EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

CERN-TH-6076/91

An Integrable Hierarchy From Vector Models

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Abstract

Completely analogous to the matrix model case, $O(N)$ vector models of arbitrary potential can also be related to an integrable hierarchy of equations. We derive the finite- N lattice equations, and construct operators L_n and V_n fulfilling the zero curvature condition. In the continuum limit this hierarchy of lattice equations becomes identical to the Burgers hierarchy, which has been derived earlier for these models in the double scaling limit.

CERN-TH-6076/91

April 1991

1) Address after June 1, 1991.

Starting from matrix formulations of random surfaces, discretized versions of string theory [1], a remarkable mathematical richness has recently been uncovered. The discovery of a computable summation of all genera through an appropriately defined “double scaling limit” [2] thus led Douglas [3] to express this continuum limit in terms of certain representations of the Heisenberg algebra. This as well as other studies [2, 4], hinted at an interesting connection to the KdV hierarchy, and certain classes of integrable models in general. A more direct way of understanding this connection between the KdV hierarchy and matrix models comes from the study of Schwinger-Dyson equations [5].

But perhaps the most transparent way of seeing the relation between integrable models of the KdV type and random surfaces, is by considering the matrix models at finite N (the size of the matrix), *before the double scaling limit is taken* [6, 7, 8]. For example, for the 1-matrix model one finds that there is indeed an underlying discrete integrable system (in this case the one-dimensional Toda hierarchy) already at finite N . In the continuum limit this turns into the KdV hierarchy. This approach also explicitly identifies the partition function of the 1-matrix model with the so-called “ τ -function” of the Toda lattice. The same technique yields slightly different discrete integrable systems when applied to different matrix ensembles [8]. It can also be extended to the general multi-matrix case, where the result is a multi-component 2D Toda hierarchy [9].

In almost all cases the derivation of discrete integrable systems behind matrix models at finite N relies on seemingly magical properties of the orthogonal polynomials that are usually used in solving these models. Since the existence of an orthogonal polynomial technique is intimately connected to the matrix measure, which again relates directly to the way random surfaces are constructed (orientable, non-orientable etc.), one might be led to believe that the existence of these underlying discrete or continuous integrable systems have deep roots of origin in string theory itself. But other features of the resulting integrable hierarchies suggest that this may not necessarily be so. Let us, as an example, return again to the simplest (hermitean) 1-matrix case. Here the resulting discrete Toda equation is expressed in terms of e^{Φ_N} , given simply by the normalization of the orthogonal polynomials [8]:

$$\int d\mu P_m(x) P_n(x) \equiv e^{\Phi_N(x)} \delta_{m,n} \quad (1)$$

where $d\mu$ is the matrix Boltzmann measure (including some potential $V(x)$). Although this still makes direct reference to the matrix nature of the model, and to the orthogonal polynomials, the partition function (up to an overall, and for this purpose irrelevant, normalization factor) itself can be written as

$$Z_N = \prod_{i=1}^N e^{\Phi_{i-1}} \quad (2)$$

In fact it is from this equation that one immediately gets an identification between the partition function Z_N and a certain τ -function of the Toda lattice, since we then have $e^{\Phi_N} = Z_{N+1}/Z_N$. Having the fundamental Toda variable expressed directly in terms of a *ratio of partition functions*, we may here find a hint of a more general origin, not necessarily related to matrix models.

The purpose of the present letter is to show that similar discrete integrable structures arise also in the case of $O(N)$ symmetric vector models. Figuratively speaking, such vector models relate to the theory of *branched polymers* [10, 11, 12] in much the same way that matrix models relate to the theory of random surfaces. Essentially, because of having one

index less, vector models are particle-like while matrix models are string-like. A non-trivial double scaling limit is also known to exist in this case [10, 11, 12].

