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Stability and Instability of Renormalization Group Flows

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Abstract

Viewed as a dynamical system, a renormalization group transformation can in principle exhibit quite complex behavior, such as limit cycles, intermittency and chaos. We investigate some of the physical consequences of this type of behavior, restricted here for simplicity to real space renormalization group transformations. Conditions for non-singular behavior of given thermodynamic functions are derived.

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1. Introduction

Consider a renormalization group (RG) transformation defined as a discrete operation linking two Hamiltonians H and H' that are defined at different spatial distances a and ab . Denote it by $\mathcal{R}$: $\mathcal{R}_{[b]}[H] = H'$.

In the space of coupling constants $\{K_i\}$ this gives a relation which, with a slight abuse of notation, we can write as

$$K'_i = \mathcal{R}_{[b]}^i[\{K_j\}] \quad (1)$$

Upon iterating this relation we obtain the renormalization group *flow* in the space of couplings. Of special interest in this connection is obviously the set of fixed points $\{K_i^*\}$ where

$$K_i^* = \mathcal{R}_{[b]}^i[\{K_j^*\}] \quad (2)$$

The study of critical phenomena focuses on the critical (unstable) fixed points (see e.g. ref. [1] for a comprehensive review of real space renormalization group methods), but as is well known also non-critical (totally attractive) fixed points are of importance for the thermodynamical behavior of any given theory. Such “sinks” can generally be thought of as characterizing the different *phases*, while critical fixed points characterize the universal properties of (continuous) *phase transitions*.

Viewed as an iterative procedure eq. (1) describes a discrete map between a (perhaps infinite dimensional) set of coordinates. Although the general study of such iterative maps has experienced quite a renaissance within the last decade, little input has so far been felt on the theory of renormalization group transformations in statistical mechanics. Is this because such non-trivial phenomena of complex dynamical systems as *bifurcations*, *limit cycles*, *intermittency*, *chaos* etc. in iterative maps necessarily cannot occur in RG transformations for realistic systems? Or are such phenomena simply of no significance for *physical observables*?

The purpose of this paper is to renew an investigation into these questions. Let us make it clear from the start that we by no means claim priority. Before the advent of QCD, Wilson [2] considered coupling constant renormalizations with limit cycles as possibly of importance for strong interactions. And even in one of the classic papers on renormalization group theory in condensed matter physics [3] the idea of more complex renormalization group solutions is briefly commented upon. Apart from the appearance of a period two limit cycle in an RG solution of the Kondo problem [4], the first explicit examples of such phenomena were as far as we know discovered within the context of real space RG solutions on hierarchical lattices [5, 6]. Inspired by the observed periodic orbits and chaotic flows in hierarchical models, a general study of some of the physical consequences was presented in ref. [7]. Our work shall to a large extent make use of the observations already made there. Limit cycle behavior has later been observed in momentum space RG studies of abelian Higgs models [8]. Chaos has also been seen in truncated versions of Kadanoff’s variational real space RG when applied to Lie group valued spin models [9].

Before embarking on any detailed analysis of these issues, it is important to first clarify a few points. The complex behavior associated with iterative sequences of the general kind (1) is normally triggered by a set of free parameters $\{a_i\}$ with which the coefficients of the iterative equations can be varied. Such free parameters do in fact naturally appear in RG studies as well. To give an example, let us first write a more explicit

real space realization of the RG transformation $\mathcal{R}$ on the space of Hamiltonians. We do this by means of a projection operator $\mathcal{P}(s', s; a_i)$ and a partial spin summation:

$$e^{-H'(s')} \equiv \int [ds] \mathcal{P}(s', s; a_i) e^{-H(s)} \quad (3)$$

In this manner the map $\mathcal{R}_{[b]}[H] = H'$ has been achieved by first linking old spins s of H to renormalized spins s' of H' by means of the projection operator $\mathcal{P}$, followed by a spin summation which has removed some of the degrees of freedom. As it stands, eq.(3) is an exact equation (by definition). One requirement on $\mathcal{P}(s', s; a_i)$ is that the free energy (or the partition function) is preserved during the renormalization process. This is trivially achieved by the normalization condition

$$\int [ds'] \mathcal{P}(s', s; a_i) = 1 \quad (4)$$

Due to the appearance of a large set of free parameters $\{a_i\}$ in $\mathcal{P}$, the RG equations (1) for this particular transformation are in fact explicitly dependent on those parameters. By tuning them one might think that one could reach all the different complex phenomena adhered to earlier. This is, however, most likely *not* the case. The reason is that all physical predictions of eq.(3), when treated without any approximations, are completely *independent* of the parameters $\{a_i\}$. It is therefore only possible for bifurcations, chaos etc. to occur in these equations if they miraculously conspire to leave all physical predictions invariant. If this is so, then such complex behavior in the RG equations is in our opinion completely uninteresting.

This forces us to distinguish sharply between complex behavior induced in the RG equations as “gauge artifacts” (the addition of redundant operators to the fixed point Hamiltonian [10] could perhaps be taken as one example), and complex behavior which could possibly have physical consequences.

