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Integrable Systems and Gauge Theories*

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Abstract

We present a general concept of integrability for systems of field theory valid in any dimension, and we use the algebraic structures underlying it to define generalized notions of gauge fields and gauge theories

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1. INTRODUCTION

The questions I would like to address in this lecture concern the problem of defining a generic concept of integrability, valid in any dimension, for systems of Field Theory and Statistical Mechanics and the possible use of the corresponding algebraic structures underlying it in defining generalized notions of Gauge Theories.

In the last twenty years considerable progress has been made in our understanding of complete integrability properties for systems with large or infinite number of degrees of freedom (see for reviews [1,2,3,4]). Most of these results may in turn be traced back to the discovery of very powerful algebraic structures leading to the exact solvability of such models, i.e., the Yang-Baxter relations [5,4] and the Lax or Zakharov-Shabat [6,7] equations. Moreover, a very powerful concept has emerged as the basic algebraic structure for $(1+1)$ -dimensional integrability, namely, the notion of the quantum group that is now the subject of intensive studies [8]-[26]. However, and notwithstanding the great amount of success obtained along these lines, the class of systems involved and their integrable structures are still mainly confined to low space or space-time dimensions ($d \leq 2$ or $d \leq 1+1$).

From the point of view of Field Theory, Lax pairs are at present the main tool for constructing and solving $1+1$ -dimensional classical integrable systems. In that case the Lax or Zakharov-Shabat equations may

be identified with a zero-curvature condition for some two-dimensional connection A parametrized by the dynamical fields of the system in such a way that the constraint $F(A) = 0$ is equivalent to their equations of motion. The interest in this situation is that it allows for an easy construction of a large set of constants of motion.

Unfortunately this picture is very specific to two-dimensional systems. If one were to try to apply it to a higher-dimensional situation, say $2+1$, then the corresponding model would exhibit rather peculiar features. For example, the conserved quantities constructed from the corresponding Lax equations would appear to be given as line-integrals rather than integrals all over a space-like surface, i.e., such conserved quantities would not "scan" the total space-structure (i.e., initial data) of the dynamics, unless the Lax pair were given in terms of non-local functionals of the fields. Even though such a case might be interesting, it would be desirable to have a more generic tool, giving the true analogue of the Lax equations for higher-dimensional dynamical systems that may, in particular, be described through local fields.

It is the purpose of this note to give a preliminary account of new concepts and algebraic structures leading to a generic extension of the notion of Lax pairs and Yang-Baxter equations into any dimensions, and to exhibit a simple geometrical interpretation for them that will be shown to generate an extended type of gauge

fields and gauge theories.

Most of the material presented here is new. A more detailed description of these concepts together with related results and applications will be published separately. This paper is organized as follows.

In section 2, we define what we call the Fundamental Simplex Relations (FSR's) that give a natural geometrical and algebraic extension of the notion of zero-curvature equations. These considerations enable us to make a brief incursion into the problem of defining, along these lines, the structures of a non-Abelian cohomology type theory. For the low-dimensional FSR's, we make contact with the Lax and Yang-Baxter structures, therefore providing a new geometrical point of view for them.

In section 3, we consider the application of the above results to the definition of multi-dimensional integrable dynamical systems. In particular we show how to obtain results for the lattice integrable case previously published in [27,28,29] from this present more fundamental approach. We then give very recently constructed 3-dimensional lattice dynamical systems that are examples of the above algebraic structures.

Section 4 is devoted to the extension of the results of sections 2 and 3 and to some of their applications. We give, for example, a geometrical interpretation of the Yang-Baxter equations and of their associated Hopf algebras. In particular we describe a geometrical and algebraic setting for the so-called universal R -matrix associated to the quantum group that follows from the structures we described in section 2. This construction is related to the local infinitesimal limit of the integrability equations for 3-dimensional systems given in sections

2 and 3 that we also describe here. These considerations lead us to define new types of gauge fields and gauge theories. We call them n -simplex type gauge fields and gauge theories. The FSR's of section 2 are shown to be zero-curvature equations for these extended gauge fields. The 1-simplex situation reproduces usual gauge theories. We describe the example of the 2-simplex extended gauge theories both on the lattice and in the continuum, together with their possible applications.

It is a pleasure to dedicate this paper to Raymond Stora on the occasion of his 60th birthday.

2. THE FUNDAMENTAL SIMPLEX RELATIONS

As we just recalled it in the Introduction, for low-dimensional systems of Field Theory ($d = 1 + 1$), the integrability properties of the dynamics are intimately related to the existence of a Lax pair. In such a situation, the equations of motion may be written as the flatness condition for a certain two-dimensional gauge field A parametrized by the dynamical variables of the system. This fact enables us, in non-degenerate cases, to solve the corresponding model by providing enough independent constants of motion.

In order to be able to extend this picture to higher-dimensional systems, it is useful to get a more geometrical and algebraic understanding of this two-dimensional situation.

The above zero-curvature condition for the gauge field A may be thought of as a localized or gauged version of a commutativity condition for operators. This formulation will be called "cubic-type" in the following for reasons to become clear later on. Another point of view is to say that when going from the point (i) to

the point (j) , we have a transition matrix g_{ij} (given as $\mathcal{P} \exp(\int_{C_{ij}} A)$, C_{ij} being an oriented path connecting (j) to (i)), mapping a vector Ψ_j defined at point (j) to its value in (i) in such a way that the result is not dependent on the path C_{ij} . In other words, if we choose three arbitrary points i, j, k , we have the relation

$$g_{ij} \cdot g_{jk} = g_{ik} \quad (1)$$

which is another way of writing the flatness condition. It should be noted, however, that the integrated relation (1) contains in fact more information than the equation $F(A) = 0$, i.e., it encodes also the holonomy behaviour of the field A . From a solution g_{ij} of these equations, we can easily construct commutativity-type relations by choosing, for example, four points i, j, k, l to be the successive four summits of a square plaquette. We obtain from eq.(1)

$$g_{ij} \cdot g_{jk} = g_{il} \cdot g_{lk} \quad (2)$$

It is this type of relation that is used in building Lax equations for dynamical systems, in particular for lattice systems. However, let us stress that (1) is to be considered as a more fundamental relation in the sense that it gives a minimal way of writing the flatness condition, i.e., using only three points and three paths connecting them that define a minimal simplex in $d = 2$. For that reason we will call relations like those in (1) "simplex-type" and relations as in (2) "cubic-type" that may always be considered as consequences of the simplex ones (1). Note that in this picture, we are using integrated relations rather than lattice or discretized ones, in particular, the points i, j, k, l in eqs.(1,2) are arbitrary points in $\mathbf{R}^2$. Of course discretized or lattice structures follow easily from these by restricting the set of possible points.

Let us now reformulate this picture in a more algebraic and geometrical language. We consider points (i) in some space (say $\mathbf{R}^d$) and oriented paths C_{ij} going

from (j) to (i) (the orientation arrow being from (j) to (i) also). Then, we define a geometrical product of two oriented paths C_{ij} and C_{jk} (when the starting point of the first is equal to the ending point of the second) as the oriented total path C_{ik} going from (k) to (i) , i.e.,

$$C_{ij} \circ C_{jk} = C_{ik} \quad (3)$$

To any point (i) we associate a vector space V_i . To any oriented curve C_{ij} , we associate line-objects $g(C_{ij}) \in V_i \otimes V_j^*$ (where V_j^* is the dual space to V_j), in such a way that the above geometrical product of two oriented paths C_{ij} and C_{jk} is mapped onto a product for the corresponding g 's realized through the duality bracket (evaluation) at point (j) between V_j^* and V_j , namely,

$$g(C_{ij}) \circ_{V_j} g(C_{jk}) = g(C_{ij} \circ C_{jk}) \quad (4)$$

Here, and in the following, we denote by $x \circ_{V_j} y$ the product between two elements $x \in V_j^*$ and $y \in V_j$ obtained by the evaluation of x on y given by the duality bracket between V_j^* and V_j . We define point-like objects $\Psi_i \in V_i$, such that their values at any points (i) and (j) are related by a line-element $g(C_{ij})$ as

$$\Psi_i = g(C_{ij}) \circ_{V_j} \Psi_j \quad (5)$$

where the product between g and Ψ is again given by the evaluation of $g(C_{ij})$ on Ψ_j (duality between V_j^* and V_j). If we require the Ψ 's to be functions of the points only, we obtain the following (sufficient) compatibility condition for the corresponding g

$$g(C_{ij}) = g(C'_{ij}) \quad (6)$$

for any oriented paths C_{ij} and C'_{ij} . This is an integrated flatness condition for g . If g is parametrized by the fields of some dynamical system, this relation leads, for suitably chosen points (i) and (j) and paths C_{ij} and C'_{ij} , to constants of the motion given as line integrals. In $d = 2$, this situation gives rise, in generic cases, to

integrability properties of the corresponding system.

Let us now show how to set the above equations in the minimal (simplicial) way. We consider now only straight oriented paths C_{ij} that we denote by (ij) , g_{ij} being the associated element $g(C_{ij})$. Then the above zero-curvature equation may be written in a minimal way, by using only three points i, j, k and the paths associated to them, as

$$g_{ij} \circ_{V_j} g_{jk} = g_{ik} \quad (7)$$

This is again the 2-simplex relation. There is a gauge invariance of this equation (and also of eqs.(4,5,6)). We consider invertible elements $u_i \in \text{End}(V_i)$ that we also may interpret as elements of $V_i \otimes V_i^*$, and denote by u_i^* the corresponding dual endomorphism, such that their action leaves invariant the duality bracket between V_i and its dual, i.e., we have for two elements $a \in V_i^*$ and $b \in V_i$

$$u_i^*(a) \circ_{V_i} u_i(b) = a \circ_{V_i} b$$

Then the u -transformed quantities, namely,

$$\Psi_i^{(u)} = u_i(\Psi_i) = u_i \circ_{V_i} \Psi_i$$

and

$$g^{(u)}(C_{ij}) = (u_i \otimes u_j^*) \circ_{(V_i \otimes V_j^*)} g(C_{ij})$$

still satisfy the above equations if Ψ and g do.

I shall now explain how it is possible to generalize this two-dimensional picture. In going to higher-dimensional systems of Field Theory, we would like to avoid the constants of motion being given as simply line-integrals, i.e., following from an equation of the above type (6) or (7). Moreover, we would like the corresponding integrability equations generalizing the above ones, for integrable systems in dimension d , to be possibly written only in dimensions higher than or equal to d ,

and not in dimensions lower than d in order to avoid the above problem and trivial dimensional reductions of the system.

Let us first discuss how this program may be realized in the case $d = 2 + 1$. For $(2 + 1)$ -dimensional integrable systems, we would like to obtain the constants of motion from functionals of the dynamical variables on a space-like surface (initial data), i.e., following from a conservation law of the type

$$\mathcal{U}(\Sigma_C) = \mathcal{U}(\Sigma'_C)$$

for some functional quantities $\mathcal{U}$, where Σ_C and Σ'_C are $2-d$ surfaces having the same closed loop C as boundary.

The idea to obtain such a picture is given by the $2-d$ case. In fact, the relation (5) might be considered as a non-Abelian obstruction (given by g) to the equality $\Psi_i = \Psi_j$, the integrability relations (6,7) being such that Ψ is still a function of a point. Now, we consider objects $h(C_{ij})$ that are effectively dependent on oriented curves C_{ij} , i.e., they no longer satisfy eq.(4), but such that two such objects, say $h(C_{ij})$ and $h(C'_{ij})$ are connected by some mapping, $F(\Sigma_{CC'})$, depending a priori on a surface $\Sigma_{CC'}$ having the oriented closed curve $C_{ij} \circ C'_{ji}$ as boundary. The requirement that the h 's are still functionals of curves only is implemented by a (sufficient) consistency condition for the F 's of the type $F(\Sigma_{CC'}) = F(\Sigma'_{CC'})$. In other words, the F 's will realize an obstruction to eq.(6) that preserves the functionality of the h 's with respect to oriented curves. To explain this scheme more precisely, we need to introduce some notations.

To any oriented path C_{ij} we associate a vector space denoted by $\mathcal{A}_{C_{ij}}$ (to distinguish it from those attached to its boundary points (i) and (j) , i.e., V_i and V_j^*). We

note C_{ij}^* the path C_{ji} (i.e., the same path as C_{ij} but with reversed orientation), the vector space $\mathcal{A}_{C^*}$ being the dual space to $\mathcal{A}_C$, i.e., $\mathcal{A}_{C^*} = \mathcal{A}_C^*$. Furthermore, to the geometrical product of two curves C_{ij} and C_{jk} as in (3), we assign the vector space $\mathcal{A}_{C_{ij} \circ C_{jk}} = \mathcal{A}_{C_{ij}} \otimes \mathcal{A}_{C_{jk}}$. We will also make use, in the following, of another geometrical product between two oriented paths C_{ij} and C_{kl} having a path in common, say C_{ab} , on which their respective orientations are opposite. More precisely if we have $C_{ij} = C_{ia} \circ C_{ab} \circ C_{bj}$ and $C_{kl} = C_{kb} \circ C_{ba} \circ C_{al}$, such that $C_{ia}, C_{bj}, C_{kb}, C_{al}$ are all disjoint, we define the star product of C_{ij} with C_{kl} as the disjoint union of $C_{il} = C_{ia} \circ C_{al}$ and $C_{kj} = C_{kb} \circ C_{bj}$, i.e.,

$$C_{ij} * C_{kl} = C_{il} \cup C_{kj} \quad (8)$$

To any oriented curve C_{ij} , we associate line-elements $h(C_{ij}) \in \mathcal{A}_{C_{ij}} \otimes V_i \otimes V_j^*$ that verify the relation (4) with the natural property that $h(C_{ij} \circ C_{jk}) \in \mathcal{A}_{C_{ij} \circ C_{jk}} \otimes V_i \otimes V_k^*$, i.e., there is a product on the V -type indices, and a tensor product for the $\mathcal{A}$ -type ones. Now, we require two h 's associated to two paths C_{ij} and C'_{ij} to be related by the action of an object $F(\Sigma_{CC'}) \in \mathcal{A}_{C_{ij}} \otimes \mathcal{A}_{C'_{ij}}$ depending a priori on a surface $\Sigma_{CC'}$ having $C_{ij} \circ C'_{ji}$ as a closed loop boundary, (F could in general depend on the precise decomposition of this boundary into C_{ij} and C'_{ij} and therefore also on the points (i) and (j)), i.e.,

$$h(C_{ij}) = F(\Sigma_{CC'}) *_{\mathcal{A}_{C'}} h(C'_{ij}) \quad (9)$$

where the star action of F on h is given, again, by evaluation (duality bracket) between an element of $\mathcal{A}_{C'}$ and an element of its dual. The surface $\Sigma_{CC'}$ being by no means unique, the fact that the h 's are only dependent on paths C_{ij} is guaranteed if the following (sufficient) consistency condition is satisfied by the F 's

$$F(\Sigma_{CC'}) = F(\Sigma'_{CC'}) \quad (10)$$

for two surfaces $\Sigma_{CC'}$ and $\Sigma'_{CC'}$ having the same closed oriented loop boundary $C_{ij} \circ C'_{ji}$. It follows that the

F 's are only dependent on a closed oriented contour $C_{ij} \circ C'_{ji}$, and on the points (i) and (j) that are left fixed in the transformation induced by F . Moreover, if two closed contours, say C_1 and C_2 have a common part on which their respective orientations are opposite we may define another closed oriented contour $C_3 = C_1 * C_2$ by using the geometrical star product of eq.(8). Then this geometrical product is implemented on the F 's by using the star product of (9) between two F 's provided the points left fixed by F , i.e., (i) and (j) exist and are the same for the two above curves. In that case we have

$$F(\Sigma_{C_1}) *_{\mathcal{A}_{C_1 \cap C_2}} F(\Sigma_{C_2}) = F(\Sigma_{C_3}) \quad (11)$$

the product again being given by the duality bracket in $\mathcal{A}_{C_1 \cap C_2}$. This relation for the star product of the F 's (that implement the star product for the curves (8)) ensures that the action of F as in eq.(9) is compatible with the product for the h 's as given in (4). Hence, eq.(11) is another way of writing a zero-curvature condition on the space of functional of loops as the h -objects. Indeed, if we consider three oriented paths C_{ij} , C'_{ij} and C''_{ij} and their associated h 's, eq.(11) guarantees that if we first transform $h(C'')$ to $h(C')$ and then to $h(C)$ we obtain the same result as when going from $h(C'')$ directly to $h(C)$. Hence eqs.(9,11) are the natural generalizations of eqs.(5,6).