The zero dimensional partition function of a general $O(N)$ symmetric vector model can be written

$$\begin{aligned} Z_N(\lambda) &= \int \prod_{i=1}^N dn_i e^{-\sum_{j=1}^{\infty} \lambda_j (\vec{n}^2)^j - V(\vec{n}^2)} \\ &\equiv f_N \int dR R^{N-1} e^{-\sum_{j=1}^{\infty} \lambda_j R^{2j} - V(R^2)} \\ &\equiv f_N \int d\mu(R) R^{N-1} e^{-\sum_{j=1}^{\infty} \lambda_j R^{2j}} \end{aligned} \quad (3)$$

where we have simply changed to polar coordinates, and denoted the surface area of an N -sphere by f_N (its explicit form is irrelevant for what follows). The potential term $V(R)$ can be chosen at will, in particular such as to always render the resulting integral finite.

Note that

$$\frac{\partial Z_N(\lambda)}{\partial \lambda_i} = -f_N \int d\mu(R) R^{N-1} R^{2i} e^{-\sum_{j=1}^{\infty} \lambda_j R^{2j}} = \frac{f_N}{f_{N+2i}} Z_{N+2i}(\lambda) \quad (4)$$

and we are thus, in analogy with the matrix model case, naturally led to define a new variable $\phi_N(\lambda)$ by

$$e^{\phi_N(\lambda)} \equiv f_N^{-1} Z_N(\lambda) \quad (5)$$

In terms of this, eq. (4) takes the following form:

$$\frac{\partial \phi_N(\lambda)}{\partial \lambda_i} = -e^{\phi_{N+2i}(\lambda) - \phi_N(\lambda)} \quad (6)$$

From now on the explicit dependence on the coupling constants λ_i will be omitted, in order to simplify the notation. Consider first the case $i = 1$. By a change of variables $\varphi_N \equiv \phi_N - \phi_{N-2}$, we find

$$\frac{\partial \varphi_N}{\partial \lambda_1} = -e^{\varphi_{N+2}} + e^{\varphi_N} \quad (7)$$

or, introducing $u_N \equiv e^{\varphi_N}$, and relabelling $t \equiv -\lambda_1$,

$$\frac{\partial u_n}{\partial t} = (u_{n+2} - u_n)u_n \quad (8)$$

This resulting equation is quite similar to, but different from, the Volterra equation

$$\frac{\partial u_n}{\partial t} = (u_{n+1} - u_{n-1})u_n \quad (9)$$

obtained from the 1-matrix model with even potential. But note that in contrast to the Volterra equation, the system (8) couples only between next-nearest neighbours on the lattice. It thus completely decouples into two identical sectors corresponding to n even and n odd. For simplicity, consider now only the case n even, and let $v_n \equiv u_{2m}$, so that eq. (8) becomes

$$\frac{\partial v_n}{\partial t} = (v_{n+1} - v_n)v_n \quad (10)$$

The Volterra equation is an almost classic example of an integrable system. Also for our lattice equation (10) can we find the discrete analogue of a Lax pair¹⁾. That is, we

1) See *e.g.* the textbook [13].

can show the existence of a matrix L_n (describing parallel transport from site n to $n+1$) and a matrix V_n (describing parallel transport in time t) such that the zero curvature condition

$$\frac{\partial L_n(t, \lambda)}{\partial t} + L_n(t, \lambda)V_n(t, \lambda) - V_{n+1}(t, \lambda)L_n(t, \lambda) = 0 \quad (11)$$

is fulfilled if and only if eq. (10) is satisfied. Here λ is the spectral parameter (not to be confused with the coupling constants λ_i of eq. (3)).

After a little algebra, we find

$$L_n = V_n = \begin{pmatrix} u & 0 \\ \lambda & 1 \end{pmatrix} \quad (12)$$

The fact that the solution happens to give L_n equal to V_n is not unexpected. In this case the zero curvature representation

$$\begin{aligned} F_{n+1} &= L_n(t, \lambda)F_n \\ \frac{\partial F_n}{\partial t} &= V_n(t, \lambda)F_n \end{aligned} \quad (13)$$

reduces to $\partial F_n / \partial t = F_{n+1}$. This condition is automatically satisfied by the vector valued function (of identical entries) $\hat{F}_n \equiv F_{2m}$ with $F_n \equiv f_n^{-1}Z_n$, as follows from eq. (4).

