frac{1}{2}C_{ijk}q_jq'_k + q_i H(\mathbf{q}', \mathbf{p}) + k_i(\mathbf{q}', \mathbf{p})$$

for some functions k_i of $\mathbf{q}'$ and $\mathbf{p}$, and some constants C_{ijk} antisymmetric in j and k .

d) Using (1.10), and the same type of argument as in step b), we show that $C_{i,jk} = 0$, and k_i are rotational covariant. So, we have in (2.14):

$$K_i(\mathbf{q}, \mathbf{q}', \mathbf{p}) = q_i H(\mathbf{q}', \mathbf{p}) + k_i(\mathbf{q}', \mathbf{p})$$

e) From the last relation, using the “manifest covariance” condition (2.12) it is easy to show that k_i must be $\mathbf{p}$ -independent. So, we have:

$$K_i(\mathbf{q}, \mathbf{q}', \mathbf{p}) = q_i H(\mathbf{q}', \mathbf{p}) + k_i(\mathbf{q}'). \quad (2.24)$$

where the function $\mathbf{k}$ is rotational covariant:

$$k_i(R\mathbf{q}') = R_{ij} k_j(\mathbf{q}'). \quad (2.25)$$

f) Using the lemma in the Appendix, (1.11) can be shown to be equivalent with the existence of some functions F_i of $\mathbf{q}$, $\mathbf{q}'$ and $\mathbf{p}$ such that:

$$\left(\frac{\partial K_i}{\partial q_j} - \frac{\partial^2 K}{\partial q'_j \partial q_k} q'_k \right) \cdot \frac{\partial H}{\partial p_j} - p_i = \frac{\partial F_i}{\partial q_j} q'_j. \quad (2.26)$$

$$\frac{\partial^2 K_i}{\partial q'_j \partial q'_k} \cdot \frac{\partial H}{\partial p_k} - \frac{\partial K_i}{\partial p_k} \cdot \frac{\partial^2 H}{\partial q'_j \partial q'_k} = - \frac{\partial F_i}{\partial q'_j}. \quad (2.27)$$

$$\frac{\partial^2 K_i}{\partial p_j \partial q'_k} \cdot \frac{\partial H}{\partial p_k} - \frac{\partial K_i}{\partial p_k} \cdot \frac{\partial^2 H}{\partial p_j \partial q'_k} = - \frac{\partial F_i}{\partial p_j}. \quad (2.28)$$

We substitute (2.24) into these three relations. From (2.28) it follows that F_i does not depend on $\mathbf{p}$. The other two become:

$$\frac{\partial^2 k_i}{\partial q'_j \partial q'_k} \cdot \frac{\partial H}{\partial p_k} = - \frac{\partial F_i}{\partial q'_j}. \quad (2.29)$$

$$H \frac{\partial H}{\partial p_i} - q'_i \frac{\partial H}{\partial q'_j} \cdot \frac{\partial H}{\partial p_j} - p_i = \frac{\partial F_i}{\partial q_j} q'_j. \quad (2.30)$$

If we differentiate (2.29) with respect to q_i we get

$$\frac{\partial^2 F_i}{\partial q_i \partial q'_j} = 0,$$

i.e. F_i is of the form:

$$F_i(\mathbf{q}, \mathbf{q}') = f_i(\mathbf{q}) + F_i(\mathbf{q}')$$

So, in (2.29) we can substitute:

$$F_i \longrightarrow f_i$$

and in (2.30) we must substitute

$$F_i \longrightarrow f_i$$

Now, we differentiate (2.30) with respect to q_k and we get $\frac{\partial^2 f_i}{\partial q_j \partial q_k} = 0$, i.e. f_i is of the form $f_i(\mathbf{q}) = f_{ij}q_j + f_i$ for some constants f_{ij} and f_i . So, (2.30) becomes:

$$H \frac{\partial H}{\partial p_i} - q'_i \frac{\partial H}{\partial q'_j} \cdot \frac{\partial H}{\partial p_j} - p_i = f_{ij}q'_j$$

But, using (2.23) it is easy to show that f_{ij} is a second-order rotational invariant tensor, so it must be of the form $f_{ij} = C\delta_{ij}$ for some constant C . The preceding relation becomes:

$$H \frac{\partial H}{\partial p_i} - q'_i \frac{\partial H}{\partial q'_j} \cdot \frac{\partial H}{\partial p_j} - p_i = Cq'_i. \quad (2.31)$$

g) We impose now (1.12). Using (2.29) and (2.30) we get after some simple computations:

$$\int d\sigma \left[Cq'_i(\sigma)q_j(\sigma) + \frac{\partial F_i}{\partial q'_k}(\mathbf{q}'(\sigma))q'_j(\sigma)q''_k(\sigma) - (i \leftrightarrow j) \right] = 0$$