Note that the above relations for the F 's exhibit, like the 2-simplex ones, the following local invariance. Let $u(C_{ij})$ be an invertible endomorphism of $\mathcal{A}_{C_{ij}}$ that we can represent as an element of $\mathcal{A}_{C_{ij}} \otimes \mathcal{A}_{C_{ij}}^*$ such that it verifies the factorisation

$$u(C_{ij}) \otimes u(C_{jk}) = u(C_{ij} \circ C_{jk}) \quad (12)$$

Moreover we ask the dual endomorphism of $u(C_{ij})$ to be given by $u(C_{ji})$ in such a way that the duality bracket between $\mathcal{A}_{C_{ij}}$ and its dual $\mathcal{A}_{C_{ji}}$ is invariant under the

action of u , namely, if $a \in \mathcal{A}_{C_{ji}}$ and $b \in \mathcal{A}_{C_{ij}}$ we have

$$a *_{\mathcal{A}_{C_{ij}}} b = u^*(a) *_{\mathcal{A}_{C_{ij}}} u(b)$$

where

$$u^*(a) = u(C_{ji}) *_{\mathcal{A}_{C_{ji}}} a$$

and

$$u(b) = u(C_{ij}) *_{\mathcal{A}_{C_{ij}}} b$$

Then we may define u -transformed h 's and F 's as

$$F^{(u)}(\Sigma_{C_{ij} \circ C'_{ji}}) = u(C_{ij} \circ C'_{ji}) *_{(\mathcal{A}_{C_{ij}} \otimes \mathcal{A}_{C'_{ji}})} F(\Sigma_{C_{ij} \circ C'_{ji}}) \quad (13)$$

and similarly

$$h^{(u)}(C_{ij}) = u(C_{ij}) *_{\mathcal{A}_{C_{ij}}} h(C_{ij}) \quad (14)$$

These u -transformed quantities satisfy eqs.(9,10,11) if h and F do.

We give now the minimal (simplicial) setting for the above structure. To derive it we consider three oriented paths going from a point (l) to a point (i) built out of straight paths only. The first one is just the straight oriented path C_{il} , the two other paths being $C'_{il} = C_{ij} \circ C_{jl}$ and $C''_{il} = C_{ik} \circ C_{kl}$, where all the paths C are straight oriented lines. So in total we need only four independent points, i, j, k, l . In general position, these points determine a tetrahedron (3-simplex) that needs a three-dimensional space to be drawn. We simply denote by (ij) the oriented straight path C_{ij} , and by $\mathcal{A}_{ij}$ the associated vector-space. All the paths composed of straight paths are given simply by the ordered set of points giving the straight lines building them. The oriented closed paths $C_1 = (kji)$, $C_2 = (lki)$, $C'_1 = (lji)$, $C'_2 = (lkj)$, associated to the four faces of the tetrahedron are such that $C_1 * C_2 = C'_1 * C'_2$. To each triangle delimited by these four oriented loops we associate an F we denote

by F_{kji} , F_{lki} , etc. defined by their action on h -objects associated to the straight paths as

$$F_{kji} *_{(\mathcal{A}_{ij} \otimes \mathcal{A}_{jk})} (h_{ij} \circ_{V_j} h_{jk}) = h_{ik} \quad (15)$$

where $h_{ij} \in \mathcal{A}_{ij} \otimes V_i \otimes V_j^*$ is a line element attached to the straight oriented path C_{ij} with vector space $\mathcal{A}_{ij} = \mathcal{A}_{C_{ij}}$ and $F_{kji} \in \mathcal{A}_{ik} \otimes \mathcal{A}_{kj} \otimes \mathcal{A}_{ji}$. Note here that the fixed points of F_{kji} are (i) and (k) . Note also that in general those F 's are not invariant under cyclic permutations of the points i, j, k just because they are dependent on given fixed points (i) and (k) . Hence, there are in general several F 's associated to a given oriented closed contour. Then the consistency condition for an obstruction to the 2-simplex fundamental relation (7) for the h 's as in eq.(15) is given by eqs.(10,11), i.e.,

$$F_{lki} *_{\mathcal{A}_{ik}} F_{kji} = F_{lji} *_{\mathcal{A}_{jl}} F_{lkj} \quad (16)$$

This equation is obtained from the fact that the product $h_{ij} \circ h_{jk} \circ h_{kl}$ can be transformed into h_{il} by using two different combinations of the F 's, that are just the l.h.s. and the r.h.s. of eq.(16), hence leading to this equation as a (sufficient) consistency condition. We may also define dual mappings $\tilde{F}_{ijk}$, whenever they exist, by their action on the h 's as

$$\tilde{F}_{ijk} *_{\mathcal{A}_{ik}} h_{ik} = (h_{ij} \circ_{V_j} h_{jk}) \quad (17)$$

They also lead to consistency conditions as in eq.(16) that are always derived by considering two different geometrical ways of mapping an h associated to a curve C_{ij} to an h corresponding to another curve with the same boundary points (i) and (j) , say C'_{ij} . For example, if the mappings $\tilde{F}_{jkl}$ and $\tilde{F}_{ikl}$ are defined, we obtain two different ways of transforming $h_{ij} \circ h_{jl}$ into $h_{ik} \circ h_{kl}$, leading to the following consistency condition (that might be seen as a consequence of eq.(16))

$$\tilde{F}_{ikl} *_{\mathcal{A}_{il}} F_{lji} = F_{kji} *_{\mathcal{A}_{jk}} \tilde{F}_{jkl} \quad (18)$$

Equations such as (16,18) will be called the 3-simplex fundamental relations. They give the minimal simplicial analogue of eqs.(10,11). It is also interesting to study the particular case where the F 's are only dependent on a closed oriented curve, say $C_{ij} \circ C'_{ji}$, but not on particular fixed points (i) and (j). In that case to any closed oriented loop C we may associate an F . In the simplicial setting it means that to any oriented loop (ijk) defining a 2-simplex we have only one F which is invariant under cyclic permutation of the points (ijk). Hence such an F will be invariant under reparametrisation of the closed oriented loop. It will obey eqs.(16,18) for any closed oriented loops C_1 and C_2 since in that case we do not have to specify fixed points. The 3-simplex fundamental relations further simplify in that case. In particular we do not need to define $\tilde{F}$ objects; or, more precisely, they are given by the F 's associated to the loops with reversed orientations. We present at the end of this section an example having this property.

We now describe the relations generalizing eqs.(4,5, 6) ($n = 2$) and eqs.(9,10,11) ($n = 3$) for arbitrary $n \geq 2$ from which the higher FSR's may be obtained. Let p be any positive integer. We denote by Σ_p and Σ'_p two p -dimensional oriented manifolds having the same closed, compact, oriented boundary $\partial\Sigma_p = \partial\Sigma'_p = \Sigma_p \cap \Sigma'_p$, and by Σ_p^* the manifold Σ_p with opposite orientation. We further require $\partial\Sigma_{p+1} = \partial\Sigma'_{p+1} = \Sigma'_p \circ \Sigma_p^*$ where $\circ$ stands for the geometrical product of Σ'_p with Σ_p^* obtained by gluing them on their common boundary on which they have opposite orientations ($\partial\Sigma_p^* = (\partial\Sigma_p)^*$). To any manifold Σ_p we associate a real vector-space $\mathcal{A}_{\Sigma_p}$ with $\mathcal{A}_{\Sigma_p^*} = \mathcal{A}_{\Sigma_p}^*$ (dual vector-space) and $\mathcal{A}_{\Sigma'_p \circ \Sigma_p^*} = \mathcal{A}_{\Sigma'_p} \otimes \mathcal{A}_{\Sigma_p^*}$. Let $n \geq 2$. We define objects h to be functionals of $(n-2)$ -dimensional oriented manifolds Σ_{n-2} such that $h(\Sigma_{n-2}) \in \mathcal{A}_{\Sigma_{n-2}} \otimes \mathcal{A}_{\partial\Sigma_{n-2}}$. We require the values of such h 's corresponding to the above-defined manifolds

Σ_{n-2} and Σ'_{n-2} , (with $\partial\Sigma_{n-2} = \partial\Sigma'_{n-2}$) to be related by a mapping F , depending a priori on the $(n-1)$ -dimensional manifold Σ_{n-1} that connects Σ'_{n-2} to Σ_{n-2} , i.e., $\partial\Sigma_{n-1} = \Sigma'_{n-2} \circ \Sigma_{n-2}^*$, and $F(\Sigma_{n-1}) \in \mathcal{A}_{\Sigma'_{n-2}} \otimes \mathcal{A}_{\Sigma_{n-2}^*}$, as

$$h(\Sigma'_{n-2}) = F(\Sigma_{n-1}) *_{\mathcal{A}_{\Sigma_{n-2}}} h(\Sigma_{n-2}) \quad (19)$$

where the $*$ -product between F and h is given again by evaluating F on h using the duality bracket between $\mathcal{A}_{\Sigma_{n-2}}$ and its dual vector-space. This equation is the generalization of eqs.(5,9) for all n .

Then we require the h 's to be still functionals of manifolds Σ_{n-2} only, leading to the following (sufficient) consistency condition for the F 's

$$F(\Sigma_{n-1}) = F(\Sigma'_{n-1}) \quad (20)$$

for two manifolds Σ_{n-1} and Σ'_{n-1} defined as above, i.e., having in particular the same oriented boundary. It means that we can write $F(\Sigma_{n-1}) = F(\Sigma'_{n-2}, \Sigma_{n-2})$. For $n = 2$, it reduces to eq.(6) and for $n = 3$ it gives eq.(10). For arbitrary n it leads to the general FSR's.

Let $\Sigma_p^{(1)}$ and $\Sigma_p^{(2)}$ be two p -dimensional manifolds with respective boundaries $\partial\Sigma_p^{(1)} = \Sigma_{p-1}^{(1)'} \circ \Sigma_{p-1}^{(1)*}$ and similarly for $\Sigma_p^{(2)}$, such that $\Sigma_{p-1}^{(1)} = \Sigma_{p-1}^{(2)'} = \Sigma_p^{(1)} \cap \Sigma_p^{(2)}$. We define $\Sigma_p^{(12)} = \Sigma_p^{(1)} * \Sigma_p^{(2)}$ to be the oriented manifold obtained by gluing $\Sigma_p^{(1)}$ and $\Sigma_p^{(2)}$ along their common boundary $\Sigma_{p-1}^{(1)}$. Note that we have $\partial\Sigma_p^{(12)} = \Sigma_{p-1}^{(1)'} \circ \Sigma_{p-1}^{(2)*}$ and $\mathcal{A}_{\Sigma_p^{(12)}} = \mathcal{A}_{\Sigma_p^{(1)}} \otimes \mathcal{A}_{\Sigma_p^{(2)}}$. Now we ask this geometrical $*$ -product to be implemented on the h 's and on the F 's, namely

$$h(\Sigma_p^{(12)}) = h(\Sigma_p^{(1)}) *_{\mathcal{A}_{\Sigma_{p-1}^{(1)}}} h(\Sigma_p^{(2)}) \quad (21)$$

and similarly,

$$F(\Sigma_p^{(12)}) = F(\Sigma_p^{(1)}) *_{\mathcal{A}_{\Sigma_{p-1}^{(1)}}} F(\Sigma_p^{(2)}) \quad (22)$$

From eq.(20), we have the following (extended) flatness relation

$$F(\Sigma_{n-2}, \Sigma''_{n-2}) = \frac{F(\Sigma_{n-2}, \Sigma'_{n-2}) * \mathcal{A}_{\Sigma'_{n-2}}}{F(\Sigma'_{n-2}, \Sigma''_{n-2})} \quad (23)$$

whenever $\partial\Sigma_{n-2} = \partial\Sigma'_{n-2} = \partial\Sigma''_{n-2}$. Remark again that in general F will depend not only on the closed oriented boundary $\Sigma'_{n-2} \circ \Sigma''_{n-2}$ but also on its decomposition into these two pieces, i.e., on the common boundary Σ_{n-3} of Σ_{n-2} and Σ'_{n-2} . To obtain a minimal simplicial setting of these relations it is sufficient to give a minimal discrete version of the above manifolds, that leads in turn to the decomposition of the above F 's and h 's in terms of $*$ -products of F and h objects associated to simplices. In that case the above relation turns into the n -simplex fundamental relation. We shall describe the details of these structures elsewhere.

To conclude this general setting of the FSR's we show that the above relations exhibit the following local (gauge) invariance. Let $u \in \text{End}(\mathcal{A}_{\Sigma_{n-2}})$ and denote by $u^* \in \text{End}(\mathcal{A}_{\Sigma_{n-2}}^*)$ the dual endomorphism such that their action leaves invariant the duality bracket between $\mathcal{A}_{\Sigma_{n-2}}$ and its dual space. Then we may define u -transformed quantities F and h as

$$\frac{F^{(u)}(\Sigma_{n-1})}{F(\Sigma_{n-1})} = \frac{u(\Sigma'_{n-2} \circ \Sigma''_{n-2}) * (\mathcal{A}_{\Sigma_{n-2}} \otimes \mathcal{A}_{\Sigma_{n-2}}^*)}{F(\Sigma_{n-1})} \quad (24)$$

where $\partial\Sigma_{n-1} = \Sigma'_{n-2} \circ \Sigma''_{n-2}$, and similarly

$$h^{(u)}(\Sigma_{n-2}) = u(\Sigma_{n-2}) * \mathcal{A}_{\Sigma_{n-2}} h(\Sigma_{n-2}) \quad (25)$$

These u -transformed quantities satisfy eqs.(19,20,23) if h and F do.

