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The Aharonov-Casher effect and Berry's phase

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ABSTRACT

We show that the topological phase of the Aharonov-Casher effect can be derived as a special case of Berry's phase.

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In 1984 Berry [1] showed that the wave function $|\psi(t)\rangle$ of a quantum system develops a non-trivial phase when the system undergoes an adiabatic evolution if certain conditions are satisfied. The wave function satisfies the Schrodinger equation,

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H(R(t))|\psi(t)\rangle, \quad (1)$$

where $R(t)$ indicates the parameters which determine the Hamiltonian of the system. At any instant, there is an eigenstate basis $|n(R)\rangle$ of $H(R)$, such that,

$$H(R)|n(R)\rangle = \epsilon(R)|n(R)\rangle. \quad (2)$$

Adiabatically, a system prepared in one of these states $|n(R(0))\rangle$ will evolve with $H(R(t))$ and so be in the state $|n(R(t))\rangle$. The wave function of the system at time t is,

$$|\psi(t)\rangle = \exp\left\{-i \int_0^t dt' \epsilon(R(t'))\right\} \exp(i\gamma(t)) |n(R(t))\rangle. \quad (3)$$

If there is a time T such that $R(T) = R(0)$, the excursion of the system between time $t = 0$ and $t = T$ can be pictured as transport round a closed path $C = \{R(t) : t \in (0, T)\}$ in parameter space. The phase developed for such a excursion is given by,

$$\gamma(C) = i \oint_C \langle n(R) | \nabla_R n(R) \rangle \cdot dR, \quad (4)$$

where $\nabla_R n(R)$ is the gradient of $n(R)$ respect to the parameters R and the integral is over the path C . In fact if during the time interval T , the state is always in a energy eigenstate, the adiabatic approximation is not necessary.

The phase $\gamma(C)$ (Berry's phase) is a fundamental prediction of quantum mechanics. Several quantum phases have been shown to be special cases of Berry's phase[1]. For example, the topological phase of the Aharonov-Bohm (AB)effect [2] is a special case of Berry's phase. In this note we show that the topological phase of the Aharonov-Casher (AC) effect [3] is also a special case of Berry's phase. While we are writing up this note, we received a preprint by Mignani [4], who also studied the same problem. However, in Mignani's study a non-relativistic approximation was used. In the following we treat the problem without making any approximation.

The AC effect was pointed out in 1984 by Aharonov and Casher[3]. They found that in the non-relativistic limit the wave function of a neutral particle with non-zero magnetic dipole moment μ develops a topological phase when the particle is transported along a closed path which encircles an infinitely long filament carrying a uniform charge density. A relativistic treatment of this effect was given by us in Ref. [5]. The AC phase has been measured experimentally [6].

The equation of motion for a neutral spin-half particle with a non-zero magnetic dipole moment moving in a static electric field $\mathbf{E}$ is

$$(\not{\partial} + \frac{1}{2}\mu\sigma_{\mu\nu}F^{\mu\nu} + m)\psi = 0, \quad (5)$$

which can be written as

$$(\not{\partial} + \mu\boldsymbol{\gamma} \cdot \mathbf{E}\gamma_4 + m)\psi = 0, \quad (6)$$

where $F^{\mu\nu}$ is the field strength of the electric field.

Certain conditions must be satisfied in order to develop a topological AC phase. These conditions are[5]:

i) a neutral spin half particle with a non zero magnetic dipole moment is moving in a plane, say the $x - y$ plane, in a external electric field $\mathbf{E}$;

ii) On this plane, $E_z = 0, \partial_z E_z = 0$; and $\partial_z \psi = 0$, and

iii) It is possible for this particle to travel a closed path in the $\nabla \cdot \mathbf{E} = 0$ region which is such that a surface whose boundary is the path intersects regions where $\nabla \cdot \mathbf{E} \neq 0$.

When conditions i) and ii) are satisfied, the equation of motion can be written as

$$(\not{\partial}' + m + \mu(\gamma_1 E_1 + \gamma_2 E_2)\gamma_4)\psi = 0, \quad (7)$$

where $\not{\partial}' = \gamma_4 \partial_4 + \gamma_1 \partial_1 + \gamma_2 \partial_2$. We can further written is as,

$$(\not{\partial}' + m)\psi' = 0, \quad (8)$$

with $\psi' = e^{-i \int^r (\mu\sigma_{12}\gamma_4(\mathbf{E} \times \hat{k}) \cdot d\mathbf{r})} \psi$.

If one now transports the particle around a closed path satisfying the condition iii), a phase will be developed in the wave function ψ . This phase is: $\phi_{AC} = \mu\sigma_{12}\gamma_4 \oint (\mathbf{E} \times \hat{k}) \cdot d\mathbf{r} =$

$-\mu\sigma_1 2\gamma_4 \lambda$ with λ being the linear charge density. Notice that the convention of the phase is defined as the phase of the wave function ψ respect to the wave function ψ' of a free particle and is opposite in sign as the convention used in Ref.[5].