Applying the lemma in the Appendix we get some functions G_{ij} of $\mathbf{q}$ and $\mathbf{q}'$ antisymmetric in i and j and verifying:

$$C(q'_i q_j - (i \leftrightarrow j)) = \frac{\partial G_{ij}}{\partial q_k} q'_k. \quad (2.32)$$

$$\frac{\partial F_i}{\partial q'_k} q'_j - (i \leftrightarrow j) = -\frac{\partial G_{ij}}{\partial q'_k}. \quad (2.33)$$

Differentiating (2.33) with respect to q_l we get: $\frac{\partial^2 G_{ij}}{\partial q_k \partial q'_l} = 0$, i.e. G_{ij} is of the form: $G_{ij}(\mathbf{q}, \mathbf{q}') = G_{ij}(\mathbf{q}) + g_{ij}(\mathbf{q}')$. So, in (2.32) we can consider G_{ij} to be independent of $\mathbf{q}$, and in (2.33) we must change $G_{ij} \rightarrow g_{ij}$. Then (2.32) is equivalent to:

$$\frac{\partial G_{ij}}{\partial q_k} = C(\delta_{ik} q_j - (i \leftrightarrow j))$$

Applying *FCI* we get $C = 0$ so, in fact, G_{ij} are some constants. Next, we write (2.33) in the equivalent form:

$$\frac{\partial}{\partial q'_k} (F_i q'_j - F_j q'_i + g_{ij}) = \delta_{jl} F_i - \delta_{il} F_j$$

Applying *FCI* it is easy to prove that F_i are also some constants. Summing up, (2.29) and (2.31) become:

$$\frac{\partial^2 k_i}{\partial q'_j \partial q_k} \cdot \frac{\partial H}{\partial p_k} = 0. \quad (2.34)$$

and

$$H \frac{\partial H}{\partial p_i} - q'_i \frac{\partial H}{\partial q'_k} \cdot \frac{\partial H}{\partial p_k} - p_i = 0. \quad (2.35)$$

h) We now exploit completely the rotational invariance of H , i.e. (2.23). It is easy to prove that there must exist a real function h depending on three variables (x, y, z) , defined on the domain $z^2 \leq xy$, such that:

$$H(\mathbf{q}', \mathbf{p}) = h(\mathbf{p}^2, \mathbf{q}'^2, \mathbf{p} \cdot \mathbf{q}'). \quad (2.36)$$

Inserting this relation into (2.35) we easily get the following two relations:

$$2h \frac{\partial h}{\partial x} = 1. \quad (2.37)$$

$$h \frac{\partial h}{\partial z} - 4z \frac{\partial h}{\partial x} \cdot \frac{\partial h}{\partial y} - 2y \frac{\partial h}{\partial y} \cdot \frac{\partial h}{\partial z} - 2x \frac{\partial h}{\partial x} \cdot \frac{\partial h}{\partial z} - z \left(\frac{\partial h}{\partial z} \right)^2 = 0. \quad (2.38)$$

From (2.37) it follows that we have:

$$[h(x, y, z)]^2 = x + M(y, z). \quad (2.39)$$

where the function M verifies:

$$2M \frac{\partial M}{\partial z} - 4z \frac{\partial M}{\partial y} - 2y \frac{\partial M}{\partial y} \cdot \frac{\partial M}{\partial z} - z \left(\frac{\partial M}{\partial z} \right)^2 = 0. \quad (2.40)$$

i) We impose the condition that the right-hand side of (2.13) is well defined. This means that for any $\phi \in \text{Diff}(S^1)$ we must have (see (2.6)):

$$\int d\sigma H(\mathbf{q}(\sigma), \mathbf{q}'(\sigma), \mathbf{p}(\sigma)) = \int d\sigma H(\mathbf{q} \circ \phi(\sigma), \phi'(\sigma) \mathbf{q}' \circ \phi(\sigma), \phi'(\sigma) \mathbf{p} \circ \phi(\sigma))$$