Let us give some examples of *physical* parameters that conceivably *could* trigger non-trivial behavior of the RG flow as these parameters are varied: In $O(n)$ vector models it could be the parameter n , in q -state Potts models it could be the number of states q , or it could simply be the dimensionality of space d . Once given for arbitrary n (or q , or d), the RG transformations of such models should be defined by analytic continuation to $n, q, d \in \mathbb{R}$. We believe that it necessarily must be generally true that periodic orbits instead of absolutely stable fixed points, chaotic flows etc. can only occur or not in *any given model*. These phenomena should not arise in any “dynamical” setting within a given model or universality class. This should be contrasted to the formal analogies between certain aspects of the period-doubling route to chaos, and the approach toward a critical point in statistical mechanics (see e.g. ref. [11] for some reviews). Such analogies, which are simply related to the scaling form (self-similarity) of certain functions, have nothing to do with the possibly nontrivial facets of RG transformations we shall be concerned with in this paper.

Except in very rare and special cases, exact real space renormalization group transformations for models on Bravais lattices are not known. By construction certain models on hierarchical lattices [12, 5, 7] can be solved exactly by a finite set of real space RG equations. Such hierarchical models are in many respects fundamentally different from models on Bravais lattices, but they do offer the tantalizing possibility of studying all the above effects in an exact framework. Our restriction to discrete real space renormalization

groups should, if treated exactly, and if not involving *e.g.* singularities of the kind suggested by Griffiths [13], not represent a limitation. Indeed, there seems *a priori* no reason to think that similar complex phenomena could not occur in momentum space renormalizations, also when perhaps expressed in differential form. But we should of course always be aware of the danger of introducing complex non-linear behavior solely due to inappropriate approximations.

In the special case of $d = 2$ dimensions, the additional assumption of *conformal invariance* at fixed points leads to what is close to prohibiting the kind of behavior we wish to investigate here. This is due to Zamolodchikov's *c*-theorem [14], which, briefly stated, guarantees the existence of a monotonic function C defined along an RG flow. This function is stationary only at (conformally invariant) fixed points, where it takes on unique values c (the central charges). Zamolodchikov's *c*-theorem thus appears to forbid in 2 dimensions any of the complex phenomena one could conceive for non-trivial RG flows, - at least for unitary theories, and for theories for which some continuum limit with a stress-energy tensor exists. (Hierarchical models presumably do not belong to this class). The so far unsuccessful search for a higher dimensional analogue of the *c*-theorem [15, 16, 17, 18] seems to still leave the question rather open for systems with $d > 2$.

Although in many respects very special, we shall start in section 2 by reanalyzing the case of just one coupling constant. The approach of this part of the paper is rather similar to that of ref. [7], but we hope to shed some new light on the problem. Section 3 is devoted to a discussion of finite volume scaling relations with *e.g.* chaotic RG flows. Here we also present some examples of how limit cycles and chaos in higher dimensional coupling constant spaces might arise. Section 4 contains our conclusions, as well as some comments on the possible existence of a *c*-theorem (or some appropriate generalization thereof) forbidding this kind of behavior in higher dimensions.

2. The Free Energy From RG Transformations

To simplify the discussion (while still keeping most of the essential ingredients) we shall first restrict ourselves to RG transformations in d dimensions *with only one coupling constant*. We call this coupling constant K .

Let $f(K)$ denote the free energy per spin. The central equation

$$f(K) = g(K) + b^{-d} f(\mathcal{R}_{[b]}(K)) \quad (5)$$

relates this free energy to the constant term $G(K) \equiv b^d g(K)$ of the Hamiltonian, and the RG transformation $\mathcal{R}_{[b]}(K)$. This equation shall form the basis of our analysis.

Consider first a standard *critical* fixed point K_0 at which

$$\lambda \equiv \left. \frac{\partial \mathcal{R}_{[b]}(K)}{\partial K} \right|_{K=K_0} > 1 \quad (6)$$

As we shall see, such a critical fixed point will always be associated with thermodynamic singularities in sufficiently high derivatives of $f(K)$. How do such singularities arise out of the iterative equation (5)? At the n th RG step the length scale increases $b^{n-1} \rightarrow b^n$. In the beginning, for small length scales, the constant term $g(K)$, which is assumed to be analytic around K_0 , cannot contribute to the singularities. The singular part $f_s(K)$ close to K_0 is then found by solving the homogenous equation

$$f_s(K) \sim b^{-d} f_s(K_0 + \lambda(K - K_0)) \quad (7)$$

Assuming a singular behavior of the form $f_s(K) \sim A(K - K_0)^\psi$, this gives $\psi = d \ln b / \ln \lambda$. It is obvious [19, 1] that one can multiply this solution $f_s(K) \sim A(K - K_0)^\psi$ by any function $C(K - K_0)$ satisfying $C(K - K_0) = C(\lambda(K - K_0))$, or, alternatively, by a periodic function $H(\ln(K - K_0)/\ln \lambda)$ of period 1. Indeed, the existence of such a periodic function modulating the standard singular behavior can also be understood from a physical point of view. At each RG step the correlation length decreases by a factor b : $\xi \rightarrow \xi/b$. Close to a standard 2nd order phase transition this correlation length diverges like $\xi \sim (K - K_0)^{-\nu}$, where $\nu = \ln b / \ln \lambda$. Near K_0 one thus expects the modulation to occur whenever $\xi \sim b^n$, i.e. precisely when $n = \ln(K - K_0)/\ln \lambda$.