Before using the above FSR for the purpose of multi-dimensional integrability, let us make a short incursion into the possibility of defining, using the above lines of

thought, a non-Abelian cohomology type theory.

In such a picture, we would like to interpret the above fundamental simplex relations (7,16,18) as the non-Abelian analogues of the usual cocycle equations. Therefore we have to show, in particular, that we may build maps $\delta_1, \delta_2, \dots$ such that from any object $v^{(0)}, v^{(1)}, \dots$ (of a type to be defined in the following) associated respectively to 0-simplices, 1-simplices, ... we may assign objects $\delta_1 v^{(0)}, \delta_2 v^{(1)}$ associated respectively to 1-simplices, 2-simplices, ... such that $\delta_n v^{(n-1)}$ solves identically the above $(n+1)$ -simplex relations. We begin by considering elements $v_i^{(0)} \in V_i \otimes V$ associated to points (i) , such that there exists an element $w_i^{(0)} \in V_i^* \otimes V^*$ that verifies

$$(w_i^{(0)} \circ_{V_i} v_i^{(0)}) = Id_V \quad (26)$$

where Id_V is the representation of the identity endomorphism in V written as a canonical element of $V \otimes V^*$ and we have denoted by $\circ_{V_i}$ the product given by the duality bracket between V_i^* and V_i as usual. Then, if we define

$$(\delta_1 v^{(0)})_{ij} = (w_j^{(0)} \circ_V v_i^{(0)}) \quad (27)$$

which is an element of $V_i \otimes V_j^*$ associated to the 1-simplex (ij) , it solves identically eq.(7). If $V_i = V_j = V$, eq.(26) just states that v is an invertible element of $\text{End}(V)$ and we have $w = v^{-1}$, such that $(\delta_1 v)_{ij} = v_i v_j^{-1}$ which obviously solves (7) or more precisely (1).

Let us show that this scheme extends to the next FSR's, i.e., to the 3-simplex relations. We consider elements $v_{ij}^{(1)} \in \mathcal{A}_{ij} \otimes V_i \otimes V_j^*$ associated to 1-simplices (ij) such that we have the relation

$$(v_{ji}^{(1)} \circ_{\mathcal{A}_{ij}} v_{ij}^{(1)}) = Id_{V_i \otimes V_j} \quad (28)$$

$Id_{V_i \otimes V_j}$ being the identity endomorphism in $V_i \otimes V_j$ written as the canonical element of $V_i \otimes V_i^* \otimes V_j \otimes V_j^*$. Now

we define

$$(\delta_2 v^{(1)})_{ijk} = ((v_{ij}^{(1)} \circ_{V_j} v_{jk}^{(1)})) \circ_{(V_i \otimes V_k)} v_{ki}^{(1)} \quad (29)$$

which may be thought of as a cyclic product around the closed curve (ijk) , given by the duality brackets associated to V_i, V_j, V_k . This is an element of $\mathcal{A}_{ij} \otimes \mathcal{A}_{jk} \otimes \mathcal{A}_{ki}$. Using (28) it is easy to show that $(\delta_2 v^{(1)})_{ijk}$ solves identically the 3-simplex relations given in (16,18). This picture extends to the higher FSR's.

Let $n \geq 2$ and $v^{(n-2)}(\Sigma_{n-2}) \in \mathcal{A}_{\Sigma_{n-2}} \otimes \mathcal{A}_{\partial\Sigma_{n-2}}$ with the normalization

$$v^{(n-2)}(\Sigma_{n-2}^*) *_{\mathcal{A}_{\Sigma_{n-2}}} v^{(n-2)}(\Sigma_{n-2}) = Id_{\mathcal{A}_{\partial\Sigma_{n-2}}} \quad (30)$$

where $Id_{\mathcal{A}_{\partial\Sigma_{n-2}}} \in \mathcal{A}_{\partial\Sigma_{n-2}} \otimes \mathcal{A}_{\partial\Sigma_{n-2}}^*$ is the identity endomorphism of $\mathcal{A}_{\partial\Sigma_{n-2}}$ that we may represent canonically in $\mathcal{A}_{\partial\Sigma_{n-2}} \otimes \mathcal{A}_{\partial\Sigma_{n-2}}^*$. Then we define the co-boundary operator δ_{n-1} as

$$(\delta_{n-1} v^{(n-2)})(\Sigma_{n-1}) = \frac{v^{(n-2)}(\Sigma'_{n-2}) *_{\mathcal{A}_{\partial\Sigma_{n-2}}} v^{(n-2)}(\Sigma_{n-2}^*)}{v^{(n-2)}(\Sigma_{n-2}^*)} \quad (31)$$

where $\partial\Sigma_{n-1} = \Sigma'_{n-2} \circ \Sigma_{n-2}^*$ is a closed, compact, oriented boundary with $\partial\Sigma_{n-2} = \partial\Sigma'_{n-2}$.

Then $F^{(0)}(\Sigma_{n-1}) = (\delta_{n-1} v^{(n-2)})(\Sigma_{n-1}) \in \mathcal{A}_{\Sigma'_{n-2}} \otimes \mathcal{A}_{\Sigma_{n-2}^*}^*$ solves identically the n -simplex relations (20,23). The simplicial version is obtained by choosing suitable decomposed manifolds $\Sigma_n, \Sigma_{n-1}, \dots$ in terms of simplices. The corresponding $\delta_{n-1} v^{(n-2)}$ solves identically the n -simplex relations that might be thought of as a non-Abelian $(n-1)$ -cocycle relation.

To get some more understanding of these FSR's let me go back to our 3-simplex relations (16,18) and derive their associated "cubic-type" relations, in the same way as eq.(2) was associated to eq.(1). For that purpose, we consider a closed curve $C_P = (kjil)$, where the 4 points

i, j, k, l define a square plaquette, say Σ_P , having C_P as closed oriented boundary. Then we want to map an h associated to the oriented curve (ijk) to the one corresponding to (ilk) . Hence the points (i) and (k) remain fixed. To this plaquette we associate an object $F(\Sigma_P)$. It is easy to see that such a plaquette can be decomposed into two triangles (ikj) and (ilk) . Hence one may decompose $F(\Sigma_P)$ in a product of two F 's associated to the corresponding triangles. We define $R_{kji} = F(\Sigma_P)$, i.e.,

$$R_{kji} = \tilde{F}_{ilk} *_{\mathcal{A}_k} F_{kji} \quad (32)$$

where we now suppose the dual mapping $\tilde{F}_{kil}$ of F_{kli} to exist. We consider a cube (where (i, j, k, l) is a face and (i', j', k', l') the parallel face) and the six objects R attached to its six faces, constructed as above from the F 's. As a consequence of eqs.(16,18) for F and $\tilde{F}$ or more directly of the equation (10), we obtain

$$\begin{aligned} R_{li'li'} * R_{k'kll'} * R_{kji} &= \\ &= R_{k'j'j'i'} * R_{j'ji'} * R_{k'kji'} \end{aligned} \quad (33)$$

the products between the R 's being given by duality, on the l.h.s. on the tensor product of three spaces, $\mathcal{A}_{jk} \otimes \mathcal{A}_{lk} \otimes \mathcal{A}_{kk'}$, and on the r.h.s. on the tensor product of three other spaces, $\mathcal{A}_{i'i} \otimes \mathcal{A}_{i'l'} \otimes \mathcal{A}_{i'j'}$, i.e., on the spaces attached to the common links with opposite orientations of the corresponding plaquettes. This equation may be obtained by considering the two different ways of mapping the product $h_{ij} \circ_{V_j} h_{jk} \circ_{V_k} h_{kk'}$ into the product $h_{i'i'} \circ_{V_{i'}} h_{i'l'} \circ_{V_{l'}} h_{l'k'}$.

Let us show that a particular case of (33) is the Yang-Baxter equation. To see this, we set $\mathcal{A}_{ij} = \mathcal{A}_{lk} = \mathcal{A}_{l'k'} = \mathcal{A}_{i'j'} =: \mathcal{A}_3$ and similarly $\mathcal{A}_{jk} = \mathcal{A}_{il} = \mathcal{A}_{j'k'} = \mathcal{A}_{i'l'} =: \mathcal{A}_2$, $\mathcal{A}_{ii'} = \mathcal{A}_{jj'} = \mathcal{A}_{kk'} = \mathcal{A}_{ll'} =: \mathcal{A}_1$, meaning that we associate the same vector-space to two curves that may be mapped onto each other by translation. In that case we can identify the R 's as endomorphisms

acting in the tensor product of two vector-spaces $\mathcal{A}$. For example R_{kji} may be considered as an element of $End(\mathcal{A}_2 \otimes \mathcal{A}_3)$ that we denote by $R_{23}^{(kji)}$, where the indices 23 stand for the vector-spaces R_{23} is acting upon, and (kji) its a priori functional dependence on the elementary plaquette (kji) . If we further require those objects to depend only on the orientation of the plaquette through some parameters denoted by λ_{23} for $R_{23}^{(kji)}$, but not on its position in space, eq.(33) reduces to the quantum Yang-Baxter equation,

$$\begin{aligned} R_{12}(\lambda_{12}) R_{13}(\lambda_{13}) R_{23}(\lambda_{23}) &= \\ &= R_{23}(\lambda_{23}) R_{13}(\lambda_{13}) R_{12}(\lambda_{12}) \end{aligned} \quad (34)$$

where, however, the possible constraints between the λ_{ab} might be more sophisticated than just $\lambda_{13} = \lambda_{12} + \lambda_{23}$. Moreover, in the case where the plaquette objects $R_{(kji)}$ are also dependent on the position of the plaquette in space-time and not only on its orientation, we obtain a dynamical Yang-Baxter equation. In particular, if we restrict the points $i, j, k, l, i', j', k', l'$, etc., to belong to a cubic three-dimensional lattice, we also obtain, as a particular case of eq.(33), the integrability equations, generalizing the two-dimensional Lax equations, for three-dimensional integrable lattice models of ref. [27]. However, eq.(33) and of course the 3-simplex fundamental relations (16,18), are more general; in particular, they are valid in the continuum where all the point $i, j, k, \dots$ are arbitrary points in $\mathbf{R}^3$.

Hence what we have shown here is that our 3-simplex fundamental relations just lead, in the static case, to the quantum Yang-Baxter equations, with spectral parameters, providing a nice geometrical interpretation for them. In fact, the original 3-simplex fundamental relations (16,18) lead also, to the decomposition of the R -matrices solution of the quantum Yang-Baxter equations in two objects F and $\tilde{F}$ (c.f. eq.(32)) that was

considered by Drinfel'd in [8] in the static case, obtained here from a quite different point of view. In our picture, it is just the consequence of the consistent decomposition of a square into two triangles implemented on the F -functionals. There we not only obtain such a decomposition in the static case (leading to solutions of the usual quantum Yang-Baxter equation) but also in the localized case where the F 's are fully dependent on closed arbitrary curves.

In connection with our setting for a non-Abelian cohomology theory, a very nice picture arises for the classification problem of the solutions of the quantum Yang-Baxter equation : they will be given by the analogue of the cohomology classes corresponding to our 3-simplex fundamental relations with coboundary operator δ_2 .

Our present formulation leads also to a general setting of integrable models of Field Theory in $d = 2 + 1$ as we will show in the following. Note in particular that the corresponding conserved quantities can then be obtained from the F associated to a constant-time surface. We shall give some more details on this in the next section.

We end this section by giving a very simple solution to the three-simplex fundamental relations (16,18), and hence to eq.(33), which is typical of their loop-space interpretation. We consider the particular case where the vector-spaces $\mathcal{A}_{ij}$ can be decomposed into the tensor product of two spaces as $\mathcal{A}_{ij} = \mathcal{A}_i \otimes \mathcal{A}_j$. Then an F associated to a triangle (ijk) , noted F_{ijk} , can be thought of as an endomorphism of $\mathcal{A}_i \otimes \mathcal{A}_j \otimes \mathcal{A}_k$, $\dim \mathcal{A}_k = N$. The following F is a solution to eqs.(16,18)

$$(F_{ijk}^{(0)})_{abc}^{def} = N^{-\frac{1}{2}} \delta_a^d \delta_b^e \delta_c^f \quad (35)$$

This F is invariant under cyclic permutation of the

points i, j, k . Hence there are only two F 's of this type associated to a given triangle (ijk) corresponding to the two possible orientations (ijk) and (kji) . It determines a constant solution of the quantum Yang-Baxter equation (34) that may be written in terms of the tensors $P_{ab,cd} = \delta_{ad}\delta_{bc}$ (permutation matrix), and $\Delta_{ab,cd} = \delta_{ab}\delta_{cd}$. This solution is typical of the above loop-space setting of the Yang-Baxter equation. It is in fact cohomologically trivial in the sense that it is the δ_2 of the identity element, or in other words it is the basic object that defines δ_2 . Similar "elementary" solutions may be given for the higher FSR's, leading in particular to solutions of the so-called $(d-1)$ -simplex equations that are the static limit of the "cubic-type" relations associated to our above d -simplex fundamental relations, and therefore that can be derived from them (see next section).

3. MULTIDIMENSIONAL INTEGRABLE SYSTEMS

The aim of this section is to apply the above results and algebraic structures to multi-dimensional integrable systems, with a particular focus on the $2+1$ -dimensional case.

At this point I would like to make some general comments on the possible interpretations and uses of the FSR's we defined in section 2 with regards to the notion of integrable systems of Field Theory.

The first point of view, that in fact we have taken to define the FSR's, is that they give natural generalizations of the concept of Lax or Zakharov-Shabat zero-curvature type equations for integrable systems in dimensions $d \geq 3$. We will indeed develop this aspect in the following.

Another point of view is suggested by the last result of section 2 that shows the quantum Yang-Baxter equation to be a particular case of our 3-simplex fundamental relations or more precisely of their associated cubic-type relations. The meaning of this is that while the 2-simplex fundamental relations lead to classical integrable field equations in $d = 1 + 1$, the 3-simplex ones describe the consistent quantization of such systems, preserving, in the anomaly-free case, their integrability properties. Moreover, as we shall see in the following, this picture seems likely to be extended into higher-dimensions ($d \geq 3$) where the integrable field equations should be given by our d -simplex fundamental relations, their integrable quantization, if it exists, being governed by the $(d+1)$ -simplex ones.

What are the reasons for these quite intriguing features? Even if I cannot give a complete answer to this question, let me briefly describe some of my ideas on this problem. We first consider the case of 2-dimensional Field Theories, but our picture could be easily extended to higher-dimensional integrable systems.

