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MANIFEST COVARIANCE CONDITION IN A RELATIVISTIC LAGRANGIAN STRING THEORY

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ABSTRACT

The manifest covariance condition of Currie and Hill for a relativistic Lagrangian N-particle theory is generalised for a Lagrangian relativistic string theory. The corresponding Currie-Hill equations are derived. We can analyse the emerging problem only in a particular case, and we obtain only a trivial solution for the Lagrangian. This seems to indicate that a non-constrained Lagrangian formalism is not very suitable for the description of a relativistic string.

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1. Introduction

The purpose of this paper is to analyse the physical implications of the so-called “manifest covariance” (or “world line”) condition for a theory of a relativistic string, in the framework of the Lagrangian formalism.

First, we illustrate the use of this condition in the Lagrangian theory of a relativistic particle. We follow essentially the presentation in [1].

1. For a point particle evolving in R^3 , we take as the configuration space $Q = R^3$. We identify as usual $T(Q) \cong R^3 \times R^3$. In the configuration space Q we have a natural action of the special Euclidian group $SE(3)$:

$$\varphi_{R,\mathbf{a}}(\mathbf{q}) \equiv R\mathbf{q} + \mathbf{a}. \quad (1.1)$$

This action lifts to an action of $SE(3)$ on $T(Q)$:

$$\dot{\varphi}_{R,\mathbf{a}}(\mathbf{q}, \mathbf{v}) = (R\mathbf{q} + \mathbf{a}, R\mathbf{v}). \quad (1.2)$$

2. We search now for an action Φ of the special orthochronous Poincaré group $P_+^\uparrow$ on $T(Q)$ which extends $\dot{\varphi} : \Phi|_{SE(3)} = \dot{\varphi}$. Let us denote by X_ξ the infinitesimal generator of the transformation corresponding to the element ξ in the Lie algebra $Lie(P_+^\uparrow)$ of $P_+^\uparrow$. The generators of time translation, space translations, rotations and boosts in $Lie(P_+^\uparrow)$ are denoted by H, P_i, J_i, K_i respectively.

3. The Euler-Lagrange equations are some algebraic identities imposed on X_H . Explicitly, if $L : T(Q) \rightarrow R$ is the Lagrangian of the system, one defines the 1-form θ_L by:

$$\theta_L \equiv \frac{\partial L}{\partial v_i} dq_i. \quad (1.3)$$

(with the convention summation over $i = 1, 2, 3$), the 2-form ω_L by:

$$\omega_L \equiv d\theta_L. \quad (1.4)$$

and then the “energy” function $E_L : T(Q) \rightarrow R$ by:

$$E_L \equiv i_{X_H} \theta_L - L. \quad (1.5)$$

Then, the Euler-Lagrange equations are:

$$i_{X_H} \omega_L = -dE_L. \quad (1.6)$$

4.A sufficient condition for the Poincaré transformations to be symmetries of our Lagrangian system is:

$$\Phi_g^* \omega_L = \omega_L. \quad (1.7)$$

for any $g \in P_+^\uparrow$.

5. Finally, one imposes the “manifest covariance” condition i.e. a natural way to transform the variables in $T(Q)$ with respect to boosts. A standard argument of Currie and Hill [2,3] shows that the infinitesimal generators of the boost transformations must be of the form:

$$X_{K_i} = q_i v_j \frac{\partial}{\partial q_j} + (v_i v_j - \delta_{ij} + q_i A_j) \frac{\partial}{\partial v_j}. \quad (1.8)$$

with $\mathbf{A}$ a function of $\mathbf{q}$ and $\mathbf{v}$ (the acceleration).