Conversely, after very many RG steps almost all spins have been integrated out, and the singular behavior shifts from the last term in eq. (5) to the first. As $n \rightarrow \infty$ the first term indeed equals the logarithm of the partition function, and all thermodynamic functions can be derived from it. This is seen in detail by writing the solution to eq. (5) as

$$f(K) = \sum_{n=0}^{N-1} b^{-dn} g(\mathcal{R}_{[b]}^{(n)}(K)) + b^{-dN} f(\mathcal{R}_{[b]}^{(N)}(K)) \quad (8)$$

Provided $f(\mathcal{R}_{[b]}^{(N)}(K))$ remains sufficiently bounded, the full free energy per spin is then given by

$$f(K) = \sum_{n=0}^{\infty} b^{-dn} g(\mathcal{R}_{[b]}^{(n)}(K)) \quad (9)$$

At any fixed point K^* the solution $f(K)$ is readily found to be

$$f(K^*) = (1 - b^{-d})^{-1} g(K^*) \quad (10)$$

which in particular holds at the critical point K_0 itself.

Consider next the m th derivative of $f(K)$, evaluated at the critical point K_0 . Provided the sum converges, i.e. provided $\lambda < b^{d/m}$, we find the solution

$$\begin{aligned} f^{(m)}(K_0) &= \sum_{n=0}^{\infty} (b^{-d} \lambda^m)^n g^{(m)}(K_0) \\ &= (1 - b^{-d} \lambda^m)^{-1} g^{(m)}(K_0) \end{aligned} \quad (11)$$

When the sum

$$S_N^{(m)} = \sum_{n=0}^N (b^{-d} \lambda^m)^n g^{(m)}(K_0) \quad (12)$$

no longer converges as $N \rightarrow \infty$ (which happens for sufficiently large m), we can, strictly speaking, not identify this divergence with a singularity in $f^{(m)}(K_0)$. (The interchange of summation and differentiation, which is implicit in eq.(11), is no longer permissible). Still, if we are not dealing with pathological cases, we can definitely take this as an indication that a singularity may have occurred in $f^{(m)}(K_0)$ itself. Further, if the sum converges, then we know with certainty that $f^{(m)}(K_0)$ is finite and well defined. Convergence vs. divergence of the sum in eq.(11) thus does give a *bound* on the order of a possible phase transition at K_0 .

Indeed, we see that $S_N^{(m)}$ diverges for

$$\lambda \geq b^{d/m} \quad (13)$$

and it is clearly of interest to compare this with the special case of a 2nd order phase transition, where $f^{(2)}(K) \sim |K - K_0|^{-\alpha}$. From the scaling relation

$$2 - \alpha = d\nu \quad (14)$$

we see that the criterion for divergence of $f^{(2)}(K_0)$, *i.e.* $\alpha \geq 0^2$) becomes $\nu \leq 2/d$. Inserting the definition

$$\nu \equiv \frac{\ln(b)}{\ln(\lambda)} \quad (15)$$

this precisely yields

$$\lambda \geq b^{d/2}, \quad (16)$$

in agreement with eq. (13) in this special case of a 2nd order phase transition, where $m = 2$.

Curiously, even when the solution (12) yields a singularity in the m th derivative of $f(K)$ at K_0 , a finite “ghost” solution to the original equation (5) of precisely the same form as eq. (12) still exists. In fact, whenever the original expansion (12) diverges, a *convergent* solution of eq. (5) in the form of

$$f^{(m)}(K_0) = - \sum_{n=1}^{\infty} b^{dn} \lambda^{-mn} g^{(m)}(K_0) \quad (17)$$

appears. This derivative solution, which can be viewed as an analytic continuation of the sum (12) into the domain where it diverges, can obviously *not* be interpreted as the derivative of the physical free energy of the given system.