If on the right-hand side we make the change of variable $\sigma \mapsto \phi^{-1}(\sigma)$ we get:

$$\int H(\mathbf{q}(\sigma), \mathbf{q}'(\sigma), \mathbf{p}(\sigma)) d\sigma = \int H(\mathbf{q}(\sigma), \alpha(\sigma) \mathbf{q}'(\sigma), \alpha(\sigma) \mathbf{p}(\sigma)) \frac{1}{\alpha(\sigma)} d\sigma$$

where $\alpha(\sigma) \equiv \phi' \circ \phi^{-1}(\sigma)$. Applying now the lemma in the Appendix with $x = (\mathbf{q}, \mathbf{p}, \alpha)$, we get easily that we must have:

$$H(\mathbf{q}, \alpha \mathbf{q}', \alpha \mathbf{p}) = \alpha H(\mathbf{q}, \mathbf{q}', \mathbf{p}). \quad (2.41)$$

(for any α in R^*). This in turn implies for the function M the following identity:

$$M(\lambda y, \lambda z) = \lambda M(y, z). \quad (2.42)$$

(for any λ in R_+). So, M must be of the form:

$$M(y, z) = y f\left(\frac{z}{y}\right). \quad (2.43)$$

Substituting this expression into (2.40) we get the following differential equation:

$$[f'(\xi)]^2 - 4f(\xi) + 4\xi f'(\xi) = 0. \quad (2.44)$$

If we differentiate this relation we end up with:

$$f''(\xi)[f'(\xi) + 2\xi] = 0$$

so, necessarily f is of the form $f_1(\xi) = c - \xi^2$ or $f_2(\xi) = a_0 + a_1 \xi$ for any real constants c, a_0, a_1 . Inserting this back into (2.44) we get the following solutions: $f_1(\xi) = -\xi^2$ and $f_2(\xi) = 2a\xi + a^2$ (for any real a). To these solutions correspond the following Hamiltonian densities H :

$$H_1(\mathbf{q}', \mathbf{p}) = \frac{1}{\mathbf{q}'^2} \sqrt{\mathbf{p}^2 \mathbf{q}'^2 - (\mathbf{p} \cdot \mathbf{q}')^2}. \quad (2.45)$$

and

$$H_2(\mathbf{q}', \mathbf{p}) = |\mathbf{p} + a \mathbf{q}'|. \quad (2.46)$$

Let us remark that if we define:

$$\mathbf{p}_\perp \equiv \mathbf{p} - \frac{(\mathbf{p} \cdot \mathbf{q}')}{q'^2} \mathbf{q}'. \quad (2.47)$$

then (2.45) can be written as:

$$H_1(\mathbf{q}', \mathbf{p}) = |\mathbf{p}_\perp|. \quad (2.48)$$

In principle, one must exclude from the phase space M some states $[\mathbf{q}, \mathbf{p}]$ such that H is not too singular and as a consequence the integral (2.13) is finite and has functional derivatives of arbitrary order. This seems to be a more subtle problem which we do not analyse here.

j) Finally we analyse (2.25)+(2.34). From (2.25) it follows that $\mathbf{k}$ is of the form:

$$\mathbf{k}(\mathbf{q}') = \mathbf{q}' f(q'^2)$$

We denote

$$v_k \equiv \frac{\partial H}{\partial p_k}$$

and observe from (2.45) and (2.46) that in both cases $\mathbf{v} \neq \mathbf{0}$. Then it is easy to prove that (2.34) implies $\mathbf{k} = \mathbf{0}$. So, finally we have:

$$K_i(\mathbf{q}, \mathbf{q}', \mathbf{p}) = q_i H(\mathbf{q}', \mathbf{p}). \quad (2.49)$$

k) We analyse here the Hamilton equations of motion for H_1 and H_2 . In general these equations are:

$$\dot{q}_i(t, \sigma) = \frac{\partial H}{\partial p_i}(\mathbf{q}'(t, \sigma), \mathbf{p}(t, \sigma)). \quad (2.50)$$

and

$$\dot{p}_i(t, \sigma) = \frac{\partial^2 H}{\partial q'_i \partial q'_j}(\dots) q''_j(t, \sigma) + \frac{\partial^2 H}{\partial q'_i \partial p_j}(\dots) p'_j(t, \sigma). \quad (2.51)$$

Computing explicitly the right-hand sides, one can prove that for *both* H_1 and H_2 the most general solution of (2.50) and (2.51) is of the form:

$$\mathbf{q}(t, \sigma) = \mathbf{f}(2at + \sigma) + \mathbf{Q}(\sigma). \quad (2.52)$$

$$\mathbf{p}(t, \sigma) = a\mathbf{f}'(2at + \sigma) - a\mathbf{Q}'(\sigma). \quad (2.53)$$

Here $\mathbf{f}$ and $\mathbf{Q}$ are smooth periodic functions with period 2π . For H_2 the analysis is rather elementary. For H_1 , one must first prove that the quantity:

$$a \equiv -\frac{(\mathbf{p} \cdot \mathbf{q}')}{\|\mathbf{q}'\|^2}. \quad (2.54)$$

is independent of t and σ .