To prove that the AC phase is a special case of Berry's phase, we need to correctly identify the eigenstate $n(R)$ of H and calculate Eq.(4). Before doing this we note that Eq.(4) is in one component form, while the AC phase has different components. One has either to modify Eq.(4) to incorporate different components or to formulate the AC phase into a different form such that Eq.(4) can be used. We choose to do the latter in the following. To this end, we rewrite Eq.(7) into two-component form for spin up and spin down. We have

$$[\partial_0 + m\sigma_3 + \frac{1}{i}\sigma' \cdot \nabla - s\mu\sigma' \cdot \mathbf{A}']\psi(s) = 0, \quad (9)$$

where $s = \pm 1$ for spin up and down respectively, $\sigma' = (\sigma_1, s\sigma_2)$, and $\mathbf{A}' = (E_2, -E_1)$. In the charge free region, the above equation can be written as

$$[\partial_0 + m\sigma_3 + \frac{1}{i}\sigma' \cdot \nabla]\psi(s)' = 0, \quad (10)$$

with $\psi' = e^{-is\mu \int^r \mathbf{A}' \cdot d\mathbf{r}'} \psi$.

When written in this form, even the equation of motion is a two-component form, the phases for the two components are the same and we can use Eq.(4) directly. We can identify in our case,

$$H = m\sigma_3 + \frac{1}{i}\sigma' \cdot \nabla - s\mu\sigma' \cdot \mathbf{A}'. \quad (11)$$

Once this has been done, the rest of the proof is similar to the proof that the Aharonov-Bohm effect is a special case of Berry's phase as shown in Berry's original paper[1]. Let the particle to be confined to a box centred at $\mathbf{R}$ and $\mathbf{r}$ be the position operator. In the absence of electric field, H depends on the momentum $\mathbf{p}$ and positions, that is, $H = H(\mathbf{p}, \mathbf{r} - \mathbf{R})$ and the wave functions have the form $\psi'(\mathbf{r} - \mathbf{R})$ with energy ϵ . We can now identify $\mathbf{R}$ with the R in Eq.(4). We normalize this wave function to have no additional phase except the dynamical one, that is, $\oint \langle \psi'(\mathbf{R}) | \nabla_{\mathbf{R}} \psi'(\mathbf{R}) \rangle \cdot d\mathbf{R} = 0$. When the electric field is turned on, the Hamiltonian becomes $H(\mathbf{p} - s\mu\mathbf{A}', \mathbf{r} - \mathbf{R})$. We then need to find the state

$n(\mathbf{R})$ which satisfies,

$$H(\mathbf{p} - s\mu\mathbf{A}', \mathbf{r} - \mathbf{R})|n(\mathbf{R})\rangle = \epsilon|n(\mathbf{R})\rangle. \quad (12)$$

From Eq.(10) we immediately find the state $n(\mathbf{R})$ is given by,

$$\langle \mathbf{r}|n(\mathbf{R})\rangle = \exp\left\{is\mu \int_{\mathbf{r}-\mathbf{R}}^{\mathbf{r}} \mathbf{A}' \cdot d\mathbf{r}'\right\} \psi'(\mathbf{r} - \mathbf{R}). \quad (13)$$

When the particle is transported along a closed path during a time interval T , we have $\mathbf{R}(0) = \mathbf{R}(T)$, so we have a Berry's phase given by,

$$\begin{aligned} \gamma(C) &= i \oint_C \langle n(\mathbf{R}) | \nabla_{\mathbf{R}} n(\mathbf{R}) \rangle \cdot d\mathbf{R} \\ &= i \oint_C d\mathbf{R} \int d\mathbf{r} \psi'^*(\mathbf{r} - \mathbf{R}) [-is\mu\mathbf{A}'(\mathbf{R}) + \nabla_{\mathbf{R}}] \psi'(\mathbf{r} - \mathbf{R}). \end{aligned} \quad (14)$$

We then, find,

$$\gamma(C) = i \oint_C d\mathbf{R} (-is\mu\mathbf{A}') = -s\mu \int_S \nabla \cdot \mathbf{E} dS = -s\mu\lambda. \quad (15)$$

So Berry's phase $\gamma(C)$ is exactly the AC phase ϕ_{AC} found before. We thus have proved that the AC phase is a special case of the Berry's phase.

There are situations where the AB and AC effects exist at the same time. This can happen if one lets a charged particle with a non-zero anomalous magnetic dipole moment μ_k , for example a proton, move in the $x - y$ plane in a vector potential $\mathbf{A}$ produced by an infinitely long solenoid carrying a total magnetic flux Φ and is placed along the z axis, and a electric field $\mathbf{E} = -\nabla V$ produced by an infinitely long filament carrying an uniform charge density parallel to the solenoid (not necessarily located at the same place as the solenoid). The topological phase developed is different depending on how the particle is transported along a closed path around the solenoid and the charged filament. These topological phases are also special cases of Berry's phase. The proof is along the same line as above. In this case,

$$\langle \mathbf{r}|n(\mathbf{R})\rangle = \exp\left\{i \int_{\mathbf{r}-\mathbf{R}}^{\mathbf{r}} (\mathbf{A} + s\mu_k\mathbf{A}') \cdot d\mathbf{r}'\right\} \psi'(\mathbf{r} - \mathbf{R}), \quad (16)$$

where ψ' is the wave function of a 'imaginary' proton with $\mu_k = 0$ moving in the potential V . One can always normalize $\gamma(C)$ of ψ' to be zero. Evaluating Eq.(4), we find Berry's

phase $\gamma(C)$ to be: i) $e\Phi$, if the path includes only the solenoid; ii) $-s\mu_k\lambda$, if the path includes only the charged filament; and iii) $e\Phi - s\mu_k\lambda$, if the path includes both the solenoid and the charged filament. These phases are the same as the findings in Ref.[5].

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