In $d = 1 + 1$, the classical integrable field equations of motion follow from a relation of the type (cf. eq.(6))

$$g(C_{ij}) = g(C'_{ij})$$

When quantizing such a system, we expect this equality to be broken by quantum corrections. Indeed, now $g(C_{ij}) \in V_i \otimes V_j^* \otimes \mathcal{A}_{C_{ij}}$, the fields of the system contained in $g(C_{ij})$ taking values in some (a priori non-commutative) operator algebra $\mathcal{A}_{C_{ij}}$. Hence, let C'_{ij} be obtained from a small local deformation of C_{ij} at some point k . Then the value $g(C'_{ij})$ is given, at first order in $\hbar$, by $g(C_{ij})$ corrected by the insertion of the curvature $F(A)$ at point k . Since the components of A are parametrized by the quantum fields of the system, their

matrix-elements are now a priori non-commuting operators, and consequently, the curvature $F(A)$ no longer vanishes, leading to an obstruction to the above equality. However, we know how to compute the first non-trivial obstruction term for classically integrable cases. It is given by some operator r acting only on the quantum indices of the A 's. This property guarantees the symplectic structure to be compatible with the integrable dynamics. If we ask this feature to hold to all orders in $\hbar$, then the obstruction to the above equality for the g 's will be given as the action of some operator, say $F(C_{ij}, C'_{ji})$, on the $\mathcal{A}$ -indices (and only on them) of $g(C_{ij})$ as

$$g(C_{ij}) = F(C_{ij}, C'_{ji}) *_{\mathcal{A}_C} g(C'_{ji})$$

But this is just the analogue of eq.(9). Hence we will obtain consistency conditions for such an F that are just the 3-simplex fundamental relations of section 2, leading in turn to the quantum Yang-Baxter equation. So, as in the classical case, the quantum structure for the fields will be compatible with the dynamics, leading in particular to a set of constants of motion that are all in involution. Moreover the operator F is intimately related to the so-called universal R -matrix associated to such a dynamical system. Obviously this argument extends to the higher FSR's.

Let us now turn to the applications of our ideas to the definition of the notion of integrability in dimensions higher than or equal to three. As we have just seen, in order to be able to give the algebraic structure of integrable systems it is useful not only to analyze the FSR's of section 2 but also to derive their associated cubic-type relations that will generate, in their static limit, the higher-dimensional analogues of the quantum Yang-Baxter equations (QYBE) that are associated to the 3-simplex fundamental relations. We also know that

if we are interested in a lattice integrable system we will have to deal with the cubic-type relations rather than with the minimal simplicial ones.

Let $n \geq 2$ be an integer. The n -cubic relations associated to the FSR's of section 2 may be derived from the general relations given in eqs.(19,20,23) by choosing suitable oriented manifolds $\Sigma_n, \Sigma_{n-1}, \Sigma_{n-2}, \dots$ (Note that these n -cubic relations may also be derived from the n -simplex fundamental relations by decomposing any n -cube in $n!$ elementary simplices.). To describe them we need to introduce some notations.

We define an n -cube C_n by giving a point, y , and a set of n free vectors $u_i, i = 1, \dots, n$. Let $C_n = \Sigma_n$. Points in C_n are given by n real coordinates $x_i, 0 \leq x_i \leq 1$ in the basis u_i . The oriented boundary of $C_n, \partial C_n$, is composed of $2n$ $(n-1)$ -cubes we denote by $C_{n-1}^{(i)}(\varepsilon), \varepsilon = 0, 1$, given by the point $(y + \varepsilon u_i)$, and the set of vectors $u_j, j \neq i$. Let $\varepsilon_i = \frac{1}{2}(1 + (-1)^i)$. We divide ∂C_n into two oriented $(n-1)$ -dimensional manifolds Σ_{n-1} and Σ'_{n-1} having the same oriented boundary, $\Sigma_{n-1}^* = \bigcup_i C_{n-1}^{(i)}(\varepsilon_i)$ and $\Sigma'_{n-1} = \bigcup_i C_{n-1}^{(i)}(1 - \varepsilon_i)$, as $\partial C_n = \Sigma'_{n-1} \circ \Sigma_{n-1}^*$. The boundary $\partial C_{n-1}^{(i)}(\varepsilon)$ of $C_{n-1}^{(i)}(\varepsilon)$ is composed of $2(n-1)$ $(n-2)$ -cubes we denote by $C_{n-2}^{(ij)}(\varepsilon, \varepsilon'), \varepsilon' = 0, 1, j \neq i$, given by the point $(y + \varepsilon u_i + \varepsilon' u_j)$, and the set of $(n-2)$ vectors $u_k, k \neq i, k \neq j$. Note that $C_{n-2}^{(ji)*}(\varepsilon', \varepsilon) = C_{n-2}^{(ij)}(\varepsilon, \varepsilon')$, i.e., they are the same manifolds but with opposite orientations. We decompose $\partial C_{n-1}^{(i)}(\varepsilon_i)$ into two pieces as $\partial C_{n-1}^{(i)}(\varepsilon_i) = \Sigma_{n-2}^{(i)'}(\varepsilon_i) \circ \Sigma_{n-2}^{(i)*}(\varepsilon_i)$ with

$$\begin{aligned} \Sigma_{n-2}^{(i)}(\varepsilon_i) &= \bigcup_k C_{n-2}^{(ijk)}(\varepsilon_i, \varepsilon_k), \\ \Sigma_{n-2}^{(i)'}(\varepsilon_i) &= \bigcup_k C_{n-2}^{(ijk)*}(\varepsilon_i, 1 - \varepsilon_k), \end{aligned} \quad (36)$$

where $1 \leq j_k < j_{k+1} \leq n, j_k \neq i, 1 \leq k \leq n-1$ is a renumbering of the set of vectors u_j for $j \neq i$. Note that for $j = j_k < i$ $\varepsilon_k = \varepsilon_j$, and for $j = j_k > i$, $\varepsilon_k = 1 - \varepsilon_j$.

In the same way we have $\partial C_{n-1}^{(i)}(1 - \varepsilon_i) = \Sigma_{n-2}^{(i)'}(1 - \varepsilon_i) \circ \Sigma_{n-2}^{(i)*}(\varepsilon_i)$ with

$$\begin{aligned}\Sigma_{n-2}^{(i)*}(1 - \varepsilon_i) &= \bigcup_k C_{n-2}^{(ijk)}(1 - \varepsilon_i, \varepsilon_k), \\ \Sigma_{n-2}^{(i)'}(1 - \varepsilon_i) &= \bigcup_k C_{n-2}^{(ijk)}(1 - \varepsilon_i, 1 - \varepsilon_k)\end{aligned}\quad (37)$$

and $\partial \Sigma_{n-2}^{(i)}(\varepsilon) = \partial \Sigma_{n-2}^{(i)'}(\varepsilon)$. From these results we also obtain

$$\partial \Sigma_{n-1}^* = \bigcup_{i \neq j} C_{n-2}^{(ij)}(\varepsilon_i, 1 - \varepsilon_j)$$

and also

$$\partial \Sigma_{n-1}' = \bigcup_{i \neq j} C_{n-2}^{(ij)}(1 - \varepsilon_i, \varepsilon_j)$$

with $\partial \Sigma_{n-1}' = \partial \Sigma_{n-1}$. We divide $\partial \Sigma_{n-1}$ into two pieces as $\partial \Sigma_{n-1} = \partial \Sigma_{n-1}' = \Sigma_{n-2}' \circ \Sigma_{n-2}^*$ with $\partial \Sigma_{n-2} = \partial \Sigma_{n-2}'$ and

$$\begin{aligned}\Sigma_{n-2} &= \bigcup_{i < j} C_{n-2}^{(ij)}(\varepsilon_i, 1 - \varepsilon_j) \\ \Sigma_{n-2}' &= \bigcup_{i < j} C_{n-2}^{(ij)}(1 - \varepsilon_i, \varepsilon_j)\end{aligned}\quad (38)$$

To each oriented $(n-2)$ -cube $C_{n-2}^{(ij)}(\varepsilon, \varepsilon')$ we associate a real vector-space $\mathcal{A}_{ij}(\varepsilon, \varepsilon')$. Note that we have the relation $\mathcal{A}_{ij}(\varepsilon, \varepsilon') = \mathcal{A}_{ji}^*(\varepsilon', \varepsilon)$. Finally we also need to describe the boundary of $C_{n-2}^{(ij)}(\varepsilon, \varepsilon')$ which is composed of $2(n-2)$ $(n-1)$ -cubes that we denote by $C_{n-3}^{(ijk)}(\varepsilon, \varepsilon', \varepsilon'')$, $\varepsilon'' = 0, 1$, and $k \neq i, k \neq j$. We have in addition the duality relations, $C_{n-3}^{(ijk)}(\varepsilon, \varepsilon', \varepsilon'') = C_{n-3}^{(ikj)*}(\varepsilon, \varepsilon'', \varepsilon') = C_{n-3}^{(kij)}(\varepsilon'', \varepsilon, \varepsilon')$ and so on. To each $C_{n-3}^{(ijk)}(\varepsilon, \varepsilon', \varepsilon'')$ we associate a real vector-space $\mathcal{A}_{ijk}(\varepsilon, \varepsilon', \varepsilon'')$ with corresponding duality properties under permutation of their indices and arguments.

We define functionals F of $(n-1)$ -cubes as

$$F_i(\varepsilon_i) = F(C_{n-1}^{(i)}(\varepsilon_i))$$

and

$$F_i(1 - \varepsilon_i) = F(C_{n-1}^{(i)}(1 - \varepsilon_i))$$

So $F_i(\varepsilon_i) \in \mathcal{A}_{\Sigma_{n-2}^{(i)'}}(\varepsilon_i) \otimes \mathcal{A}_{\Sigma_{n-2}^{(i)*}}^*(\varepsilon_i)$ is a mapping from

$$\otimes_{k=1}^{n-1} \mathcal{A}_{ijk}(\varepsilon_i, \varepsilon_k)$$

to

$$\otimes_{k=1}^{n-1} \mathcal{A}_{ijk}^*(\varepsilon_i, 1 - \varepsilon_k)$$

and $F_i(1 - \varepsilon_i) \in \mathcal{A}_{\Sigma_{n-2}^{(i)'}}^*(1 - \varepsilon_i) \otimes \mathcal{A}_{\Sigma_{n-2}^{(i)}}(1 - \varepsilon_i)$ is a mapping from

$$\otimes_{k=1}^{n-1} \mathcal{A}_{ijk}^*(1 - \varepsilon_i, \varepsilon_k)$$

to

$$\otimes_{k=1}^{n-1} \mathcal{A}_{ijk}(1 - \varepsilon_i, 1 - \varepsilon_k)$$

We define functionals h of $(n-2)$ -cubes as

$$h_{ijk}(\varepsilon_i, \varepsilon_k) = h(C_{n-2}^{(ijk)}(\varepsilon_i, \varepsilon_k))$$

$$h_{ijk}(\varepsilon_i, 1 - \varepsilon_k) = h(C_{n-2}^{(ijk)*}(\varepsilon_i, 1 - \varepsilon_k))$$

$$h_{ijk}(1 - \varepsilon_i, \varepsilon_k) = h(C_{n-2}^{(ijk)*}(1 - \varepsilon_i, \varepsilon_k))$$

$$h_{ijk}(1 - \varepsilon_i, 1 - \varepsilon_k) = h(C_{n-2}^{(ijk)}(1 - \varepsilon_i, 1 - \varepsilon_k))$$

By duality we have several relations among the h 's. For example if $j = j_k < i$ then $h_{ijk}(1 - \varepsilon_i, \varepsilon_k) = h_{ji}(\varepsilon_j, 1 - \varepsilon_i)$ and if $j = j_k > i$ then $h_{ijk}(\varepsilon_i, 1 - \varepsilon_k) = h_{ji}(\varepsilon_j, \varepsilon_i)$ and so on. Any $h(\Sigma_{n-2})$ is to be considered as a mapping from Σ_{n-3} to Σ_{n-3}' , where $\partial \Sigma_{n-2} = \Sigma_{n-3}' \circ \Sigma_{n-3}^*$, $\partial \Sigma_{n-3} = \partial \Sigma_{n-3}'$. Hence to compute $h(\Sigma_{n-2}^{(i)}(\varepsilon_i))$ or $h(\Sigma_{n-2})$ following the general formula (21), we need to make $*$ -products of the above h_{ijk} in a definite order. For example, $h_{ijk}(\varepsilon_i, \varepsilon_k)$ is a mapping from $\otimes_{s=1}^{n-2} \mathcal{A}_{ijl_s}^*(\varepsilon_i, \varepsilon_k, \varepsilon_s)$ to $\otimes_{s=1}^{n-2} \mathcal{A}_{ijl_s}^*(\varepsilon_i, \varepsilon_k, 1 - \varepsilon_s)$ with $1 \leq l_s < l_{s+1} \leq n$, $l_s \neq i, l_s \neq j$ and $1 \leq s \leq n-2$, and similarly for the other h 's defined above. It can be shown that ($\varepsilon = 0, 1$)

$$\begin{aligned}h(\Sigma_{n-2}^{(i)}(\varepsilon)) &= \prod_k^> h_{ijk}(\varepsilon, \varepsilon_k) \\ h(\Sigma_{n-2}^{(i)'}(\varepsilon)) &= \prod_k^< h_{ijk}(\varepsilon, 1 - \varepsilon_k)\end{aligned}\quad (39)$$

where the sign ($>$) means that we order the product backwards, i.e., the terms with the lowest k acting first, whereas the sign ($<$) gives the opposite ordering and the products are $*$ -products. We also obtain

$$\begin{aligned}h(\Sigma_{n-2}) &= \prod_{j_k < i}^{>>} h_{ijk}(1 - \varepsilon_i, \varepsilon_k), \\ h(\Sigma_{n-2}') &= \prod_{j_k > i}^{<<} h_{ijk}(1 - \varepsilon_i, 1 - \varepsilon_k),\end{aligned}\quad (40)$$

where the double ordering signs refer to i and j_k , namely we first order the terms with fixed i on the j_k 's and then we order the obtained expressions on i . We have ($\varepsilon = 0, 1$),

$$\prod_k^< h_{ij_k}(\varepsilon, 1 - \varepsilon_k) = F_i(\varepsilon) * \prod_k^> h_{ij_k}(\varepsilon, \varepsilon_k) \quad (41)$$

Hence the F_i 's act on the corresponding products of h 's by reversing the order of their products while at the same time shifting their arguments. Note also that if we make a complete reflection of the ordering of the set of vectors u_i , i being transformed in $n+1-i$, we obtain the objects similar to $\tilde{F}$ (see section 2 for the $n=3$ case).

From the above definitions it is quite easy to see that there are two ways of mapping $h(\Sigma_{n-2})$ to $h(\Sigma'_{n-2})$ by successive action of respectively the $F_i(\varepsilon_i)$ or the $F_i(1 - \varepsilon_i)$. They correspond to $F(\Sigma_{n-1})$ and $F(\Sigma'_{n-1})$ that we obtain as the following ordered products

$$\begin{aligned} F(\Sigma_{n-1}) &= \prod_i^> F_i(\varepsilon_i), \\ F(\Sigma'_{n-1}) &= \prod_i^< F_i(1 - \varepsilon_i) \end{aligned} \quad (42)$$

From our general framework (section 2) we require, as a sufficient condition for the h_{ij} ' to be functionals of the $C_{n-2}^{(ij)}$ only, these two mappings to be equal, leading to the following n -cubic relation

$$\prod_i^> F_i(\varepsilon_i) = \prod_i^< F_i(1 - \varepsilon_i) \quad (43)$$

This is a local equation, i. e., an equation for objects F_i that are functionals dependent in particular on the space-time points $(y + \varepsilon_i u_i)$ and $(y + (1 - \varepsilon_i) u_i)$. Furthermore each F_i is also dependent on the set of $(n-1)$ real vectors u_j with $j \neq i$ that determine the $(n-1)$ -cube $C_{n-1}^{(i)}$. Such parameters will be called spectral parameters in the following by analogy with the case $n=2$ and $n=3$. Eq.(43) is the general n -cubic relation associated

to the FSR's described in section 2.