Having established these elementary results for the magnitude of the free energy and its derivatives at fixed points (and in particular at critical points), we can now immediately make the generalization to more complex situations. Suppose that instead of fixed points we have a *periodic orbit* of period N . Be it unstable or stable, we can write down an exact matrix equation for the free energy on this periodic orbit, here denoted by the set $\{K_i \mid i = 1, 2, \dots, N\}$:

$$f(K_i) = g(K_i) + b^{-d} \sum_{j=1}^N M_{ij} f(K_j) \quad (18)$$

where $M_{ij} = 1$ for $j = i + 1 \pmod{N}$, and zero otherwise. This equation is readily solved to give

$$f(K_i) = \sum_{j=0}^N \frac{b^{-dj} g(K_{j+i \pmod{N}})}{1 - b^{-dN}} \quad (19)$$

at any point K_i of the orbit. Similarly for the derivatives of the free energy on this orbit we find

$$f^{(m)}(K_i) = \sum_{i=0}^{\infty} \sum_{j=Ni}^{N(i+1)-1} b^{-dj} \lambda_{j+i \pmod{N}}^{mj} g^{(m)}(K_{j+i \pmod{N}}) \quad (20)$$

A more general case which also turns out to be of importance is that of a (stable or unstable) fixed point K^* that can be reached in a *finite number of iterations* n from a *preimage* $K_{pre(n)}$. (Clearly all intermediate points $\mathcal{R}_{[b]}^{(m)}(K_{pre(n)})$ with $m < n$, denoted by

2) The case $\alpha = 0$ is, as usual, understood in the logarithmically divergent sense.

$K_{pre(n-m)}$, are also preimages of K^*). Using the defining equation (5) it is straightforward to solve for the free energy at any preimage:

$$f(K_{pre(n)}) = \sum_{i=1}^n b^{(i-n)d} g(K_{pre(i)}) + \frac{b^{-dn}}{1 - b^{-d}} g(K^*) \quad (21)$$

Likewise for the m th derivative of the free energy $f^{(m)}(K_{pre(n)})$ at any of the preimages, we find

$$f^{(m)}(K_{pre(n)}) = \sum_{i=1}^n b^{(i-n)d} g^{(m)}(K_{pre(i)}) + \frac{(b^{-d}\lambda^m)^n}{1 - b^{-d}\lambda^m} g^{(m)}(K^*) \quad (22)$$

provided $\lambda < b^{d/m}$, i.e. provided the infinite sum converges. For $\lambda \geq b^{d/m}$ the sum diverges, and we have the possibility of a thermodynamic singularity at all preimages.

At this point it is instructive to make some contact to the explicit example discussed in ref. [7] in connection with an unstable fixed point and its preimages. To illustrate the singularities appearing at each preimage, a simple example with $g(K) = K$ and $K' = \mu K(1 - K)$ [the logistic map] was considered with $b^d = 1.01$. Letting $b^d \rightarrow 1$ clearly enhance the singularities. Indeed, at the fixed point $K^* = 1 - \mu^{-1}$ the derivative equals

$$\lambda = \left. \frac{\partial K'}{\partial K} \right|_{K=K^*} = 2 - \mu \quad (23)$$

and the condition for a singularity at the m th derivative of the free energy becomes

$$|2 - \mu| \geq 1.01^{1/m} \simeq 1 + \frac{1}{100m} \quad (24)$$

(We take the absolute value on the l.h.s. since the fixed point actually becomes unstable with $\lambda \leq -1$; the convergence criterion is obviously identical for $\lambda \leq -1$ and $\lambda \geq 1$). Except for a very narrow interval $3 < \mu < 3.01$, the singularities occur for all $\mu > 3.01$ already in the *first derivative* of $f(K)$.

The preimages of the non-trivial fixed point of the logistic map are easily found. The first occurs at $K = \mu^{-1}$. By linearizing around the other fixed point at $K = 0$ it follows that an infinite number of preimages $K_{pre(n)} \simeq \mu^{-n}$ accumulate geometrically toward $K = 0$, and, on account of the $K \leftrightarrow 1 - K$ symmetry, likewise toward $K = 1$. One of the cases considered in ref. [7] is $\mu = 3.326$, for which a period 2 stable orbit exists around the unstable fixed point. In fig. 1 we show $f(K)$, which indeed clearly has divergent first derivative at K^* and all its preimages, which here, to emphasize the geometric accumulation toward the endpoints of the interval, have been plotted on a logarithmic scale. The linear rise in $f(K_{pre(n)})$ is an immediate consequence of eq. (21). If we take the limit $b^d \equiv 1 + \epsilon \rightarrow 1$, the last term in that equation will dominate, and $f(K_{pre(n)})$ should, on approximately linear spacing (as in fig. 1), rise linearly with known slope and normalization:

$$f(K_{pre(n)}) = \epsilon^{-1}(1 - \epsilon n)g(K^*) + \dots \quad (25)$$

which in this particular case corresponds to

$$f(K_{pre(n)}) = 100(1 - 0.01n)(1 - \mu^{-1}) + \dots \quad (26)$$

We have plotted this simple prediction as the dashed line in fig. 1. The peaks (where the first derivative of f diverges) are too narrow to be followed up to their maximum values on the figure. For example, the peak at $\ln(t) = -0.3576\dots$ (the original fixed point) reaches up to $70.633\dots$, slightly above the straight line. Similarly for the preimages. Thus in fact not only the slope comes out correctly, the normalization is also given by the asymptotic formula (21).