3. Conclusions

One may ask what relation is between our model for a relativistic string and the usual Nambu Lagrangian. Another way to put this question is to ask if the usual string theory verifies the manifest covariance

condition (2.12). It is easy to prove using the standard formulae (see e.g. [13],(II03) and (II12)) that this condition is not verified. So, it follows that our model is different from the usual one. At this stage it is hard to judge if our model is more suitable for the description of a relativistic string.

There are some other points which deserve further attention. First, it would be interesting to analyse the physical implications of the “manifest covariance” condition in a Lagrangian theory of a relativistic string. This will be done elsewhere. Also we note that it would be interesting to investigate if there exist solutions of our problem of the type (2.13)+(2.14), but with the densities on the right-hand sides depending on higher order derivatives. This problem can be solved in principle, but seems to be very complicated from the computational point of view. On the other hand, it seems hopeless to analyse completely the general situation when H and K_i are arbitrary functionals.

Appendix

Lemma: Let F be a functional over $C^\infty(S^1, R^n)$ of the following type:

$$F[x] \equiv \int F(x(\sigma), x'(\sigma), x''(\sigma)) d\sigma$$

for $F : R^n \times R^n \times R^n \longrightarrow R$, such that

$$\frac{\delta F}{\delta x_i(\sigma)} = 0$$

for any i . Then F is of the following form:

$$F(x, x', x'') = \frac{\partial G}{\partial x_i}(x, x') x'_i + \frac{\partial G}{\partial x'_i}(x, x') x''_i$$

for some function $G : R^n \times R^n \longrightarrow R$ (i.e. a “total derivative”).

Remark: The result is well known if F does not depend on x'' . Because a similar result for higher order derivatives it is not known to the author, we include the proof here for the convenience of the reader.

Proof: We have for any function $\sigma \longrightarrow x(\sigma)$:

$$\frac{\delta F}{\delta x_i(\sigma)} = \frac{\partial F}{\partial x_i}(x(\sigma), x'(\sigma), x''(\sigma)) - \frac{d}{d\sigma} \left(\frac{\partial F}{\partial x'_i}(\dots) \right) + \frac{d^2}{d\sigma^2} \left(\frac{\partial F}{\partial x''_i}(\dots) \right) = 0. \quad (A1)$$

The coefficient of $x'''(\sigma)$ must be identically zero i.e.

$$\frac{\partial^2 F}{\partial x''_i \partial x''_j} = 0$$

So we must have:

$$F(x, x', x'') = f(x, x') + F_i(x, x') x''_i. \quad (A2)$$

We insert that in (A1) and equate to zero the coefficient of $x'''(\sigma)$; we get:

$$\frac{\partial F_i}{\partial x'_j} = \frac{\partial F_j}{\partial x'_i}$$

. We use now *FCI* and we get that F_i are of the form:

$$F_i(x, x') = \frac{\partial G}{\partial x'_i}(x, x'). \quad (A3)$$

for some function $G : R^n \times R^n \longrightarrow R$. Finally, we insert (A2) and (A3) into (A1) and we end up with the following relations:

$$\frac{\partial^2}{\partial x'_i \partial x'_j} \left(\frac{\partial G}{\partial x'_k} x'_k - f \right) = 0. \quad (A4)$$

and

$$\frac{\partial f}{\partial x_i} - \frac{\partial^2 f}{\partial x'_i \partial x_j} x'_j + \frac{\partial^3 G}{\partial x'_i \partial x_j \partial x_k} x'_j x'_k = 0. \quad (A5)$$

From (A4) we infer that f is of the form:

$$f(x, x') = \frac{\partial G}{\partial x_i}(x, x') x'_i + G_i(x) x'_i + g(x). \quad (A6)$$

for some functions G_i and g of x . If we substitute (A6) into (A5) we get easily that g is constant and:

$$\frac{\partial G_i}{\partial x_j} = \frac{\partial G_j}{\partial x_i}$$

Applying *FCI* we get that G_i is of the form:

$$G_i = \frac{\partial H}{\partial x_i}$$

for some function H of x . Using this information in (A2), (A3) and (A6) and redefining

$$G \longrightarrow G + H,$$

we get the desired result.

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