One of the main characteristics of eq.(43) is that on the r.h.s and the l.h.s. two arbitrary F 's, say F_i and F_j , are connected to each other by their $*$ -product in one and only one vector-space $\mathcal{A}_{ij}(\varepsilon, \varepsilon')$. Moreover this n -cubic relation may be thought of as a kind of generalized commutativity equation for n -fold products.

In order to gain some insight into these new structures let us first derive the static limit of the n -cubic relation. For that purpose we simplify the above framework by making the following identifications of vector-spaces, $\mathcal{A}_{ij}(\varepsilon, \varepsilon') = \mathcal{A}_{ij}^*(1 - \varepsilon, \varepsilon') = \mathcal{A}_{ij}^*(\varepsilon, 1 - \varepsilon') = \mathcal{A}_{ij}(1 - \varepsilon, 1 - \varepsilon') = \mathcal{A}_{ji}^*(\varepsilon', \varepsilon)$. We also define $\mathcal{A}_{ij} = \mathcal{A}_{ij_k}(\varepsilon_i, \varepsilon_k)$ for $j = j_k$. Note that $\mathcal{A}_{ij} = \mathcal{A}_{ji}$. In the same way we identify $\mathcal{A}_{ijk}(\varepsilon, \varepsilon', \varepsilon'') = \mathcal{A}_{ijk}^*(\varepsilon, \varepsilon', 1 - \varepsilon'')$ and so on by duality. We define $\mathcal{A}_{ijl} = \mathcal{A}_{ij_k l}(\varepsilon_i, \varepsilon_{j_k}, \varepsilon_l)$. Note that $\mathcal{A}_{ijl}$ is completely symmetric in (ijl) .

Now it is quite easy to see that we may interpret the F_i 's as endomorphisms of $\otimes_{j \neq i} \mathcal{A}_{ij}$. In the static case, where we require the h 's and the F 's not to depend on the space-time variables, we have $F_i(\varepsilon_i) = F_i(1 - \varepsilon_i) = S_i$, leading to the following static n -cubic relation

$$\prod_i^> S_i = \prod_i^< S_i \quad (44)$$

where the products between the S 's are now products of endomorphisms acting in $\otimes_{i < j} \mathcal{A}_{ij} = \mathcal{A}$, S_i acting non-trivially only in $\otimes_{j \neq i} \mathcal{A}_{ij} = \mathcal{A}_i$, its action in the other vector-spaces in $\mathcal{A}$ being equal to the identity. Again let us stress that on the r.h.s and the l.h.s. of eq.(44) any two S , say S_i and S_j , are connected by their product as endomorphisms in one and only one vector-space, i. e., $\mathcal{A}_{ij}$. Note that even in this static limit, the S_i are still functions of the set of $(n-1)$ vectors u_j for $j \neq i$, i. e., they are functions of the spectral parameters.

It is quite easy to show that this static n -cubic relation gives exactly the so-called $(n-1)$ -simplex equation (as described for example in [27]), together with a new interpretation of the spectral parameter dependence of the S_i 's. For $n = 2$ it gives just the commutativity relation for matrices depending on complex parameters, for $n = 3$ we obtain the quantum Yang-Baxter equation with full spectral parameter dependence, for $n = 4$ we have the tetrahedron equation of Zamolodchikov [30], and so on (for related works on this static case see [27,28,29,30,31,32,33,34,35,36]). To write them down explicitly from eq.(44) in their usual form (see for example [27]), let us relabel the vector-spaces $\mathcal{A}_{ij}$ as $\mathcal{A}_{a_{ij}}$ where $a_{ij} = \frac{1}{2}(n-i)(n-i+1) + i - j + 1$ for $i > j$ and $a_{ij} = a_{ji}$ otherwise. Define $S_i = S_i^{(a_1 \dots a_{n-1})}$ where the set of a 's for S_i , given by a_{ij} , $j \neq i$, and ordered as $a_k < a_{k+1}$, denote the corresponding vector-spaces S_i is acting upon non-trivially. In the same way we define a relabelling of the vector-spaces $\mathcal{A}_{ji}$ by associating to (ijl) with $i > j > l$ a number b_{ijl} symmetric in (ijl) such that $b_{n,n-1,n-2} = 1$ and $b_{i,jl} < b_{i-1,j'l'} < b_{i-1,j'-1,l'}$ and $b_{ijl} = b_{ij,l+1} + 1$. The spectral parameter dependence of the S 's being only implicitly understood, we obtain the low-dimensional static n -cubic relations,

$$\begin{aligned} n=2 : S_2^{(1)} S_1^{(1)} &= S_1^{(1)} S_2^{(1)} \\ n=3 : S_3^{(12)} S_2^{(13)} S_1^{(23)} &= S_1^{(23)} S_2^{(13)} S_3^{(12)} \\ n=4 : S_4^{(123)} S_3^{(145)} S_2^{(246)} S_1^{(356)} &= \\ &= S_1^{(356)} S_2^{(246)} S_3^{(145)} S_4^{(123)} \end{aligned} \quad (45)$$

Let us now come back to the general n -cubic relations (43) and further analyze their main properties.

We first show that the general n -cubic relation can be reduced to the static one by defining new objects $\hat{F}_i$

given from the F_i by dressing them with suitable shift operators. Let us denote by $\sigma_j^{(ij)}$ the shift (of one unit in direction u_j) acting on the space $\mathcal{A}_{ij}(\varepsilon_i, \varepsilon_{j_k})$ with $j = j_k$ and by $\sigma_j^{(ij)-1}$ the shift in the opposite direction. We define

$$\hat{F}_i = \prod_{k=1}^{n-1} (\sigma_{j_k}^{(ij_k)})^{\varepsilon_k-1} F_i(0) \prod_{k=1}^{n-1} (\sigma_{j_k}^{(ij_k)})^{\varepsilon_k} \quad (46)$$

If F_i satisfies the n -cubic equation (43), then $\hat{F}_i$ satisfies its static version, i. e.,

$$\prod_i^> \hat{F}_i = \prod_i^< \hat{F}_i \quad (47)$$

We now state the principle that enables us to generate all the n -cubic relations recursively starting from the $n = 2$ case. Let us first remark that if the F 's are trivial endomorphisms, i. e., equal to the identity, then eq.(41) reduces to the $(n-1)$ -cubic relation for the h 's. Hence non-trivial F_i 's realize an obstruction to the $(n-1)$ -cubic equation as given in (41) and have to satisfy, as a sufficient consistency condition, the n -cubic relation (43). In other words, an obstruction to the $(n-1)$ -cubic relation satisfies the n -cubic one. Hence we may construct all these equations recursively. However the general structure of the FSR's enables us to give them directly as we just showed. In fact this property is the direct consequence of the similar property for the FSR's.

Another important feature of the n -cubic relations is their self-similarity behaviour under suitable products and tensor products. Having at hand any solution to these relations we may indeed produce an infinite series of other solutions by means of ordered products of our first elementary solution. This result is well known for the case $n = 2$, i. e., if two families of matrices are commuting then their respective powers also commute. There is an analogous property for the quantum Yang-Baxter equation that we will explain briefly at the end of this section. The general setting and proof of this

property are given by the equations (22) and (20) of section 2. It indeed follows from the decomposition of any F attached to a given manifold Σ_{n-1} in terms of star products of F 's attached to a consistent decomposition of the initial manifold Σ_{n-1} . For example we may divide any n -cube C_n into 2^n smaller n -cubes whose edges will have half length compared to the one of C_n . In that case the F_i attached to the $C_{n-1}^{(i)}$ will be given as ordered products of 2^{n-1} elementary F_i 's, and all satisfying n -cubic relations, in such a way that the total F_i 's will also satisfy the same equation as a consequence of the 2^n elementary n -cubic equations. For $n = 3$ these features are at the basis of the link between quantum groups and the quantum Yang-Baxter equation. What we obtained here is the generalization of it for all the higher FSR's.

Note that the objects h may be thought of as quantized F 's, whereas the F 's play a role completely analogous to the so-called universal R -matrix for quantum groups, i. e., in the case $n = 3$, F will precisely reduce to that universal object.

Let us now turn to specific examples of the above algebraic and geometrical structures for the cases $n = 2$ and $n = 3$, showing in particular that the FSR's are the genuine generalizations of the Lax equations.

In order to do that and to make contact with previously published results on these particular cases, and for lattice dynamical systems in [27,28], we shall adopt the following notations for the F 's,

$$\begin{aligned} F_i(0) &= F_{l_{j_1} \dots l_{j_{n-1}}}^{(a_1 \dots a_{n-1})} \\ F_i(1) &= F_{l_{j_1} \dots l_{j_{n-1}}}^{(a_1 \dots a_{n-1})l_i} \end{aligned} \quad (48)$$

where the a 's stand for the labels of the vector-spaces $\mathcal{A}_{ij}$ upon which F_i is acting non-trivially, and the sub-

scripts l_j are giving the space-time directions associated to $C_{n-1}^{(i)}$, i. e., $l_j \neq l_i$. Any superscript l_i means that we shift the argument of the object in direction u_i by one unit u_i . Similarly, we define

$$\begin{aligned} h_{ij}(0,0) &= (a_{ij}) h_{l_{j_1} \dots l_{j_{n-2}}}^{(b_1 \dots b_{n-2})} \\ h_{ij}(0,1) &= (a_{ij}) h_{l_{j_1} \dots l_{j_{n-2}}}^{(b_1 \dots b_{n-2})l_j} \\ h_{ij}(1,0) &= (a_{ij}) h_{l_{j_1} \dots l_{j_{n-2}}}^{(b_1 \dots b_{n-2})l_i} \\ h_{ij}(1,1) &= (a_{ij}) h_{l_{j_1} \dots l_{j_{n-2}}}^{(b_1 \dots b_{n-2})l_i l_j} \end{aligned} \quad (49)$$

where (a_{ij}) indicates the vector-space $\mathcal{A}_{ij}$ in which h_{ij} is taking values (as a vector) and the b_j 's correspond to the set of vector spaces $\mathcal{A}_{ijk}$ upon which h_{ij} is acting as an endomorphism. The subscripts l_j have the same meaning as for the F 's.

We begin with the $n = 2$ case. For $n = 2$ Σ_n is a square-domain, and there are four $\Sigma_{n-1}^{(i)} = \Sigma_1^{(i)}$, i. e., the edges of the square (note that in general we do not need to impose the vectors u_i to form an orthonormal basis). Then the $C_{n-2}^{(ij)}$'s reduce to the four points defining the square. Hence, there is only one vector-space $\mathcal{A}_{ij} = \mathcal{A}$, and there is no vector-space of the $\mathcal{A}_{ijk}$ -type. So we have two F 's, corresponding to the two vectors u_i , $i = 1, 2$, which are matrix-valued functions acting in $\mathcal{A}$ and might be considered as line-elements, while there is only one h which is a vector-valued function in $\mathcal{A}$. Eq.(41) reads

$$({}^{(1)}h)^l = F_l^{(1)} * ({}^{(1)}h) \quad (50)$$

with $l = 1, 2$. The compatibility relation for the F 's in order for this equation to have non-trivial h -solutions is given by considering the two possible ways of mapping $h(\Sigma_{n-2}^{(ij)}(0,0)) \equiv ({}^{(1)}h)$ to its value shifted in direction 1 and 2, namely $h(\Sigma_{n-2}^{(ij)}(1,1)) = ({}^{(1)}h)^{ll'}$. It is given also by eq.(43) as

$$F_l^{(1)l'} * F_{l'}^{(1)} = F_{l'}^{(1)l} * F_l^{(1)} \quad (51)$$

The above relations (50,51) are the usual Lax equations for 2-dimensional integrable systems, or more precisely their integrated form. Restricting the vectors u_i to have fixed length and direction, we also obtain the Lax equation for lattice dynamical systems. The infinitesimal limit of these equations leads to the usual Lax equations expressed as the flatness condition for a 2-dimensional gauge field A . Conserved quantities can be obtained from traces of powers of ordered products of the F along a constant-time line as follows from the general considerations of section 2. Defining as in eq.(46) $\hat{F}_l^{(1)} = \sigma_l^{(1)-1} F_l^{(1)}$ we have, using eq.(47),

$$\hat{F}_l^{(1)} * \hat{F}_{l'}^{(1)} = \hat{F}_{l'}^{(1)} * \hat{F}_l^{(1)} \quad (52)$$

which is just the commutativity relation for two operators, i. e., the static 2-cubic (or 1-simplex) relation. In the infinitesimal limit $\hat{F}$ reduces at first order to the covariant derivative associated to the 2-dimensional gauge field A . Note that these F 's are dependent not only on the direction l but also (in general) on one vector u_i , leading to the appearance of a spectral parameter. Note that we have, from eqs.(24,25) for $n = 2$, the following local (gauge) invariance of the above relations, namely,

$$F_l \longmapsto g^l F_l g^{-1} \quad h \longmapsto g h$$

Under this transformation the object $\hat{F}$ transforms covariantly as expected. Hence our scheme reproduces in $d = 1 + 1$, i. e., $n = 2$ the well-known Lax structure for integrable systems. Let us now give a very simple but important example of these structures in the lattice case. Let $F_l = Id \cdot \lambda_l + g^l g^{-1}$, where as above g is an invertible matrix depending on space-time variables. Then eq.(51) defines equations of motion for g which are just the lattice σ -model ones (provided we have the spectral parameter relation $\lambda_l + \lambda_{l'} = 0$).

We now consider the case $n = 3$ that will correspond to $d = 2 + 1$ integrable systems. For $n = 3$ Σ_n

is a cubic-type-domain, and there are six $\Sigma_{n-1}^{(i)} = \Sigma_2^{(i)}$, i. e., the six faces of the cube. Then the $\Sigma_{n-2}^{(ij)}$'s reduce to the eight edges defining the cube. Hence, there are three vector-spaces $\mathcal{A}_{ij}$, and there is one vector-space $\mathcal{A}_{ijk}$. So we have three F 's, corresponding to the three vectors u_i , $i = 1, 2, 3$, which are 2-tensor-valued functions acting in the $\mathcal{A}_{(i)}$'s and that might be considered as plaquette-type objects, while there are three h 's (plus shifted quantities) which are vector valued-functions in the $\mathcal{A}_{(i)}$'s and are acting as endomorphisms in $\mathcal{A}_{ijk}$, these types of space being denoted by latin superscripts in the following ($a, b, \dots$). Eq.(41) reads

$$F_{ll'}^{(12)} * ({}^{(1)}h_l^{(b)l'} {}^{(2)}h_{l'}^{(b)}) = {}^{(2)}h_{l'}^{(b)l} {}^{(1)}h_l^{(b)} \quad (53)$$

with $l, l' = 1, 2, 3$.