Then it is easy to prove the following facts. First using only the action property of Φ in the infinitesimal form:

$$[X_\xi, X_\eta] = X_{[\xi, \eta]}$$

for any $\xi, \eta \in \text{Lie}(P_+^\uparrow)$, we can show that in fact $\mathbf{A} = \mathbf{0}$ (see [1]). Next, using (1.6) and (1.7) one can show that if L is defined all over $T(Q)$, then L is a total derivative. If L is defined only on some open subset of $T(Q)$, then we can have non-trivial solutions. Namely, the subset

$$C_< \equiv \{(\mathbf{q}, \mathbf{v}) \mid |\mathbf{v}| < 1\}$$

admits only solutions of the usual form:

$$L(\mathbf{q}, \mathbf{v}) = -m\sqrt{1 - |\mathbf{v}|^2}. \quad (1.9)$$

and the set

$$C_> \equiv \{(\mathbf{q}, \mathbf{v}) \mid |\mathbf{v}| > 1\}$$

admits solutions of the form:

$$L(\mathbf{q}, \mathbf{v}) = m\sqrt{|\mathbf{v}|^2 - 1}. \quad (1.10)$$

(the tachionic case). In (1.9) and (1.10) $m \in R^*$.

We note that these results can be generalised to the case of the N-particle relativistic system and we get a “no-interaction” type theorem [4,5,1].

As we have said at the beginning of this section, we will investigate the same type of problem in the case of a relativistic string. For simplicity we study only the case of a closed string. We will use the configuration space identified in [6] (where the same problem was studied in the framework of the Hamiltonian formalism) and we extend naturally the above framework to a string system. In Section 2 we will study the accelerations and derive the corresponding Currie-Hill equations. A general analysis of these equations seems very difficult. We conjecture that, in analogy with the N-particle problem, one has only the trivial solution, i.e. the acceleration is zero, but we are able to prove it only in a particular case.

In Section 3 we try to determine the Lagrangian for the case when the acceleration is zero. We assume as in [1] a particular local form for the Lagrangean which is suggested by the existing literature [7]. By contrast with the Hamiltonian analysis which has produced some non-trivial solutions, in the Lagrangian framework we find only the trivial solution. This is done in Section 3. The situation seems to be analogous to the zero mass case in the particle mechanics. As we can see from the analysis of the one-particle case outlined above, there is no Lagrangian theory for a zero mass particle. On the contrary, it is well known that there exists a Hamiltonian theory for this system.

2. The determination of the acceleration

1. As in [6],[7] we take the configuration space for a single closed string to be:

$$Q \equiv C^\infty(S^1, R^3)/Diff(S^1). \quad (2.1)$$

where the action of $Diff(S^1)$ on $\mathbf{q} : S^1 \rightarrow R^3$ is:

$$(\phi, \mathbf{q}) \mapsto \mathbf{q} \circ \phi. \quad (2.2)$$

Next, one can identify:

$$T(Q) \cong (C^\infty(S^1, R^3) \times C^\infty(S^1, R^3))/Diff(S^1). \quad (2.3)$$

with the action of $Diff(S^1)$:

$$(\phi, (\mathbf{q}, \mathbf{v})) \mapsto (\mathbf{q} \circ \phi, \mathbf{v} \circ \phi). \quad (2.4)$$

In Q we have a natural action of the special Euclidian group $SE(3)$, namely the factorization of:

$$(\varphi_{R,\mathbf{a}}(\mathbf{q}(\cdot))) = R\mathbf{q}(\cdot) + \mathbf{a}. \quad (2.5)$$

which has as the lift to $T(Q)$ the factorization of:

$$\dot{\varphi}_{R,\mathbf{a}}(\mathbf{q}(\cdot), \mathbf{v}(\cdot)) = (R\mathbf{q}(\cdot) + \mathbf{a}, R\mathbf{v}(\cdot)). \quad (2.6)$$

We suppose now that we have an action Φ of the proper orthochronous Poincaré group $P_+^\uparrow$ which extends $\dot{\varphi}$ and we will use the notations introduced in Section 1.2.