After this discussion of fixed points, and, more generally, limit cycles, let us now explore the physical consequences of *chaotic* RG transformations. If, continuing from the previous cases, we consider the bifurcation route to chaos, then this can be viewed as the limiting case (and beyond) of $N \rightarrow \infty$ in a period- N limit cycle. With such a simple analogy in mind, one could expect to some remnants of the phenomena described above, *but now at any point of the chaotic attractor*. This actually turns out to be a quite accurate description.

We return to what we take as the defining equation of the free energy, eq. (5). One of the characteristic signals for chaos is that two neighboring points K_0 and $K_0 + \delta K$ under successive iterations separate exponentially along the chaotic attractor. From the series solution

$$f(K) = \sum_{n=0}^{\infty} b^{-dn} g(\mathcal{R}_{[b]}^{(n)}(K)) \quad (27)$$

it is clear that this will typically have almost no bearing on $f(K)$: The combination of the damping factor b^{-dn} and the assumed boundedness of the function g leads to a very rapid convergence of the sum in eq.(27) [20] (although of course counterexamples do exist). The fact that $\mathcal{R}_{[b]}^{(n)}(K)$ now moves along a chaotic attractor has essentially no significance. Thus the free energy itself will normally turn out to be a continuous and finite function even in the chaotic regime.

Then how precisely *will* chaos in the underlying dynamical system manifest itself in the thermodynamic functions derived from the RG flow? Since chaos is characterized by nearby points separating exponentially with the number of iterations, one is naturally led to investigate, again, *derivatives* of the free energy.

As long as the right hand side is absolutely convergent we have

$$f^{(m)}(K) = \sum_{n=0}^{\infty} b^{-dn} \frac{\partial^m g(\mathcal{R}_{[b]}^{(n)}(K))}{\partial K^m} \quad (28)$$

but this does not give us the m th derivative of the free energy f in a very suitable form. In contrast to the behavior around fixed points and limit cycles, details of the function g in the whole chaotic coupling constant interval are now needed to determine the sum. Still, assuming again boundedness of g (the analytical properties of g can evidently be chosen completely independent of the chaotic nature of the iterative map), we can actually gain some information.

First note that the exponential separation of neighboring points in the chaotic regime can be quantified in terms of the *Liapunov exponent* $\mu(K)$ of the map. For the one-dimensional case at hand, we can define it as [21]

$$\mu(K) \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left| \frac{\partial \mathcal{R}_{[b]}^{(N)}(K)}{\partial K} \right| \quad (29)$$

Phrased differently, for N sufficiently large the derivative $\partial \mathcal{R}_{[b]}^N(K)/\partial K$ approaches an asymptotic value

$$\frac{\partial \mathcal{R}_{[b]}^{(N)}(K)}{\partial K} \sim e^{\mu(K)N} \quad (30)$$

which, in the way it has been defined here, depends on the initial point $\mathcal{R}_{[b]}^{(0)}(K) \equiv K$. In fact, although it may take varying values at an infinity of points on the chaotic interval, it will for generic K take on a unique value, which we can call the Liapunov exponent of the map itself, μ . Chaos clearly implies $\mu > 0$.

Substituting the large- N behavior (30) into the right hand side of eq.(28) gives us

$$\tilde{S}^{(m)} = \sum_{n=0}^{\infty} g^{(m)}(\mathcal{R}_{[b]}^{(n)}(K)) e^{(\mu m - d \ln(b))n} \quad (31)$$

which of course cannot be identified with the left hand side of eq.(28), since we have blindly inserted the limiting behavior (30) for all values of n . Still, since convergence/divergence of the series (28) precisely is determined by the behavior of the infinitely long tail, we can use eq.(31) as an estimator.

Note that when $g^{(m)}(K)$ is a bounded function, the sum $\tilde{S}^{(m)}$ converges when the Liapunov exponent is limited by

$$\mu < \frac{d}{m} \ln(b) \quad (32)$$

As one could perhaps have expected, the rôle of the derivative at the fixed point (the RG eigenvalue λ) has in this case been taken over by the exponential of the Liapunov exponent $\exp[\mu]$.

Since μ is believed to be a highly discontinuous function of the bifurcation parameters, it is of interest that there exists a continuous function h , the *topological entropy*, which gives an upper bound on the Liapunov exponent μ (see e.g. ref. [21]). If one is interested solely in a bound on the number of derivatives one can take without encountering a divergent solution (32), μ can be replaced by this quantity.

We can turn the preceding argument around, and conclude that the m th derivative of the free energy $f^{(m)}(K)$ is convergent whenever $m < d \ln(b)/\mu$. (Recall that the chaotic interval may also in general contain an infinite set of unstable fixed points and associated preimages at which the Liapunov exponent will take on locally different values). Since the Liapunov exponent μ typically is of order unity, often a high number of derivatives are needed before one reaches the point where the next derivative may diverge. Precisely at the chaos threshold $\mu = 0$ the function $f(K)$ will be infinitely many times differentiable.

So far the discussion has implicitly been restricted to coupling constant values *inside* the chaotic attractor. As long as one starts just within the *domain of attraction* of the chaotic region, this is actually no real restriction. After a finite number of RG steps one lands inside the chaotic interval, and the above asymptotic arguments go through also in this case.