The sufficient compatibility condition for this equation to have non-trivial solutions h is given, see eqs.(40, 42), by considering the two different ways of mapping

$$h(\Sigma_{n-2}) = h(\Sigma_1) = {}^{(1)}h_l^{(b)l''} {}^{(2)}h_{l'}^{(b)l''} {}^{(3)}h_{l''}^{(b)}$$

to

$$h(\Sigma'_{n-2}) = h(\Sigma'_1) = {}^{(3)}h_{l''}^{(b)l''} {}^{(2)}h_{l'}^{(b)l} {}^{(1)}h_l^{(b)}$$

It gives the following 3-cubic relation:

$$F_{ll'}^{(12)} F_{ll''}^{(13)l'} F_{l'l''}^{(23)} = F_{l'l''}^{(23)l} F_{ll''}^{(13)} F_{ll'}^{(12)l''} \quad (54)$$

Eqs.(53,54) are the equations defining integrable dynamical systems in $d = 2 + 1$. They give as particular cases the equations given in [27] for lattice systems. These lattice equations are obtained by restricting the set of possible points y and by giving fixed values to the vectors u_i . We shall discuss the infinitesimal version of these equations in the next section together with the extended notion of gauge field and gauge theories they are associated with. It is possible to show, following the general framework of section 2, that these equations

lead to the construction of an infinite set of constants of motion for dynamical systems of field theory whose equations of motion are given by eq.(54) for suitably parametrized F 's. We will give a few examples of such systems in the following in the lattice case. In that lattice situation it is possible to construct these constants of motion as traces of ordered products of F 's on a two-dimensional constant-time surface. This object is actually given by the general formula (22) by decomposing this space-like surface on the lattice in its elementary plaquette objects $F_{ll'}$, the product law to compute the F associated to the total surface being obtained by successive use of eq.(22). This amounts in fact to defining two directions on this surface and considering the double-ordered product of the $F_{ll'}^{(12)}$, i. e., to order the matrix product in space 1 according to the direction 2 and vice-versa. Then we obtain conserved quantities by choosing suitable boundary conditions on the lines $\Sigma_{n-2} = \Sigma_1$ and $\Sigma'_{n-2} = \Sigma'_1$, chosen here such that their gluing is bounding the space-domain of definition of our dynamical system and taking traces of powers of $F(\Sigma_{n-1}) = F(\Sigma_2) = F(\Sigma'_2)$. We will give an explicit computation in the following.

Now let us give the gauged version of the above equation, leading in fact to the static 3-cubic (2-simplex) relation well known as the quantum Yang-Baxter equation. We define according to (46)

$$\hat{F}_{ll'}^{(12)} = \sigma_l^{(2)-1} F_{ll'}^{(12)} \sigma_{l'}^{(1)}$$

which satisfies, following from eq.(54) or directly from eq.(47),

$$\hat{F}_{ll'}^{(12)} \hat{F}_{ll''}^{(13)l} \hat{F}_{l'l''}^{(23)} = \hat{F}_{l'l''}^{(23)} \hat{F}_{ll''}^{(13)} \hat{F}_{ll'}^{(12)} \quad (55)$$

Note that for static F 's this is exactly the quantum Yang-Baxter equation with spectral parameters, namely, each $F_{ll'}$ depends on two real vectors u_l and $u_{l'}$ that

are precisely playing the role of the usual spectral parameters. In fact any solution to the quantum Yang-Baxter equation determines a corresponding solution of the FSR's for $n = 3$. Indeed, one of the main features of eq.(54) is its resemblance to the quantum Yang-Baxter equations (QYBE's), except for the fact that here there are translations along the lattice directions which appear in a non-trivial fashion in the equation.

From eqs.(24,25), we have the following local (extended gauge) invariance of the above structures (g_l are invertible matrix-valued functions dependent on a space-time direction index l),

$${}^{(1)}h_l^{(b)} \rightarrow g_l^{(1)} {}^{(1)}h_l^{(b)}$$

and the $F_{ll'}$'s transform under local gauge transformations g_l as

$$F_{ll'}^{(12)} \rightarrow (g_l \otimes g_{l'}) \cdot F_{ll'}^{(12)} \cdot (g_l'^{-1} \otimes g_{l'}^{-1}) ,$$

and transform in a covariant way when they are replaced by the operators $\hat{F}_{ll'}$,

$$\hat{F}_{ll'}^{(12)} \rightarrow (g_l \otimes g_{l'}) \cdot \hat{F}_{ll'}^{(12)} \cdot (g_l^{-1} \otimes g_{l'}^{-1}) .$$

In [29] we investigated the algebraic structures associated with the system (54), and in particular it can be shown that (55) gives rise to a notion of a 'gauging' of a quantum group [29]. In fact, by introducing the lattice translations in (54), the quantum group structure becomes a *local* structure. All these properties follow quite obviously from our present more fundamental approach, developed in section 2, and are valid not only for lattice systems but also for continuous ones.

Let us now give some details on why the system of equations (54) gives rise to integrability. In fact, it can be shown that from these 3-cubic relations we have a means of deriving conserved objects, in much the same

way as the linear system (51) gives rise to conservation laws. We introduce the path-ordered objects J and I along an arbitrary curve C , starting from a point Q and defined by the ordered set of lattice directions $(qq'q'' \dots)$. This leads in a natural way to the introduction of line- resp. surface transfer matrices. These are objects defined as

(line objects:)

$$\begin{aligned} J_{l(qq'q'' \dots q^{(n)})}^{1(22'2'' \dots 2^{(n)})} &= F_{lq}^{(12)} F_{lq'}^{(12')} F_{lq''}^{(12'')} F_{lq'''}^{(12''')} \dots \\ I_{(q^{(n)} \dots q^{(2)} q^{(1)})l}^{(2^{(n)} \dots 2^{(2)} 2^{(1)})} &= \dots F_{q^{(n)}l}^{(2^{(n)})} F_{q^{(2)}l}^{(2^{(2)})} F_{q^{(1)}l}^{(2^{(1)})} \end{aligned} \quad (56)$$

and

(surface-ordered object:)

$$\begin{aligned} T_{(\dots p''p'p)(qq'q'' \dots)}^{(\dots 1''1'1)(22'2'' \dots)} &= I_{\hat{p}q}^{(12)} I_{\hat{p}q'}^{(12')} F_{\hat{p}q''}^{(12'')} \dots \\ &= \dots J_{p'q}^{(1'2)} J_{p\hat{q}}^{(12)} \end{aligned} \quad (57)$$

where in (56) there is a matrix product in space (1) and a tensor product on the spaces $(2, 2', 2'', \dots)$. We will denote these quantities (56) simply by $J_{l\hat{q}}^{(1\hat{2})}$ and $I_{\hat{q}l}^{(2\hat{1})}$ and (57) respectively by $T_{\hat{p}\hat{q}}^{(1\hat{2})}$. Using the definitions (56) and (57) together with (54) one has

$$\begin{aligned} F_{l\hat{p}}^{(12)} J_{l\hat{q}}^{(1\hat{3})} J_{\hat{p}\hat{q}}^{(2\hat{3})} &= J_{\hat{p}\hat{q}}^{(2\hat{3})} J_{l\hat{q}}^{(1\hat{3})} F_{l\hat{p}}^{(12)} \hat{q}, \\ I_{\hat{p}l}^{(3\hat{1})} I_{\hat{p}l'}^{(3\hat{2})} F_{l\hat{p}}^{(12)} &= F_{l\hat{p}}^{(12)} \hat{p} I_{\hat{p}l'}^{(3\hat{2})} I_{\hat{p}l}^{(3\hat{1})} l', \\ J_{l\hat{p}}^{(1\hat{3})} F_{l\hat{p}}^{(12)} \hat{p} I_{\hat{p}l'}^{(3\hat{2})} &= I_{\hat{p}l'}^{(3\hat{2})} F_{l\hat{p}}^{(12)} J_{l\hat{p}}^{(1\hat{3})} l', \end{aligned} \quad (58)$$

and

$$\begin{aligned} J_{l\hat{p}}^{(1\hat{2})} J_{l\hat{q}}^{(1\hat{3})} \hat{p} T_{\hat{p}\hat{q}}^{(2\hat{3})} &= T_{\hat{p}\hat{q}}^{(2\hat{3})} J_{l\hat{q}}^{(1\hat{3})} J_{l\hat{p}}^{(1\hat{2})} \hat{q}, \\ T_{\hat{p}\hat{q}}^{(1\hat{2})} I_{\hat{p}l}^{(1\hat{3})} \hat{q} I_{\hat{q}l}^{(2\hat{3})} &= I_{\hat{q}l}^{(2\hat{3})} \hat{p} I_{\hat{p}l}^{(1\hat{3})} T_{\hat{p}\hat{q}}^{(1\hat{2})} l, \\ I_{\hat{p}l}^{(1\hat{2})} T_{\hat{p}\hat{q}}^{(1\hat{3})} J_{l\hat{q}}^{(2\hat{3})} &= J_{l\hat{q}}^{(2\hat{3})} \hat{p} T_{\hat{p}\hat{q}}^{(1\hat{3})} I_{\hat{p}l}^{(1\hat{2})} \hat{q}. \end{aligned} \quad (59)$$

The compatibility condition on (53) leads to (54) for the F 's, whereas the compatibility condition for (59) leads to an equation for T , namely

$$T_{\hat{p}\hat{q}}^{(1\hat{2})} T_{\hat{p}\hat{r}}^{(1\hat{3})} \hat{q} T_{\hat{q}\hat{r}}^{(2\hat{3})} = T_{\hat{q}\hat{r}}^{(2\hat{3})} \hat{p} T_{\hat{p}\hat{r}}^{(1\hat{3})} T_{\hat{p}\hat{q}}^{(1\hat{2})} \hat{r}. \quad (60)$$

Let us stress again here that all these quantities and all the above relations are direct consequences of the FSR for $n = 3$, namely they all follow from the lattice or discretized version of eqs.(20,22) for suitably chosen surfaces Σ_{n-1} and Σ'_{n-1} . So, eq.(60) is nothing but another version of the 3-cubic relation, but now for surface-ordered objects $T_{\hat{p}\hat{q}}$ with indices running along the oriented loop which is bordering the surface. As we already remarked, it is in fact this object T that can be identified with the so-called universal R-matrix, here in its local version. In fact, every choice of a different loop corresponds to a specific choice of a representation of our local quantum group, cf. [29]. For our purpose it is sufficient to note that these surface-ordered objects T play the role of the monodromy matrix of the system (53). It has been shown in [27] that this object gives rise to the construction of conserved quantities for a lattice dynamical system whose equations of motion are given by eq.(54).

Let us now give a very simple but non-trivial way of getting interesting integrable lattice dynamical systems in $d = 2 + 1$ whose equations of motion will be given by eq.(54). We define

$$F_{ll'}^{(12)} = A_l^{(2)} \mathbf{P}^{(12)} A_{l'}^{(1)-1} \quad (61)$$

where $A_l^{(i)}$, $l = 1, 2, 3$, is a matrix-valued function depending on a space-time direction l , (i) standing for the vector-space label upon which this object is acting, and $\mathbf{P}_{ab,cd} = \delta_{ad}\delta_{bc}$ is the usual permutation operator. We further parametrize the A 's in terms of dynamical fields a_l that are matrix-valued functions on a 3-dimensional lattice as

$$A_l = Id + a_l$$

Then eq.(54) leads to the following equation for the A 's (we have suppressed the vector-space superscript (i) since in that equation all A 's are acting in the same

space),

$$A_l'' (A_l'')^{-1} A_l''' = A_l^l (A_l^l)^{-1} A_l''' \quad (62)$$

where as usual a superscript l means that we shift the corresponding function by one unit in the space-time direction given by l . This equation leads to highly non-trivial dynamics in terms of the a_l -fields in $d = 2 + 1$. In particular, using various parametrizations of a_l in terms of $(2 + 1)$ -dimensional dynamical fields it is possible to generate in the continuum limit of the above equations a large number of already known $(2 + 1)$ -dimensional integrable systems such as the KP and matrix KP equations, the Davey-Stewartson equation, the N -wave interaction system, together with $(2 + 1)$ -dimensional generalizations of the Heisenberg ferromagnet and Toda equations [1]. In fact we not only obtain one equation but complete hierarchies of equations following from the expansion of the lattice equations in power series of the lattice parameters. The details of these models will be published elsewhere.

We now briefly describe the $n = 4$ case without discussing its properties here. (The following structures will correspond to integrable dynamics in $d = 3 + 1$). Eq.(41) reads, for $n = 4$,

$$F_{ll''}^{(123)} * \left(\begin{matrix} (1)h_{ll'}^{(bb')} & (2)h_{ll''}^{(bb'')} & (3)h_{ll'''}^{(b'b'')} \\ (3)h_{ll'''}^{(b'b'')} & (2)h_{ll''}^{(bb'')} & (1)h_{ll'}^{(bb')} \end{matrix} \right) = \quad (63)$$

where the dot-products are products for the h 's as endomorphisms in the vector-spaces labelled as b, b', b'' and the star-product gives the action of F in the vector-spaces labelled by $(1, 2, 3)$ in which the h 's are vectors whereas F 's act as endomorphisms in such spaces. Now, as described in the general case, we obtain a consistency requirement on the F 's by considering the two possible ways of mapping

$$h(\Sigma_{n-2}) = \begin{matrix} (1)h_{l_1 l_2}^{(ab)} & (2)h_{l_1 l_3}^{(ac)} & (3)h_{l_2 l_3}^{(bc)} \end{matrix}$$

$$\begin{matrix} (4)h_{l_1 l_4}^{(ad)} & (5)h_{l_2 l_4}^{(bd)} & (6)h_{l_3 l_4}^{(cd)} \end{matrix} \quad (64)$$

to

$$h(\Sigma'_{n-2}) = \begin{matrix} (6)h_{l_3 l_4}^{(cd)} & (5)h_{l_2 l_4}^{(bd)} & (4)h_{l_1 l_4}^{(ad)} \\ (3)h_{l_2 l_3}^{(bc)} & (2)h_{l_1 l_3}^{(ac)} & (1)h_{l_1 l_2}^{(ab)} \end{matrix} \quad (65)$$

leading to, (c.f. 43)

$$\begin{aligned} F_{l_1 l_2 l_3}^{(123)} * F_{l_1 l_2 l_4}^{(145)} * F_{l_1 l_3 l_4}^{(246)} * F_{l_2 l_3 l_4}^{(356)} = \\ = F_{l_2 l_3 l_4}^{(356)} * F_{l_1 l_3 l_4}^{(246)} * F_{l_1 l_2 l_4}^{(145)} * F_{l_1 l_2 l_3}^{(123)} \end{aligned} \quad (66)$$

The static limit of this equation is the so-called tetrahedron equation. We define after eq.(46)

$$\hat{F}_{ll''}^{(123)} = \sigma_l^{(3)-1} \sigma_{l''}^{(1)-1} F_{ll''}^{(123)} \sigma_{l'}^{(2)}$$

If F satisfies the 4-cubic relation (66), then $\hat{F}$ satisfies the static 4-cubic (or tetrahedron) relation

$$\begin{aligned} \hat{F}_{l_1 l_2 l_3}^{(123)} * \hat{F}_{l_1 l_2 l_4}^{(145)} * \hat{F}_{l_1 l_3 l_4}^{(246)} * \hat{F}_{l_2 l_3 l_4}^{(356)} = \\ = \hat{F}_{l_2 l_3 l_4}^{(356)} * \hat{F}_{l_1 l_3 l_4}^{(246)} * \hat{F}_{l_1 l_2 l_4}^{(145)} * \hat{F}_{l_1 l_2 l_3}^{(123)} \end{aligned} \quad (67)$$

As in the the cases $n = 2$ and $n = 3$ and following from our general setting of section 2, it is possible to construct infinite series of conserved charges for dynamical systems having equations of motion given through eq.(66).