Finally, we impose the “manifest covariance” condition. Extending straightforwardly the expression (1.7), we will impose that X_{K_i} is of the form:

$$X_K F[\mathbf{q}, \mathbf{v}] = \int d\sigma [q_i(\sigma) v_j(\sigma) \frac{\delta F \circ \pi}{\delta q_i(\sigma)} + (v_i(\sigma) v_j(\sigma) - \delta_{ij} + q_i(\sigma) A_j(\cdots) \frac{\delta F \circ \pi}{\delta v_j(\sigma)}]. \quad (2.7)$$

Here F is an arbitrary functional on $T(Q)$, $d\sigma$ is the Lebesgue measure on the circle S^1 , π is the canonical projection which associates to $(\mathbf{q}, \mathbf{v})$ its equivalence class and the “accelerations” A_j are supposed to depend only on a finite number of derivatives of $\mathbf{q}$ and $\mathbf{v}$ with respect to σ , say derivatives up to order N . The functions A_i must also satisfy some identities in order that the right-hand side of (2.7) be well defined, i.e. independent of the representative $(\mathbf{q}, \mathbf{v})$ in the equivalence class modulo $Diff(S^1)$, denoted $[\mathbf{q}, \mathbf{v}]$.

2. We impose now the relations

$$[X_{P_i}, X_{P_j}] = 0. \quad (2.8)$$

$$[X_{J_i}, X_{J_j}] = \epsilon_{ijk} X_{J_k}. \quad (2.9)$$

$$[X_{J_i}, X_{P_j}] = \epsilon_{ijk} X_{P_k}. \quad (2.10)$$

$$[X_{K_i}, X_{P_j}] = \delta_{ij} X_H. \quad (2.11)$$

$$[X_{P_i}, X_H] = 0. \quad (2.12)$$

$$[X_{K_i}, X_H] = X_{P_i}. \quad (2.13)$$

$$[X_{K_i}, X_{K_j}] = -\epsilon_{ijk} X_{J_k}. \quad (2.14)$$

$$[X_{J_i}, X_H] = 0. \quad (2.15)$$

$$[X_{J_i}, X_{K_j}] = \epsilon_{ijl} X_{K_l}. \quad (2.16)$$

which express the action property of Φ in an infinitesimal form. Let us remark that from (2.6) one can get the expressions of the spatial translations and of the rotations:

$$X_{P_i} F[\mathbf{q}, \mathbf{v}] = - \int d\sigma \frac{\delta F \circ \pi}{\delta q_i(\sigma)}. \quad (2.17)$$

$$X_{J_i} F[\mathbf{q}, \mathbf{v}] = -\epsilon_{ijk} \int d\sigma [q_j(\sigma) \frac{\delta F \circ \pi}{\delta q_k(\sigma)} + v_j(\sigma) \frac{\delta F \circ \pi}{\delta v_k(\sigma)}]. \quad (2.18)$$

Remarks: 1) It is easy to prove that the expressions appearing on the right-hand sides of (2.17) and (2.18) are well defined, i.e. they do not depend on the representative $(\mathbf{q}, \mathbf{v})$ in the equivalence class modulo $Diff(S^1)$.

2) From (2.17) and (2.18) it follows that (2.8)-(2.10) are identically fulfilled.

From now on, we follow closely the analysis in Ref [1].

-Inserting (2.7) and (2.17) into (2.11) we get that:

$$\frac{\partial A_k}{\partial q_j} = 0. \quad (2.19)$$

for any $i, j = 1, 2, 3$, i.e. A is independent of $\mathbf{q}$, and that X_H is of the form:

$$X_H[\mathbf{q}, \mathbf{v}] = \int d\sigma [v_i(\sigma) \frac{\delta F \circ \pi}{\delta q_i(\sigma)} + A_i(\mathbf{q}', \dots, \mathbf{v}, \dots) \frac{\delta F \circ \pi}{\delta v_i(\sigma)}]. \quad (2.20)$$

-The relation (2.19) ensures that (2.12) is also true.

-Next, we insert (2.7), (2.20) and (2.17) into (2.13). After some computations we arrive at the following relation:

$$\begin{aligned} & \sum_{n=1}^N [\sum_{p=1}^n \binom{n}{p} q_i^{(p)}(\sigma) v_j^{(n-p)}(\sigma)] \frac{\partial A_k}{\partial q_j^{(n)}}(\mathbf{q}'(\sigma), \dots, \mathbf{v}(\sigma), \dots) + [v_i(\sigma) v_j(\sigma) - \delta_{ij}] \frac{\partial A_k}{\partial v_j}(\dots) + \\ & \sum_{n=1}^N [\sum_{p=0}^n \binom{n}{p} v_i^{(p)}(\sigma) v_j^{(n-p)}(\sigma) + \sum_{p=1}^n \binom{n}{p} q_i^{(p)}(\sigma) \frac{d^{n-p}}{d\sigma^{n-p}} A_j(\dots)] \frac{\partial A_k}{\partial v_j^{(n)}}(\dots) = \\ & 2v_i(\sigma) A_k(\dots) + v_k(\sigma) A_i(\dots). \end{aligned} \quad (2.21)$$

for any $\mathbf{q}, \mathbf{v} \in C^\infty(S^1, R^3)$ and any $i, k = 1, 2, 3$.