Let us now see how a continuum limit of our lattice equation (8) can be taken. We use the known technique [14] of letting $u_n \rightarrow \exp[u(x)]$, and then expanding all exponentials. With a rescaling $\partial/\partial x \rightarrow \epsilon \partial/\partial x$, $\partial/\partial t \rightarrow \epsilon^{k_1} \partial/\partial t$ and $u(x) \rightarrow \epsilon^{k_2} u(x)$, where $k_1, k_2 > 0$ are to be determined in such a way that a meaningful continuum limit can be taken, we obtain

$$\epsilon^{k_1+k_2} u_t = 2\epsilon^{1+k_2} u_x + 2\epsilon^{2+k_2} u_{xx} + \frac{4}{3}\epsilon^{3+k_2} u_{xxx} + 2\epsilon^{1+2k_2} u_x u + 2\epsilon^{2+2k_2} (u_{xx} u + u_x^2) + \dots \quad (14)$$

Since the first term on the r.h.s always can be removed by a galilean transformation in x (see later), we find the condition $k_1 = 2, k_2 = 1$. This leads to

$$u_t = \frac{2}{\epsilon} u_x + 2[u_{xx} + u_x u] \quad (15)$$

which after a shift $x \rightarrow x + 2t/\epsilon$ and, subsequently, $t \rightarrow t/2$, gives

$$u_t = u_x u + u_{xx} \quad (16)$$

This is known in the literature as the Burgers equation, one of the simplest nonlinear integrable models with the Painlevé property. It can also be defined in a Hamiltonian formalism, with an infinite number of integrals of motion in involution [15].

The Burgers equation is known to admit a hereditary recursion operator [16] of the form (let $D \equiv d/dx$):

$$\mathcal{D} = D + \frac{1}{2}u + \frac{1}{2}u_x D^{-1} \quad (17)$$

This generates an infinite hierarchy of integrable equations:

$$u_{t_j} = \mathcal{D}^j B(u) \quad (18)$$

where $B(u) \equiv u_{xx} + uu_x = D(u_x + \frac{1}{2}u^2)$. The first flow ($j = 0$) is just the original Burgers equation (16). As an illustration we also list the next:

$$\begin{aligned} u_{t_1} &= \mathcal{D}B(u) \\ &= (D^2 + \frac{1}{2}uD + \frac{1}{2}u_x)(u_x + \frac{1}{2}u^2) \\ &= u_{xxx} + \frac{3}{2}uu_{xx} + \frac{3}{2}u_x^2 + \frac{3}{4}u^2u_x \end{aligned} \quad (19)$$

By noting that the Burgers equation (16) is the integrability condition for $v_x = uv$ and $v_t = (\frac{1}{2}u^2 + u_x)v$, we can immediately construct a Lax pair for this equation. Namely, let $L\Psi = \lambda\Psi$ and $\partial\Psi/\partial t = B\Psi$ with

$$\begin{aligned} L &\equiv -\frac{\partial}{\partial x} + u \\ B &\equiv \frac{1}{2}u^2 + u_x \end{aligned} \quad (20)$$

then we obviously have

$$\frac{\partial L}{\partial t} = [B, L]. \quad (21)$$

Starting directly from the double scaling limit of the $O(N)$ vector model, Nishigaki and Yoneya [11] have derived an infinite hierarchy of equations in the form of (with our notation and normalization conventions)

$$\frac{\partial u}{\partial t_{k-2}} = D(D + \frac{1}{2}u)^k \cdot 1 \quad (22)$$

where the whole operator acts on unity, as indicated. For $k = 1$ this gives the trivial flow $u_{t_{-1}} = \frac{1}{2}u_x$, but already for $k = 2$ it gives the non-trivial Burgers equation (16). This may seem surprising, since it suggests the existence of another recursion operator (from eq. (22)) associated with the Burgers hierarchy discussed above. In fact, it turns out that the two formulations (18) and (22) are completely equivalent. To see this, simply note that starting from $k = 1$, we no longer need to act on unity if we proceed recursively. The relevant recursion operator from eq. (22) is just