4. EXTENDED GAUGE THEORIES

The purpose of this section is to extend further the results of the previous sections, and to give some applications of them, in particular to the definition of a new notion of gauge fields and gauge theories.

In $d = 1 + 1$, the 2-simplex and 2-cubic relations of section 2 that lead to the Lax equations for integrable systems have a natural interpretation as zero-curvature

equations for two-dimensional gauge fields. It is therefore very natural to ask whether an analogous picture exists for the higher FSR's, leading in particular to the question of defining the corresponding notions of gauge fields and gauge theories. This idea is supported by the existence of the local invariance (24) of the FSR's for arbitrary n that reduces to the usual gauge invariance for $n = 2$. In order to deal with these problems, we begin by deriving some additional properties associated with the functional objects F we used in constructing the FSR's.

We consider general functionals $F(\Sigma_{n-1}) \in \mathcal{A}_{\partial\Sigma_{n-1}}$, i.e., not satisfying a priori the FSR's given in eq.(20), but that obey the geometrical $*$ -product law (22). Moreover we interpret the functionals $F(\Sigma_{n-1}) \in \mathcal{A}_{\partial\Sigma_{n-1}}$ as mappings from $\mathcal{A}_{\Sigma_{n-2}}$ to $\mathcal{A}_{\Sigma'_{n-2}}$ and also $F(\Sigma_{n-1}^*) \in \mathcal{A}_{\partial\Sigma_{n-1}^*}$ as a mapping from $\mathcal{A}_{\Sigma'_{n-2}}$ to $\mathcal{A}_{\Sigma_{n-2}}$, such that we have the following normalization relations,

$$F(\Sigma_{n-1}^*) *_{\mathcal{A}_{\Sigma'_{n-2}}} F(\Sigma_{n-1}) = Id_{\mathcal{A}_{\Sigma_{n-2}}}$$

and

$$F(\Sigma_{n-1}) *_{\mathcal{A}_{\Sigma_{n-2}}} F(\Sigma_{n-1}^*) = Id_{\mathcal{A}_{\Sigma'_{n-2}}}$$

where as before $Id_{\mathcal{A}_{\Sigma_{n-2}}}$ is the identity endomorphism canonically represented in $\mathcal{A}_{\Sigma_{n-2}} \otimes \mathcal{A}_{\Sigma_{n-2}}^*$. These conditions mean that we are able to define the inverse mapping to a given F as the mapping F attached to the same manifold but with reversed orientation. Now we may consider the transport operation of a functional $h(\Sigma'_{n-2})$, using the above F 's, first along Σ_{n-1}^* , i.e., leading to a functional of Σ_{n-2} , and then back to Σ'_{n-2} by using another manifold Σ_{n-1} having the same boundary as Σ'_{n-1} . The transport operator is given in terms of the above F 's as

$$\dot{G}(\Sigma_n) = F(\Sigma_{n-1}) *_{\mathcal{A}_{\Sigma_{n-2}}} F(\Sigma_{n-1}^*) \quad (68)$$

where $\partial\Sigma_n = \Sigma'_{n-1} \circ \Sigma_{n-1}^*$ and $\partial\Sigma'_{n-1} = \partial\Sigma_{n-1} = \Sigma'_{n-2} \circ$

Σ_{n-2}^* . This operator G is in fact a generalized holonomy operator defined on h -functionals. Note that $G(\Sigma_n) \in \mathcal{A}_{\Sigma'_{n-2}} \otimes \mathcal{A}_{\Sigma_{n-2}}^*$ can be interpreted as an endomorphism of $\mathcal{A}_{\Sigma'_{n-2}}$. Asking this operator to be equal to the identity endomorphism leads to the general FSR given in eq.(20). However, in a general situation, G is non-trivial.

We also may write generalized Bianchi-type identities for G as

$$G(\Sigma_n) = G(\Sigma'_n)$$

whenever $\partial\Sigma_n = \partial\Sigma'_n$. Note that these holonomy operators G can be used to define a generalized notion of curvature by considering infinitesimal manifold Σ_n . It is important to remark here that G transforms covariantly under the local u -actions defined in eq.(24).

Before giving some explicit realization of these structures, I would like to comment on the definition of these holonomy operators and more generally on the definition of the operators F . For $n = 2$ we know that the above definitions lead to the usual gauge field, curvature, and holonomy associated to the ordered product of operators along paths. Let me stress that for arbitrary integer $n \geq 2$ the framework I just sketched defines non-ambiguously the notion of ordered products for operators along $(n-1)$ -dimensional manifolds. This result is in fact contained in the formula (22) when reinterpreting the corresponding F 's as endomorphisms or operators. An example of this is given by the formulas (56,57) for the discrete $n = 3$ case where we have been able to define ordered surface-products of the discrete version of F (57). The reason for these ordered products to be uniquely defined is that the $F(\Sigma_{n-1})$'s (think about them in their discrete version associated to $(n-1)$ -cubes) can be interpreted as operators act-

ing in the tensor product of $n - 1$ vector-spaces. So we have as many "matrix"-spaces as we have geometrical directions, i. e., $n - 1$. Hence, we may order each matrix product according to a corresponding geometrical one-dimensional direction. This is in fact what we did in defining the F 's in eq.(42). Two of the important requirements in defining such products are first, to have associativity for products in a given direction and matrix space, and second, to have "connected" associativity for products along two different directions. These two properties are always satisfied when we have matrix or tensor representations for the F 's, and are in turn related to the associativity of the $*$ -products for the F 's in eq.(22). Note that for $n = 3$ this picture gives a geometrical setting for the product and co-product axioms for Hopf algebras.

We now construct the holonomy operators in the framework of discretized manifolds $\Sigma_{n-1}, \dots$ in terms of hyper-cubes. The notations are those of section 3. We have

$$G(\Sigma_n) = \prod_i^> F_i(\varepsilon_i) \prod_i^> F_i(1 - \varepsilon_i)^{-1} \quad (69)$$

This generalized holonomy operator reduces to the identity for the integrable case where eq.(43) is satisfied. Note that in defining G it is possible to use the $\hat{F}$'s (see eq.(46)) instead of the F 's proving in particular that G transforms covariantly under the local transformations of eq.(24) since the difference between G , and say $\hat{G}$, is just a multi-shift operator that defines Σ_{n-2} . Using the notations of eq.(48), we obtain, for $n = 2$,

$$G_{ll'}^{(1)} = F_l^{(1)l'} F_l^{(1)} F_l^{(1)-1} F_l^{(1)l-1}$$

and for $n = 3$,

$$G_{ll'l''}^{(123)} = F_{ll'}^{(12)} F_{ll''}^{(13)l'} F_{l'l''}^{(23)} F_{ll''}^{(12)l-1} F_{ll''}^{(13)-1} F_{l'l''}^{(23)l-1}$$

We will consider this last case with more details in the following. Indeed, these quantities lead directly to the

definition of extended gauge theories on the lattice.

What I would like to discuss now is the problem of deriving the infinitesimal versions of the above definitions and equations. Let me first explain why the answer to that question is not completely straightforward. As can be seen from eq.(22), in the continuum case the vector-spaces $\mathcal{A}_\Sigma$ are either one-dimensional or infinite-dimensional. The essential reason for that property is that the vector-space attached to a manifold Σ is equal to the tensor product of the vector-spaces attached to any partition of the manifold. Hence in the continuum situation, any vector-space $\mathcal{A}_\Sigma$ will be a continuum tensor product over the manifold Σ of "elementary" vector-spaces essentially attached to each point of Σ . Thus, taking the infinitesimal limit of the F 's means that we want to compute the localized objects from which any F can be obtained by means of a kind of integral over the given manifold. In the $n = 2$ case we just obtain a gauge field A as a local object defining the corresponding F 's attached to open paths by means of path-ordered exponential integrals of A . However in that case both the global quantity F and the local one A are essentially of the same dimension in the sense that in a matrix representation of A , F is also a matrix of the same dimension. For $n \geq 3$ this will no longer be the case. So taking the limit of $F(\Sigma)$ when the size of the manifold Σ is going to zero changes not only the object F but also the vector-space in which it takes values. For $n = 2$ it was sufficient to attach the same vector-space to any point in space-time to solve this problem. Let me stress that for $n = 3$ this is already a non-trivial functional problem containing as a particular case the problem of properly defining string field functionals.

In order to give a preliminary discussion of this question, I will not treat that aspect in this lecture in its

full setting but rather consider a regularized framework for it. In fact it is quite clear that we may go back to a finite-dimensional vector-space situation as soon as we cannot divide any manifold indefinitely into smaller and smaller pieces. So, we consider the situation where there exist elementary cells $((n-1)$ -cubes) from which will be built any "macroscopic" manifold Σ . Hence even when the sizes of the elementary objects are small, any "macroscopic" manifold will correspond to a vector-space which will be a tensor product of the vector-spaces attached to the elementary cells, i. e., it will be a large but finite-dimensional vector-space provided those associated to the elementary cells are finite-dimensional. Then making the size of the elementary cells go to zero we have a means of getting a regularized infinitesimal limit for the F 's. It is clear that the complete treatment of this problem is to be more sophisticated than this scheme. However, we shall see that we will already obtain quite interesting results within this regularized framework.

For definiteness let me concentrate on the $n = 3$ situation which is generic in many respects. Moreover as we have shown before, this is an interesting case corresponding, for the FSR's, to the quantum Yang-Baxter equation and to quantum groups. For $n = 3$, F 's are functionals of two-dimensional surfaces having a closed curve as boundary (see section 2 for details). The elementary pieces will be taken to be small straight segments from which we will build lines and squares from which surfaces can be constructed. To each segment we associate a finite dimensional vector-space V . Hence an elementary F attached to a square built out of four segments may be interpreted as an endomorphism of $V \otimes V$. We denote by a the size of the edges of the square. This means that the two vectors u and v that determine the functional F have length a . The nota-

tions are those of section 3. We consider the following infinitesimal expansion of F in power series of a ,

$$F_{uv}^{(12)} = Id_{12} + a f_{uv}^{(12)} + O(a^2) \quad (70)$$

Note that in doing such an expansion we need in particular to choose the zero order term. We have assumed here that this term is simply the identity. Moreover, we will assume the normalization condition $F_{uv}^{(12)} = F_{vu}^{(21)-1}$ leading to $f_{uv}^{(12)} = -f_{vu}^{(21)}$. However there are other possibilities. One of them has already been used for the examples of $(2+1)$ -dimensional integrable systems at the end of section 3. There the zero order term was the permutation operator P_{12} . This last limit is in particular relevant when the vectors u and v are almost collinear.

As an application of this framework, let me now show what are the integrability equations analogous to the continuum Lax equations, but for $d = 2 + 1$ following from eqs.(19,20) with the limit (70). They are given as

$$[f_{uv}^{(12)}, f_{uv}^{(13)} + f_{vu}^{(23)}] + [f_{uv}^{(13)}, f_{vu}^{(23)}] + \partial_v f_{uv}^{(13)} - \partial_v f_{uv}^{(12)} - \partial_v f_{uv}^{(23)} = 0 \quad (71)$$

This is a localized (or gauged) classical Yang-Baxter equation. It may be obtained as the compatibility condition for the obstruction to a zero-curvature relation as can be expected from the general framework of sections 2 and 3. To show this we consider quantities $A_l^{(1)}$ that are functions in a $(2+1)$ -dimensional space, depending on a space-time direction l , with values in $End(V_1) \otimes \mathcal{G}$ where $\mathcal{G}$ is a Lie algebra with Lie bracket denoted by $\{, \}$ and we have denoted by a superscript (1) the vector space V_1 upon which $A^{(1)}$ is acting as a matrix. Then we consider the following obstruction to the zero-curvature relation for the A 's considered as (usual) gauge fields,

$$\partial_l A_l^{(2)} - \partial_l A_l^{(1)} + \{A_l^{(1)}, A_l^{(2)}\} =$$

$$= [f_{ij}^{(12)} , A_i^{(1)} + A_j^{(2)}] \quad (72)$$

The consistency relations (given in particular by the Jacobi identity of the Lie brackets) lead to eq.(71) as a sufficient condition. Hence we will interpret eqs.(72,71) as the genuine generalization of the Lax systems and Lax equations but now in $d = 2 + 1$. In this setting where eqs.(71,72) give the analogue of the Lax linear system and Lax equations in $d = 2 + 1$, the f 's have to be thought of as potentials, depending on the fields of some $(2 + 1)$ -dynamical system, whereas the A 's are the solutions of the differential system (72) with potential f . It is quite easy to obtain an integrated version of the above equations leading to quadratic-type equations for the A 's. I shall give the details of these results in a future publication.

If we now reinterpret the commutators between the f 's as Poisson brackets, the above equations describe also the integrable obstruction to the Lax equations for $(1 + 1)$ -dimensional integrable systems when quantizing them. In that case the f 's are dynamical quantities depending on the fields of the system.

Let me comment now on the geometrical interpretation of the above equations, in connection with the remarks made at the beginning of section 3. First it is easy to see that if we consider path-ordered exponential integrals of the A 's they will no longer satisfy the 2-simplex fundamental relations. The obstruction to that equation will be given by an operator F acting only on the V -indices of the A 's and given as a surface-ordered integral of the f 's on a surface mapping the initial path to another path with the same initial and final points. Now we also may understand the meaning of eq.(71). It is the infinitesimal relation ensuring that the "macroscopic" $F(\Sigma)$ will not depend on the partic-

ular surface Σ we will consider but only on the initial and final paths it is mapping onto each other. In other words, when we move infinitesimally the surface Σ at some point x , $F(\Sigma)$ would change by an infinitesimal quantity given by $F(\Sigma)$ with an insertion at point x of the l.h.s. of eq.(71). Thus, if eq.(71) is satisfied, $F(\Sigma)$ is left constant. Hence we will be able to construct objects satisfying the 3-simplex and 3-cubic relation from such an F . This picture has very interesting applications to the geometrical construction of universal R -matrices associated to quantum groups. I shall give the details of this relation elsewhere.