-The relation (2.21) ensures that (2.14) is also fulfilled.

-Finally, we insert (2.18) and (2.20) into (2.15) and we get rather easily the following relation:

$$\epsilon_{ijl} \left(\sum_{n=1}^N q_j^{(n)} \frac{\partial A_k}{\partial q_l^{(n)}} + \sum_{n=0}^N v_j^{(n)} \frac{\partial A_k}{\partial v_l^{(n)}} \right) = \epsilon_{ijk} A_j. \quad (2.22)$$

for any $i, j = 1, 2, 3$. This relation expresses the rotational covariance of $\mathbf{A}$ in an infinitesimal form:

$$R_{ij} A_j(R^{-1} \mathbf{q}', \dots, R^{-1} \mathbf{v}, \dots) = A_i(\mathbf{q}', \dots, \mathbf{v}, \dots). \quad (2.23)$$

-The relation (2.22) ensures that (2.16) is also true.

So, the action property of Φ is equivalent to the relations (2.19), (2.21) and (2.22). These relations are the Currie-Hill equations for a relativistic Lagrangian string theory.

3. The complexity of the equation (2.21) seems to make a general analysis very difficult. Guided by the result obtained in [1] we can conjecture that the only solution for which X_{K_i} is well defined is $\mathbf{A} = \mathbf{0}$, but we are able to prove this below only in the simplest case $N = 1$.

-In this case $\mathbf{A}$ depends only on $\mathbf{q}'$, $\mathbf{v}$ and $\mathbf{v}'$. Outside the hypersurface $\mathbf{q}' \cdot (\mathbf{v} \times \mathbf{v}') = 0$, these three vectors are linear independent, so $\mathbf{A}$ can be written in the form:

$$\mathbf{A}(\mathbf{q}', \mathbf{v}, \mathbf{v}') = \mathbf{q}' F(\mathbf{q}', \mathbf{v}, \mathbf{v}') + \mathbf{v} G(\dots) + \mathbf{v}' H(\dots)$$

Using (2.23) we see immediately that the functions F, G, H must be rotational invariant. This implies that they are functions of the following six invariants:

$$x \equiv \mathbf{v}^2, y \equiv \mathbf{q}'^2, z \equiv \mathbf{v}'^2, r \equiv \mathbf{v} \cdot \mathbf{q}', s \equiv \mathbf{q}' \cdot \mathbf{v}', t \equiv \mathbf{v}' \cdot \mathbf{v}. \quad (2.24)$$

So, $\mathbf{A}$ is of the following form:

$$\mathbf{A} = \mathbf{q}' f(\mathbf{v}^2, (\mathbf{q}')^2, (\mathbf{v}')^2, \mathbf{v} \cdot \mathbf{q}', \mathbf{q}' \cdot \mathbf{v}', \mathbf{v}' \cdot \mathbf{v}) + \mathbf{v} g(\dots) + \mathbf{v}' h(\dots). \quad (2.25)$$

-The equation (2.21) becomes in this case:

$$q'_i v_j \frac{\partial A_k}{\partial q'_j} + (v_i v_j - \delta_{ij}) \frac{\partial A_k}{\partial v_j} + (v_i v'_j + v'_i v_j + q'_i A_j) \frac{\partial A_k}{\partial v'_j} = 2v_i A_k + v_k A_i. \quad (2.26)$$

Inserting (2.25) into (2.26) we arrive after some computations at a relation of the following form:

$$A q'_i q'_k + B v_i v_k + C v'_i v'_k + D_1 q'_i v_k + D_2 v_i q'_k + E_1 q'_i v'_k + E_2 v'_i q'_k + F_1 v_i v'_k + F_2 v'_i v_k = \delta_{ik} g. \quad (2.27)$$