Finally, let us comment upon a phenomenon related to chaos: Intermittency. Imagine an RG flow that passes through a region in coupling constant space where it spends a long time being almost trapped, before eventually escaping. Except for the eventual escape, this situation is very reminiscent of an attractive fixed point. As expected, if the summed series for the free energy converges sufficiently rapidly, the free energy itself may look indistinguishable from the free energy of a flow ending on a fixed point inside the above

region. Differences will again only show up after a large number of derivatives have been taken. Then convergence/divergence is again determined by the infinitely long tail of the series. But this infinitely long tail arises from the RG flow in an entirely independent part of Hamiltonian space, that of the true attractor of the region.

3. Finite Volumes and Higher Dimensional Coupling Constant Spaces

The previous section dealt with systems that were in many ways unrealistic. Although hierarchical models exist for which RG equations involving just one coupling constant are exact, this will ordinarily only occur in extremely truncated RG transformations. We shall in this section briefly discuss what might occur in higher dimensional coupling constant spaces. But before moving on to this point, let us consider another idealization of the previous section: The infinite volume of the system under consideration. For ordinary critical fixed points it is known that finite size scaling theory can be just as useful a tool as the idealized infinite volume theory of critical phenomena. Can a finite size scaling theory be set up also for the more complex type of RG flows considered in the last section? To answer this question, we shall rely on a heuristic approach which is known to work in the case of ordinary critical fixed points (see *e.g.* ref. [22]). This part of the paper is for notational convenience again restricted to a space of just one coupling constant, but in this case the generalization to higher dimensional spaces is straightforward.

The idea is to consider not systems of finite size, but *systems acted upon with a finite number of real space RG steps*. (For finite size systems with periodic boundary conditions this may intuitively be seen as closely related). Furthermore, appealing again to the arguments in the beginning of section 2, we shall assume that the finite volume (or the number of RG steps) is so large that in the partially summed solution eq.(8), the most singular term of f is already contained in the first term on the right hand side. From now on we hence discard the second term.

Consider first the free energy itself around an ordinary fixed point:

$$\begin{aligned} f(K^*) &= \sum_{n=0}^N g(K^*) b^{-dn} \\ &= g(K^*) (1 - b^{-d})^{-1} (1 - b^{-d} L^{-d}) \\ &= \text{const} \cdot (1 - b^{-d} L^{-d}) \end{aligned} \quad (33)$$

where we have substituted $L^d = b^{Nd}$ in order to explicitly display the dependence on the finite volume L^d .

Suppose the m th derivative of $f(K)$ diverges at K^* when the volume is infinite. Then we expect the finite volume behavior to be of the form

$$\begin{aligned} f^{(m)}(K^*) &= \sum_{n=0}^N b^{-dn} \lambda^{mn} g^{(m)}(K^*) \\ &= g^{(m)}(K^*) (b^{-d} \lambda^m)^N \sum_{n=0}^N (b^d \lambda^{-m})^{N-n} \\ &= \frac{g^{(m)}(K^*)}{1 - b^d \lambda^{-m}} \lambda^{mN} L^{-d} (1 - b^d \lambda^{-m(N+1)} L^d) \end{aligned} \quad (34)$$

where we have made use of $b^{-d} \lambda^m > 1$, by assumption of the m th derivative of f being singular at K^* . In eq.(34) we must still eliminate the N -dependence, since this is the way

to derive the full $L = b^N$ behavior. We can achieve this by the introduction of the critical exponent ν through the definition $\nu \equiv \ln(b)/\ln(\lambda)$. Then

$$\lambda^{mN} = b^{Nm/\nu} = L^{m/\nu} \quad (35)$$

that is,

$$\begin{aligned} f^{(m)}(K^*) &= \frac{g^{(m)}(K^*)}{1 - b^d \lambda^{-m}} L^{m/\nu-d} (1 - b^d \lambda^{-m} L^{d-m/\nu}) \\ &= \text{const} \cdot L^{m/\nu-d} (1 - b^d \lambda^{-m} L^{d-m/\nu}) \end{aligned} \quad (36)$$

Since we have discarded all non-leading pieces, we cannot expect the shown correction term to be an accurate representation. But the leading term actually does give the correct behavior. In the special case $m = 2$ we thus have

$$f^{(2)}(K^*) \sim L^{2/\nu-d} \quad (37)$$

which indeed is the established leading-order finite size scaling relation for a 2nd order phase transition, where $\alpha = 2 - d\nu$.

In the above argument we have assumed $b^{-d} \lambda^m > 1$. If $\lambda^m = b^d$, we get

$$\begin{aligned} f^{(m)}(K^*) &= \sum_{n=0}^N g^{(m)}(K^*) \\ &= N g^{(m)}(K^*) \\ &= \frac{g^{(m)}(K^*)}{\ln(b)} \ln(L) \end{aligned} \quad (38)$$

i.e. the correct logarithmic dependence (as for e.g. the Ising model in $d = 2$ dimensions, where $\alpha = 0$ and $m = 2$).