Let me now turn to the subject announced in the title of the section, namely the problem of defining generalized notions of gauge fields. We start from functionals $h(\Sigma_{n-2})$ and a local action of some algebra $\mathcal{G}$ on these functionals, the representation spaces of $\mathcal{G}$ being the vector-spaces $\mathcal{A}_{\Sigma_{n-2}}$. We define the action of the elements u of $\mathcal{G}$ on the h 's as in eq.(25). Then, this action being local, i. e., depending on the manifold Σ_{n-2} , the action of the u 's and the translation in space-time of Σ_{n-2} are two non-commuting operations. So the question is whether it is possible to build the analogue of a gauge field and of its associated covariant translation whose action on the h 's results in an object that transforms covariantly under the local action of $\mathcal{G}$. As can be seen from eqs.(25,24) what we need to define as the extended gauge potential is precisely an object such as the F 's, without demanding, however, that the FSR's be satisfied. In that case the FSR's will become zero-curvature relations for these generalized gauge potentials.

For $n = 3$ the h 's will be functionals of paths and the corresponding notion of the gauge field is the one needed when defining what could be a gauge string field

theory. Note that the holonomy operators associated to such theories have been defined at the beginning of this section. To make the discussion more precise I now focus on this $n = 3$ case and first examine this theory on a square lattice of dimension greater than or equal to 3. It will be quite obvious to extend the structures I will describe to arbitrary $n \geq 4$. The elementary gauge field for $n = 3$ on a square lattice is a functional of any elementary square plaquette. Hence it is a function of the position of the plaquette (we will omit), and on the orientation of the plaquette given by an ordered couple of space-time directions denoted generically by (ll') . In addition this object takes values in the tensor product $V_1 \otimes V_2 \otimes V_1^* \otimes V_2^*$ and may be reinterpreted as an endomorphism of $V_1 \otimes V_2$. We will denote such a quantity as $B_{ll'}^{(12)}$. Moreover as in the previous section any space-time superscript (l) will stand for the shift of the argument (x) of the corresponding object by one unit in direction (l) . Let me stress here that even though the field B is associated to a plaquette (ll') , it does not mean that it is given as any kind of ordered product of some usual gauge field potential U around this plaquette. I will comment later on this possible use of the field B but then associated with *quantum* usual gauge potentials. The corresponding gauge transformations are given by the general formula in eq.(24) for $n = 3$ as,

$$B_{ll'}^{(12)} \rightarrow (g_l \otimes g_{l'}) \cdot B_{ll'}^{(12)} \cdot (g_l'^{-1} \otimes g_{l'}'^{-1}) \quad (73)$$

where the g_l are representations of invertible $\mathcal{G}$ -elements acting as endomorphisms in V_1 , etc., and depending on the curve of which they are functionals (here in this discretized case an elementary segment), given here by a point x (that we have omitted) and a space-time direction (l) .

The local curvature associated to this lattice gauge field depends on three space-time directions $(ll'l'')$ (defin-

ing an elementary cube) and acts as an endomorphism in the tensor product of three vector-spaces $V_1 \otimes V_2 \otimes V_3$, is given as

$$G_{ll'l''}^{(123)} = B_{ll'}^{(12)} B_{ll''}^{(13)} B_{l'l''}^{(23)} B_{ll'}^{(12)^{l''-1}} B_{ll''}^{(13)^{l-1}} B_{l'l''}^{(23)^{l'-1}} \quad (74)$$

We also may construct G by using the covariant shift operator $\hat{B}$ defined as follows,

$$\hat{B}_{ll'}^{(12)} = \sigma_l^{(2)-1} B_{ll'}^{(12)} \sigma_{l'}^{(1)} \quad (75)$$

It transforms in a covariant way under the above gauge action, i. e.,

$$\hat{B}_{ll'}^{(12)} \rightarrow (g_l \otimes g_{l'}) \cdot \hat{B}_{ll'}^{(12)} \cdot (g_l'^{-1} \otimes g_{l'}'^{-1}) \quad (76)$$

We have

$$\hat{G}_{ll'l''}^{(123)} = \hat{B}_{ll'}^{(12)} \hat{B}_{ll''}^{(13)} \hat{B}_{l'l''}^{(23)} \hat{B}_{ll'}^{(12)^{-1}} \hat{B}_{ll''}^{(13)^{-1}} \hat{B}_{l'l''}^{(23)^{-1}}$$

with the relation

$$\hat{G}_{ll'l''}^{(123)} = \sigma_l^{(2)-1} \sigma_{l'}^{(3)-1} \sigma_{l''}^{(1)-1} G_{ll'l''}^{(123)} \sigma_l^{(2)} \sigma_{l'}^{(3)} \sigma_{l''}^{(1)} \quad (77)$$

Hence it is obvious that the generalized curvature $G_{ll'l''}^{(123)}$ also transforms covariantly. Note that this operator is invertible as soon as the B 's are also invertible. We may write the generalized Bianchi-type identities for $\hat{G}$ as

$$\begin{aligned} \hat{G}_{123} \hat{B}_{23} \hat{B}_{13} \hat{G}_{124} \hat{B}_{24} \hat{B}_{13}^{-1} \hat{G}_{134} \hat{B}_{24}^{-1} \hat{B}_{23}^{-1} \hat{G}_{234} &= \\ = \hat{B}_{12} \hat{B}_{13} \hat{B}_{14} \hat{G}_{234} \hat{B}_{14}^{-1} \hat{B}_{13}^{-1} \hat{G}_{134} \hat{B}_{34} \hat{B}_{12}^{-1} & \\ \hat{G}_{124} \hat{B}_{24} \hat{B}_{14} \hat{G}_{123} \hat{B}_{14}^{-1} \hat{B}_{24}^{-1} \hat{B}_{34}^{-1} & \end{aligned} \quad (78)$$

The generalized holonomy operator is given as a surface-ordered product of the B 's that is replacing here the well-known Wilson loop.

Let me now give a possible action functional for such extended gauge fields that generalize the Yang-Mills action. The idea is quite simple. In the usual gauge theory

case ($n = 2$) the lattice action is given as the sum over all the lattice points and all the possible couples of lattice directions (ll') that label any elementary plaquette (ll'), of the trace of the corresponding holonomy operator, i. e., of the lattice curvature. It is quite natural to do the same here and to define the lattice action for the B -fields as

$$S[B] = \sum_{x ll' l''} tr_{123} (G_{ll' l''}^{(123)} + c.c.), \quad (79)$$

where the sum is over all the lattice points x and all the possible triples of lattice directions $ll' l''$. This action is clearly invariant under space-time translations as it should be and is also invariant under the above gauge transformations for the B 's since G transforms covariantly. We may even give the minimal coupling term to (string) matter fields Ψ_l as

$$S[\Psi] = \sum_{x ll'} tr_{12} [\Psi_l \otimes \Psi_{l'}^{\dagger} \cdot B_{ll'}^{(12)} \cdot \Psi_l'^{-1} \otimes \Psi_{l'}^{-1} + c.c.] \quad (80)$$

where the sum is over all the lattice points x and all the possible couples of lattice directions ll' . This action term is invariant under the local (gauge) transformations acting on the right of Ψ as

$$\Psi_l \rightarrow \Psi_l g_l^{-1}$$

combined with the gauge transformations acting on the B 's. The above action is exactly given by the minimal coupling of the Ψ 's to the covariant gauge field $\hat{B}$.

It is quite easy to see that our constructions extend to the higher n -simplicial and n -cubic type B objects that we may define for arbitrary n in much the same way as above simply using the results of sections 2 and 3. The n -simplex extended gauge fields are in that case associated to $(n-1)$ -cubes with the same definitions as the F 's except that they do not satisfy the n -cubic relations (43). The generalized lattice curvature is defined

as in eq.(69), and we even may give an action functional for such extended gauge fields, invariant under the local gauge transformations following from eq.(24). It reads

$$S_n[B] = \sum_{x l_1 l_2 \dots l_n} tr_{(ij)} [G_{l_1 l_2 \dots l_n}^{((ij))} + c.c.] \quad (81)$$

The minimal coupling to $((n-2)$ -brane) matter fields may also be given as in eq.(80) using the covariant shifts $\hat{B}_i$.

I will now discuss the continuum limit of this theory using the already defined continuum limit for the F for $n = 3$. In fact if we consider the case of slowly varying fields B , and for the lattice spacing a going to zero, we obtain the following infinitesimal limits for B and G ,

$$B_{ll'}^{(12)} \rightarrow Id^{(12)} + a b_{ll'}^{(12)} + O(a^2)$$

and similarly,

$$G_{ll' l''}^{(123)} \rightarrow Id^{(123)} + a^2 g_{ll' l''}^{(123)} + O(a^3)$$

where we have the following expression for the local curvature g

$$g_{ll' l''}^{(123)} = [b_{ll'}^{(12)}, b_{l'l''}^{(13)} + b_{l'l''}^{(23)}] + [b_{ll''}^{(13)}, b_{l'l'}^{(23)}] + \partial_{l'} b_{ll''}^{(13)} - \partial_{l''} b_{ll'}^{(12)} - \partial_l b_{l'l''}^{(23)} \quad (82)$$

Note that following from the inversion properties of the B 's we have the relations $b_{ll'}^{(12)} = -b_{l'l}^{(21)}$, namely the b -fields are antisymmetric under simultaneous exchange of the space-time indices (ll') and the vector-spaces (12). Hence the b -fields are not usual antisymmetric tensor fields, and reduce to this case only in the Abelian situation. In the same way the extended curvatures g are completely antisymmetric under any simultaneous exchanges of the space-time indices and of the corresponding vector-spaces superscripts. Note that the total traces of b and g are indeed antisymmetric in the space-time indices ($ll' l''$). As for the lattice case it is

possible to define a notion of a covariant derivative $\hat{b}$ (acting on string fields here)

$$\hat{b}_{ll'}^{(12)} = b_{ll'}^{(12)} + \partial_l \otimes 1 - 1 \otimes \partial_l \quad (83)$$

where, as for the shifts, the space-time derivatives act in specific vector-spaces. Then if we denote by $\langle f, f \rangle$ the (Schouten) bracket of f defined as

$$\begin{aligned} \langle f, f \rangle_{ll'rr'}^{(123)} = & [f_{ll'}^{(12)}, f_{rr'}^{(13)} + f_{r'r}^{(23)}] + \\ & + [f_{ll'}^{(13)}, f_{r'r}^{(23)}] \end{aligned} \quad (84)$$

we obtain the following compact representation for the curvature g

$$g_{ll'rr'}^{(123)} = \langle \hat{b}, \hat{b} \rangle_{ll'rr'}^{(123)} \quad (85)$$

Note that the difference between G and $\hat{G}$ vanishes for the first non-trivial order leading to g in the expansion in the lattice parameter (a). Now we may even write the Bianchi-type identities for g as

$$\langle \hat{b}, \langle \hat{b}, \hat{b} \rangle \rangle_{l_1 l_2 l_3 l_4}^{(1234)} = 0 \quad (86)$$

or more explicitly, (we set $\hat{b}_{l_1 l_2}^{(12)} \equiv \hat{b}_{12}$, and similarly $g_{l_1 l_2 l_3}^{(123)} \equiv g_{123}$)

$$\begin{aligned} & [\hat{b}_{12}, g_{234} + g_{134}] + \\ & [\hat{b}_{13}, -g_{124} + g_{234}] + \\ & - [\hat{b}_{23}, g_{124} + g_{134}] + \\ & [\hat{b}_{14}, g_{234} + g_{123}] + \\ & [\hat{b}_{24}, g_{123} - g_{134}] + \\ & [\hat{b}_{34}, g_{123} + g_{124}] = 0 \end{aligned} \quad (87)$$

which is a lengthy but straightforward computation from the definitions of the b 's and g 's. It may also be obtained from the lattice relation (78).

The (extended) gauge transformations are given by local invertible matrices g_l depending on a space-time

direction index (l). They act on the fields as

$$\begin{aligned} b_{ll'}^{(12)} & \rightarrow (g_l \otimes g_{l'}) b_{ll'}^{(12)} (g_l^{-1} \otimes g_{l'}^{-1}) \\ & + 1 \otimes (\partial_l g_{l'}) g_{l'}^{-1} - (\partial_l g_l) g_l^{-1} \otimes 1 \end{aligned} \quad (88)$$

$$\hat{b}_{ll'}^{(12)} \rightarrow (g_l \otimes g_{l'}) \hat{b}_{ll'}^{(12)} (g_l^{-1} \otimes g_{l'}^{-1}) \quad (89)$$

$$\begin{aligned} g_{ll'rr'}^{(123)} & \rightarrow \\ & (g_l \otimes g_{l'} \otimes g_{rr'}) g_{ll'rr'}^{(123)} (g_l^{-1} \otimes g_{l'}^{-1} \otimes g_{rr'}^{-1}) \end{aligned} \quad (90)$$

From the lattice case it is quite easy to derive a gauge invariant action functional for these extended gauge fields as

$$S[b] = \sum_{ll'rr'} \int_{\mathcal{M}} d\mu \operatorname{tr}_{123} ((g_{ll'rr'}^{(123)})^2 + \text{c.c.}) \quad (91)$$

This is my proposal for the extended $n = 3$ -simplex gauge field theory that generalizes the usual Yang-Mills action. As for the lattice case we may also introduce matter fields coupled minimally to the b 's. More details of this type of theories will be described elsewhere.

Now I would like to end this lecture by listing very briefly some possible (speculative) applications of these new concepts of gauge symmetries. The first question concerns the relation of these structures with string field theories. In a discretized framework (lattice case) it should be possible to make this connection more explicit. The most simple situation is probably the one of the fermionic string. Another quite interesting problem is the possible relevance of these new gauge field theories to particle physics. There is perhaps a more direct use of the b -fields in the context of usual *quantized* Yang-Mills gauge theories as a kind of *quantum* connection. In particular, if we replace the Ψ_l 's in eq.(80) by quantized Yang-Mills fields A , the B 's acting on the quantum indices of the A 's only, we obtain a modified Yang-Mills

action that reduces to the usual one when we trivialize the B 's and the corresponding quantum spaces. Note in particular that there exists a large class of non-trivial flat b -fields given simply as the solutions to the quantum Yang-Baxter equation for which the pure gauge term in the action for the B 's vanishes identically. Finally, there is certainly a very promising possibility of using these extended gauge fields in the context of the topology of low-dimensional manifolds. Note in particular that it is possible to integrate the full trace of the curvature $g_{ll''}^{(123)}$ (which is completely antisymmetric in the space-time indices (ll'')) over a 3-manifold. This quantity is gauge invariant, and this trace is the total derivative of the traces of the b 's. However the b -traces are not gauge-invariant quantities, making it possible to obtain a non-zero topological invariant of three-manifolds from the integral of the trace of g . This in fact raises the question of the analogue of the Chern classes in this context. Let me remark also that the framework I described here enables a natural three-dimensional setting for the quantum Yang-Baxter equations and more generally of the 3-simplex relations of section 2, as zero-curvature equations determining flat gauge potentials. The holonomy operator now being associated to surfaces, it would be interesting to explore the eventual topological invariant quantities to which they lead and their relations to the recently constructed invariants of knots and of three-manifolds [39,22,37,38,21,40,41,42,43] obtained precisely from solutions of the quantum Yang-Baxter equation.

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