It is easy to prove now that this implies:

$$A = B = C = D_1 = D_2 = E_1 = E_2 = F_1 = F_2 = 0. \quad (2.28)$$

and:

$$g = 0. \quad (2.29)$$

Taking into account (2.29), we can show that (2.28) implies the following equations for the functions f and h :

$$2r \frac{\partial f}{\partial y} + 2(sf + zh) \frac{\partial f}{\partial z} + (xf + sh + x) \frac{\partial f}{\partial r} + (t-1) \frac{\partial f}{\partial s} + (rf + th) \frac{\partial f}{\partial t} + h = 0. \quad (2.30)$$

$$2(x-1) \frac{\partial f}{\partial x} + 2z \frac{\partial f}{\partial r} + s \frac{\partial f}{\partial r} + r \frac{\partial f}{\partial s} + 2t \frac{\partial f}{\partial t} = 2f. \quad (2.31)$$

$$2t \frac{\partial f}{\partial z} + r \frac{\partial f}{\partial r} + (x-1) \frac{\partial f}{\partial t} = 0. \quad (2.32)$$

$$2r \frac{\partial h}{\partial y} + 2(sf + zh) \frac{\partial h}{\partial z} + (xf + sh + x) \frac{\partial h}{\partial r} + (t-1) \frac{\partial h}{\partial s} + (rf + th) \frac{\partial h}{\partial t} + h = 0. \quad (2.33)$$

$$2(x-1) \frac{\partial h}{\partial x} + 2z \frac{\partial h}{\partial r} + s \frac{\partial h}{\partial r} + r \frac{\partial h}{\partial s} + 2t \frac{\partial h}{\partial t} = h. \quad (2.34)$$

$$2t \frac{\partial h}{\partial z} + r \frac{\partial h}{\partial r} + (x-1) \frac{\partial h}{\partial t} = 0. \quad (2.35)$$

We note that the equations (2.31),(2.32) and (2.34),(2.35) have the following structure:

$$X_1 f = 2f, X_2 f = 0. \quad (2.36)$$

$$X_1 h = h, X_2 h = 0. \quad (2.37)$$

where:

$$X_1 \equiv 2(x-1) \frac{\partial}{\partial x} + 2z \frac{\partial}{\partial z} + s \frac{\partial}{\partial r} + r \frac{\partial}{\partial s} + 2t \frac{\partial}{\partial t}. \quad (2.38)$$

$$X_2 \equiv 2t \frac{\partial}{\partial z} + r \frac{\partial}{\partial r} + (x-1) \frac{\partial}{\partial t}. \quad (2.39)$$

From (2.36) and (2.37) we get immediately:

$$X_3 f = 0, X_3 h = 0. \quad (2.40)$$

where:

$$X_3 \equiv [X_1, X_2] = s \frac{\partial}{\partial r} - r \frac{\partial}{\partial s}. \quad (2.41)$$

Iterating the procedure, we have also:

$$X_4 f = 0, X_4 h = 0. \quad (2.42)$$

where:

$$X_4 \equiv [X_1, X_3] = r \frac{\partial}{\partial r} - s \frac{\partial}{\partial s}. \quad (2.43)$$

Now, from (2.40) and (2.43) we see quite easily that f and h do not depend on r and s . Then, differentiating (2.30) and (2.33) with respect to s , we get that f and h do not depend on z also. Inserting this information into (2.32) and (2.35), it follows that f and h do not depend on t . Now, differentiating (2.30) and (2.33) with respect to r , we get that f and h do not depend on y also. The preceding argument shows that f and h depend only on x . But in this case (2.30)(or (2.33)) gives:

$$h = 0. \quad (2.44)$$

Finally, (2.31) gives for f the following equation:

$$(x-1) \frac{df}{dx} = f \iff f(x) = C(x-1). \quad (2.45)$$

for some real constant C . So, the system (2.19),(2.21) and (2.22) has the following solution in the case $N = 1$:

$$A(\mathbf{q}', \mathbf{v}, \mathbf{v}') = C(\mathbf{v}^2 - 1)\mathbf{q}'. \quad (2.46)$$