We can now focus on the analogous finite size system when the RG transformation is in the chaotic regime. Assuming that the number of RG steps is so large that the asymptotic estimate (34) is applicable, we are, in the case of the m th derivative with $\mu < d \ln(b)/m$, led to consider³⁾

$$\tilde{S}_N^{(m)} = \sum_{n=0}^N g^{(m)}(\mathcal{R}_{[b]}^{(n)}(K)) e^{\mu m - d \ln(b)n} \quad (39)$$

Note the difference to the fixed point case: The summation now also extends over the n -dependent $g^{(m)}(\mathcal{R}_{[b]}^{(n)}(K))$, which in this chaotic regime covers the whole chaotic attractor. Nevertheless, as before the function $g^{(m)}$ is assumed to be analytic and bounded, which means that we can define

$$A \equiv \max_{n \in \{1, 2, \dots, N\}} g^{(m)}(\mathcal{R}_{[b]}^{(n)}(K)) \quad (40)$$

such that

$$\begin{aligned} \tilde{S}_N^{(m)} &\leq A \sum_{n=0}^N e^{(\mu m - d \ln(b))n} \\ &= \frac{A}{1 - \exp[d \ln(b) - \mu m]} L^{m\mu/\ln(b)-d} (1 + \mathcal{O}(L^{d-m\mu/\ln(b)})) \end{aligned} \quad (41)$$

3) Recall that the singular behavior arises from the tail of the series.

This heuristic argument thus gives a finite size scaling law

$$f^{(m)}(K) \leq \text{const} \cdot L^{m\mu/\ln(b)-d} \quad (42)$$

where as before μ is the positive Liapunov exponent of the map. Note that this scaling law should hold *at any point K of the chaotic attractor*, but of course with a K -dependent constant of proportionality. Also note that on account of the inequality sign this scaling law is not nearly as strong as the fixed point case.

Using the same approach one can derive related expressions for the finite size scaling law along a periodic orbit, but we shall not display this formula here.

Finally we return to the infinite volume case, but consider the modifications required if we allow the coupling constant space to be higher dimensional.⁴⁾

Let us consider the most simple case that still contains all the essentials. For such a system, with one relevant direction (say, along the temperature), the RG flow looks schematically as shown in fig. 2. There are 3 fixed points, among which one (the critical) has one eigenvalue λ_1 larger than unity, all other eigenvalues being smaller than one. The two other fixed points have all eigenvalues smaller than one; these fixed points are totally attractive. Let us now consider various perturbations of this system. Shown in fig. 3a is the case where either the high-T or low-T fixed point has become unstable, and has bifurcated in a period-2 limit cycle. This occurs when the smallest eigenvalue $\lambda_{\min}$ of the original fixed point crosses below -1. Instead of flowing directly into the original fixed point, the RG trajectories now begin to move in this direction, but eventually the negative eigenvalue $\lambda_{\min}$ causes oscillations (of period two) in the direction of its scaling field. The two new sinks are fixed points only of the RG map $\mathcal{R}_{[b]}^{(2)}(K)$.

This bifurcation sequence can obviously continue, as indicated in fig. 3b. After n bifurcations the new sinks are fixed points only of $\mathcal{R}_{[b]}^{(2^n)}(K)$. The final flow of the starting RG map can now be very complicated, jumping indefinitely around on the period- $2n$ orbit.

Since the correlation length ξ decreases by $\xi \rightarrow \xi/b$ at each RG step, we expect $\xi = 0$ all along the periodic orbit.

Perturbing the flow in this manner, we may eventually (but not necessarily) reach chaos beyond $n = \infty$. The sink has now been replaced by a higher dimensional (strange) attractor. The correlation length should vanish on this attractor. In contrast to the one-dimensional case, it requires infinite precision to land exactly on the chaotic attractor itself. Instead one will generically only approach the attractor asymptotically as the RG equations are iterated. But the behavior of the RG flow will still be governed by the Liapunov exponent of the attractor, and will for all practical purposes appear fully chaotic.

For higher dimensional RG transformations period doubling sequences and eventually chaos have been observed in truncated versions of Kadanoff's variational real space RG when applied to certain Lie group valued spin models [9, 20]. Although this permits us to check in a realistic setting some of the ideas of the present paper, it should not represent the true thermodynamic behavior of those models.