$$\begin{aligned} \tilde{\mathcal{D}} &= D(D + \frac{1}{2}u) = D^2 + \frac{1}{2}u_x + \frac{1}{2}uD \\ &= (D + \frac{1}{2}u + \frac{1}{2}u_x D^{-1})D \\ &= \mathcal{D}D \end{aligned} \quad (23)$$

which, as follows from eq. (22), always acts on $D^{-1}\mathcal{D}^{j-1}B(u)$, hence reproducing the recursion operator (18) of Olver. So the double scaling limit of vector models generates automatically the recursion operator of the Burgers hierarchy. What we have supplied here is the infinite *discrete* system underlying the vector models already at finite N . To establish the precise way in which our discrete hierarchy of equations turns into the Burgers hierarchy in the continuum limit, it is most convenient to return to the description directly in terms of partition functions. Instead of the sources $\sum_{k=1}^{\infty} \lambda_k R^{2k}$, reshuffle these into the form

$$Z_N = f_N \int d\mu(R) R^{N-1} \exp\left[\sum_{k=1}^{\infty} (-1)^k \tilde{t}_{k-2} \{(1 - R^2)^k - 1\}\right] \quad (24)$$

Defining the difference operator Δ by $\Delta F_n = F_{n+2}$, where $F_n = Z_n/f_n$, we clearly have the identity

$$\frac{\partial F_n}{\partial \tilde{t}_{k-2}} = [(1 - \Delta)^k - 1]F_n \quad (25)$$

Now shift $\partial/\partial \tilde{t}_j \rightarrow \partial/\partial t_j - 1$ (or, equivalently, let $t_j = 1 + \exp[-\tilde{t}_j]$), and consider the continuum limit $F_n \rightarrow F(x) = F(x, t_{-1}, t_0, t_1, \dots)$, $(\Delta - 1) \rightarrow \partial/\partial x$, where

$$\frac{\partial F(x)}{\partial t_{k-2}} = (-1)^k D^k F(x) \quad (26)$$

If at this stage we introduce a variable $u(x)$ by

$$F(x) \equiv \exp \frac{1}{2} \int^x u(x') dx' \equiv \exp G(x) \quad (27)$$

then we find

$$F(x) \frac{\partial G}{\partial t_{k-2}} = (-1)^k F(x) (D + \frac{1}{2}u)^k \cdot 1 \quad (28)$$

again acting on the unit, as indicated. Changing $t_j \rightarrow (-1)^j \frac{1}{2} t_j$, and using $DG(x) = \frac{1}{2}u(x)$, this gives

$$\frac{\partial u}{\partial t_{k-2}} = D(D + \frac{1}{2}u)^k \cdot 1 \quad (29)$$

which, as we have shown above, is just the Burgers hierarchy.

The change of variables which leads to the linearized formulation of the hierarchy (26) is simply the generalization of what for the Burgers equation is known as a Cole-Hopf transformation. It is perhaps not surprising, then, that a very similar linearization transformation exists also for our discrete lattice equations. Indeed, the starting point of eq. (4) can be viewed as the linearized formulation, and, running the arguments backward, the transformations between eq. (4) and eq. (8) as the discretized analogue of the Cole-Hopf transformation. Integrability of the hierarchy is then manifest.

It should be clear by now that the infinite hierarchy of integrable equations associated with matrix models are due to more general properties of zero dimensional partition functions. In a rather unexpected way we have, in the process of showing this, uncovered a discrete hierarchy of lattice equations whose continuum limit is the Burgers hierarchy, a system of equations with several applications in physics. These lattice equations have exact solutions described in terms of $O(N)$ vector model partition functions. In view of the established connection between matrix models and certain 2D topological field theories [17], it would be very interesting to see if a similar connection exists between these $O(N)$ vector models and, perhaps, some version of topological quantum mechanics.

Acknowledgement: The work of P.H.D. was partly supported by EEC Science Twinning grant no. SC1-000337.

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