-Finally, it is rather easy to prove that X_{K_i} , with $\mathbf{A}$ given by the preceding relation, is well defined iff $C = 0$, i.e.:

$$\mathbf{A} = \mathbf{0}. \quad (2.47)$$

3.The determination of the Lagrangian

Like in [1], we use (1.5) and (1.6) to find the Lagrangian. The condition (1.5) gives:

$$\int \frac{\delta^2 L}{\delta v_i(\sigma) \delta q_j(\sigma')} v_j(\sigma') d\sigma' = \frac{\delta L}{\delta q_i(\sigma)}. \quad (3.1)$$

and the condition (1.6) is better used in the infinitesimal form:

$$L_{X_\epsilon} \omega_L = 0. \quad (3.2)$$

In particular, we have:

$$L_{K_i} L = 0. \quad (3.3)$$

for any $i = 1, 2, 3$. This relation is equivalent to the following set of relations:

$$[q_i(\sigma) - q_i(\sigma')] \frac{\delta^2 L}{\delta v_j(\sigma) \delta v_k(\sigma')} = 0. \quad (3.4)$$

$$\begin{aligned} X_{K_i} \frac{\delta^2 L}{\delta v_j(\sigma) \delta v_k(\sigma')} + \delta_{ij} v_l(\sigma) \frac{\delta^2 L}{\delta v_l(\sigma) \delta v_k(\sigma')} + \delta_{ik} v_l(\sigma') \frac{\delta^2 L}{\delta v_l(\sigma') \delta v_j(\sigma)} - \\ q_i(\sigma) \left[\frac{\delta^2 L}{\delta v_k(\sigma') \delta q_j(\sigma)} - \frac{\delta^2 L}{\delta v_j(\sigma) \delta q_k(\sigma')} \right] + v_i(\sigma') \frac{\delta^2 L}{\delta v_k(\sigma') \delta v_j(\sigma)} = 0. \end{aligned} \quad (3.5)$$

$$\begin{aligned} X_{K_i} \left[\frac{\delta^2 L}{\delta v_j(\sigma) \delta q_k(\sigma')} - \frac{\delta^2 L}{\delta v_k(\sigma') \delta q_j(\sigma)} \right] - \delta_{ij} v_l(\sigma) \left[\frac{\delta^2 L}{\delta v_l(\sigma) \delta q_k(\sigma')} - \frac{\delta^2 L}{\delta v_k(\sigma') \delta q_l(\sigma)} \right] + \\ \delta_{ik} v_l(\sigma') \left[\frac{\delta^2 L}{\delta v_l(\sigma') \delta q_j(\sigma)} - \frac{\delta^2 L}{\delta v_j(\sigma) \delta q_l(\sigma')} \right] = 0. \end{aligned} \quad (3.6)$$

Like in [1], we can show that (1.5) and (1.6) are equivalent to the system (3.1)+(3.4)-(3.6). But in this case the analysis of this system seems to be a very difficult problem. So, like in [6], we search for solutions of a particular type suggested by the existing literature [8]:

$$L[\mathbf{q}, \mathbf{v}] = \int d\sigma L(\mathbf{q}(\sigma), \mathbf{q}'(\sigma), \mathbf{v}(\sigma)). \quad (3.7)$$

where the Lagrangian density L must satisfy some identity in order that the right-hand side be well defined, i.e. independent of the representative $(\mathbf{q}, \mathbf{v})$. Computing the functional derivatives and inserting into (3.1), we get after some elementary computations the following relations:

$$\frac{\partial^2 L}{\partial v_i \partial q_j} v_j = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q'_i \partial q_j} q'_j. \quad (3.8)$$

$$\frac{\partial^2 L}{\partial v_i \partial q'_j} = - \frac{\partial^2 L}{\partial q'_i \partial v_j}. \quad (3.9)$$

$$\frac{\partial^2 L}{\partial q'_i \partial q'_j} = 0. \quad (3.10)$$

for any $i = 1, 2, 3$. From (3.10) we get that L must be of the form:

$$L(\mathbf{q}, \mathbf{q}', \mathbf{v}) = q'_k L_i(\mathbf{q}, \mathbf{v}) + L_0(\mathbf{q}, \mathbf{v})$$