Bifurcations can occur not only on the totally attractive fixed points of fig. 2, but also on the critical hypersurface. This corresponds to one of the irrelevant eigenvalues

4) The actual case of interest normally involves an infinite dimensional space, but dynamical systems of infinite dimensions are not well explored. We shall thus always assume that the space is finite dimensional. From any practical point of view this restriction is immaterial.

passing through -1 from above, while the relevant eigenvalue otherwise remains larger than +1. This is illustrated in fig. 4. Here the critical behavior itself is affected. The speed away from the critical hypersurface, which by the discussion following eq. (7) yields the critical exponent, will now be determined by the largest eigenvalue $\lambda_{max}^{(2)}$ of $\mathcal{R}_{[b]}^{(2)}(K)$ on the two new fixed points on the critical hypersurface of this map. Loosely speaking, if one approaches the limit cycle on the critical hypersurface from far away, one sees just one critical fixed point. Only by tuning the flow to go very close, does one discern the split of the fixed point into a limit cycle. Specifically, then, the critical exponent ψ of the previous discussion is obtained by replacing λ of eq. (6) by $\lambda_{max}^{(2)1/2}$. This clearly generalizes to period- $2n$ limit cycles. One can also conceive situations where the critical fixed point is replaced by a completely chaotic attractor, living on the subspace of the critical hypersurface.

4. Conclusion and Comments

Limit cycles and chaos in RG equations do not seem to lead to unphysical consequences if one restricts oneself to studying the free energy $f(K)$ and its derivatives. We have found that typically there will be singularities in a high derivative of $f(K)$ if, for example, one is within the domain of attraction of a chaotic attractor. But this type of behavior is perfectly acceptable from a thermodynamic point of view. For example, it has been argued that chaotic coupling constant renormalizations may describe the physics of spin glasses [5]. (Some independent evidence in favor of the spin glass phase being chaotic has been given from a domain wall energy argument [23]). A more explicit demonstration of this is, however, still lacking.

Will such behavior actually occur in real systems? Perhaps the strongest objection to this is related to the c -theorem in $d = 2$ dimensions. An intuitive argument has been put forth [15] that the existence of a monotonically decreasing function C among the different fixed points connected by an RG flow is related to general aspects of the Renormalization Group itself, mainly the fact that this is really only a semi-group, the inverse operation not being well defined. This indeed has as one consequence a built-in irreversibility: Short distance physics is gradually being integrated out to yield effective long distance interactions. These irreversible features have led to the conjecture that perhaps an entropy-like function, monotonically increasing on any RG trajectory⁵⁾, can be defined for the RG flow in any number of dimensions, thus generalizing Zamolodchikov's original idea to systems with $d > 2$.

Arguments of the kind discussed above almost invariably rely on such features as Poincaré invariance and unitarity. But this already excludes a number of theories (and physical systems). For example, the spin glasses discussed above presumably do not fall into this class. Similarly, the limit cycle behavior found in ref. [8] when using the replica method, presumably is precisely possible because it concerns a non-unitary system. Even in $d = 2$ dimensions, "physical" systems such as random q -state Potts models lead to RG flows that do not correspond to a decreasing C -function [26], again in no conflict with Zamolodchikov's theorem since they correspond to non-unitary theories.

5) Perturbatively the RG flow of a scalar field theory in 4 dimensions may indeed be described in terms of a gradient flow of a scalar function [24]. See also [15] and [25]. A more general statement can in fact be made within the context of perturbation theory [16, 17].

Having also explicit counterexamples at our disposal, it is instructive to see why an entropy function may not be expected to exist in general. Consider as an example a fully chaotic RG flow. Also here degrees of freedom are continually being integrated out, but nevertheless one does not approach a single well defined fixed point. Instead one finds, as one probes larger and larger distances, new hierarchies of couplings arising out of the chaotic orbit. Weak and strong coupling may follow in rapid succession. On any new scale, physics may look very similar to the physics observed at a much earlier stage. In such a case it does not appear to be very useful to think of the RG transformation as associated with *loss of information*, since one perpetually wanders around on the chaotic attractor. Although theories displaying such behavior definitely are unusual, there may be nothing preventing this scenario. Certainly one may wish that other constraints (unitarity, certain field theoretic axioms etc.) can rule out such behavior in all dimensions for a large class of theories, but this program clearly lies much beyond the scope of the present work.

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Figure Captions

Fig. 1.: On a scale in which the preimages of a fixed point are linearly spaced, the free energy of these fixed points approach asymptotically the straight line indicated as $b^n \rightarrow 1$. The example shown here is that of ref. [7], in which the transformation is taken to be the logistic map, and $b^n = 1.01$. The dashed line shown is eq. (26). The peaks at the fixed point do not appear to reach completely up to the line, but this is just due to the finite grid used in preparing the curve for $f(t)$.

Fig. 2.: A typical RG flow diagram in the space of couplings, with just one relevant direction, the temperature. Shown is the critical hypersurface, the critical fixed point located on this surface, and the two trivial fixed points outside. Flows starting near the critical hypersurface move close toward the critical fixed point, but eventually deviate and finally reach the trivial sinks in a regular flow.

Fig. 3.: One of the trivial fixed points has become unstable, and a period two limit cycle has formed (a). The flow originally terminating at the fixed point now indefinitely oscillate between the two points indicated. This period two cycle can again become unstable, creating a period four limit cycle (b), and so on.

Fig. 4.: Also the critical fixed point can become unstable, now on the subspace of the critical hypersurface itself.

Fig. 1

Fig. 2

Fig. 3

Fig. 4