It is easy to show now that the condition that the right-hand side of (3.7) be well defined implies that the last term above gives a null contribution; so we can take:

$$L(\mathbf{q}, \mathbf{q}', \mathbf{v}) = q'_k L_k(\mathbf{q}, \mathbf{v}). \quad (3.11)$$

Inserting this expression into (3.8) and (3.9) we get easily:

$$\frac{\partial^2 L_k}{\partial v_i \partial q_j} v_j = \frac{\partial L_k}{\partial q_i} - \frac{\partial L_i}{\partial q_k}. \quad (3.12)$$

$$\frac{\partial L_j}{\partial v_i} = -\frac{\partial L_i}{\partial v_j}. \quad (3.13)$$

for any $i = 1, 2, 3$. From the last relation it is easy to get:

$$\frac{\partial^2 L}{\partial v_i \partial v_j} = 0. \quad (3.14)$$

i.e. L_i are of the form:

$$L_i(\mathbf{q}, \mathbf{v}) = L_{kl}(\mathbf{q}) v_l + l_k(\mathbf{q}). \quad (3.15)$$

Inserting this (3.12) and (3.13) produces:

$$\frac{\partial L_{ki}}{\partial q_j} + \frac{\partial L_{ij}}{\partial q_k} - \frac{\partial L_{kj}}{\partial q_i} = 0. \quad (3.16)$$

$$\frac{\partial l_k}{\partial q_i} = \frac{\partial l_i}{\partial q_k}. \quad (3.17)$$

$$L_{ij} = -L_{ji}. \quad (3.18)$$

From (3.17) it follows that l_i are of the form

$$l_i = \frac{\partial l}{\partial q_i}$$

for some function l of $\mathbf{q}$. So, the contribution to L of the last term in (3.15) is null, i.e. we can put $l_i = 0$. From (3.16) and (3.18) we get that L_{ij} is of the form:

$$L_{ij} = \frac{\partial \Lambda_i}{\partial q_j} - \frac{\partial \Lambda_j}{\partial q_i}$$

for some smooth function $\Lambda : R^3 \rightarrow R^3$. So, finally the relation (3.1) implies that L can be chosen of the form:

$$L(\mathbf{q}, \mathbf{q}', \mathbf{v}) = q'_i v_j \left(\frac{\partial \Lambda_i}{\partial q_j} - \frac{\partial \Lambda_j}{\partial q_i} \right). \quad (3.19)$$

Now, it is easy to prove from here that:

$$\frac{\delta^2 L}{\delta v_i(\sigma)\delta v_j(\sigma t)} = 0. \quad (3.20)$$

and:

$$\frac{\delta^2 L}{\delta v_i(\sigma)\delta q_j(\sigma t)} = \frac{\delta^2 L}{\delta v_j(\sigma t)\delta q_i(\sigma)}. \quad (3.21)$$

The relations (3.20) and (3.21) show us that the relations (3.3)-(3.6) are identically fulfilled. So, (3.19) is in fact the most general solution of the type (3.7). Unfortunately, (3.20) and (3.21) give us also:

$$\omega_L = 0. \quad (3.22)$$

so, the Lagrangian given by (3.7)+(3.19) gives trivial equations of motion.

4. Conclusions

The fact that we get only a trivial solution could be explained in two ways. One way is that maybe the limitations imposed on the form of the acceleration $\mathbf{A}$ and on the Lagrangian L have been too strong and have eliminated the non-trivial solutions. The other way, as we have said in the introduction, is that we have a phenomenon appearing also in the Lagrangian theory for systems with finite number of degrees of freedom, namely the impossibility of describing a zero-mass elementary system covariant with respect to the Poincaré group, in a non-constrained formalism.

On the other hand, one knows that one can describe zero mass systems for the Poincaré group, using the formalism of evolution spaces of Souriau [9]. So, maybe the best way to study a string in analytical mechanics is to use this formalism. One can hope that in this framework, one can get non-trivial solutions which agree with the Hamiltonian analysis performed in [6].

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