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On holomorphic factorization for free conformal fields

by

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Abstract

For free spin j conformal fields on a compact Riemann surface without boundary, we prove, locally over the space of complex structures -parametrized by Beltrami differentials- the existence of a class of partition functions which are holomorphically factorized, and whose diffeomorphism anomalies split holomorphically. For a collection of such models with vanishing total central charge, one recovers the Belavin-Knizhnik theorem.

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1 Introduction

The Belavin-Knizhnik [1]-[6] holomorphic factorization theorem for a system of free conformal fields on a compact Riemann surface without boundary is one of the main theorems in the perturbative approach to string theory. Scrutinizing the proofs of this theorem leads one to conjecture that, on the space of complex structures conveniently parametrized by Beltrami differentials, holomorphic factorization can be restored locally, even at non-vanishing central charge. This is achieved through a change of renormalization prescription implemented by a local counterterm which shifts the conventional Weyl anomaly to a chirally split form of the diffeomorphism anomaly [7]-[9].

2 Notations

We consider a b - c system of spin j over a compact Riemann surface Σ without boundary endowed with an analytic atlas with local coordinates $(z, \bar{z})$ ¹. Let $(Z, \bar{Z})$ be the local coordinates of another holomorphic atlas corresponding to the Beltrami differential μ

$$dZ = \lambda(dz + \mu d\bar{z}) , \quad (1)$$

where

$$\lambda = \partial_z Z \stackrel{\text{Def}}{=} \partial Z \quad (2)$$

is an integrating factor which fulfills²

$$(\bar{\partial} - \mu\partial) \ln \lambda = \partial\mu . \quad (3)$$

The action for the b - c system reads

$$S_j = \int_{\Sigma} \frac{dZ \wedge d\bar{Z}}{2i} b \partial_z c \quad (4)$$

where c (resp. b) is the coefficient of a j (resp. $1-j$) differential $c(dZ)^j$ (resp. $b(dZ)^{1-j}$).

The dependence of the action on the complex structure parametrized by μ is obtained by defining

$$c = \frac{\gamma}{\lambda^j} , \quad b = \frac{\beta}{\lambda^{1-j}} , \quad (5)$$

¹We shall omit the index denoting the open set where these coordinates are locally defined since all formulae will glue through holomorphic changes of coordinates.

²We shall reserve thin ∂ 's for $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$ and fat ∂ 's for the differentials $\partial \equiv dz\partial_z$, $\bar{\partial} \equiv d\bar{z}\partial_{\bar{z}}$.

which leads to

$$S_j = \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \beta (\bar{\partial} - \mu \partial - j(\partial \mu)) \gamma \quad (6)$$

after use of

$$\partial_2 = \frac{\bar{\partial} - \mu \partial}{\bar{\lambda}(1 - \mu \bar{\mu})} \quad (7)$$

and Eq.(3).

It is customary to quantize this system, and, in particular to define a partition function or vacuum functional, by introducing a metric and the corresponding Laplacian. This method has the advantage that it allows to sum up a suitably renormalized perturbative series in a compact way through e.g. the ζ -renormalized determinant formula to be described below [10]. Positivity allows to write down such a formula but holomorphy in μ is lost !

We thus introduce a metric

$$\rho_{ZZ} = \frac{\rho_{Z\bar{Z}}}{\lambda \bar{\lambda}} \equiv \frac{\rho}{\lambda \bar{\lambda}} \quad (8)$$

and the norms for $(j, \bar{j})$ differentials $a(dZ)^j(d\bar{Z})^{\bar{j}}$:

$$\begin{aligned} \|a\|^2 &\stackrel{\text{Def}}{=} \|\alpha\|^2 = \int_{\Sigma} \frac{dZ \wedge d\bar{Z}}{2i} |a|^2 \rho_{ZZ}^{1-j-\bar{j}} \\ &= \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} (1 - \mu \bar{\mu}) |\alpha|^2 \rho^{1-j-\bar{j}} \end{aligned} \quad (9)$$

where

$$a = \frac{\alpha}{\lambda^j \bar{\lambda}^{\bar{j}}} . \quad (10)$$

In particular

$$\begin{aligned} \|\bar{\partial} c\|^2 &= \int_{\Sigma} \frac{dZ \wedge d\bar{Z}}{2i} |\partial_2 c|^2 \rho_{ZZ}^{-j} \\ &= \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \frac{1}{(1 - \mu \bar{\mu})} |(\bar{\partial} - \mu \partial - j(\partial \mu)) \gamma|^2 \rho^{-j} . \end{aligned} \quad (11)$$

We define the Laplacian Δ_j through

$$\|\bar{\partial} c\|^2 = \langle \gamma | -\Delta_j \gamma \rangle . \quad (12)$$

Hence

$$\Delta_j = \frac{1}{\rho^{1-j}(1-\mu\bar{\mu})} \overline{(\bar{\partial} - \mu\partial - (1-j)(\partial\mu))} \frac{1}{\rho^j(1-\mu\bar{\mu})} (\bar{\partial} - \mu\partial - j(\partial\mu)) . \quad (13)$$

The positivity of $-\Delta_j$ stems from the factorization

$$\Delta_j = -\mathcal{T}_j^\dagger \mathcal{T}_j \quad (14)$$

with

$$\begin{aligned} \mathcal{T}_j &= \frac{1}{1-\mu\bar{\mu}} (\bar{\partial} - \mu\partial - j(\partial\mu)) \\ \mathcal{T}_j^\dagger &= \frac{-1}{\rho^{1-j}(1-\mu\bar{\mu})} \overline{(\bar{\partial} - \mu\partial - (1-j)(\partial\mu))} \frac{1}{\rho^j} = -\rho^{j-1} \bar{\mathcal{T}}_{1-j} \rho^{-j} \end{aligned} \quad (15)$$

where the adjoint is with respect to the norm of $(j, 1)$ -differentials of Eq.(11).

In view of later use, we shall choose a basis $\{\gamma_m\}$ (resp. $\{\beta_p\}$), $m = 1, \dots, N_j$ (resp. $p = 1, \dots, N_{1-j}$) in the kernel of $\mathcal{T}_j$ (resp. $\mathcal{T}_{1-j}$) holomorphic in μ ³ and independent of ρ , with, according to the Riemann-Roch theorem

$$N_j - N_{1-j} = (g-1)(2j-1) , \quad (16)$$

if Σ is of genus g .

We then define

$$\Gamma(\rho, \mu, \bar{\mu}) = -\frac{1}{2} \ln \left(\frac{\det'_\zeta(-\Delta_j)}{\det \|\langle \gamma_m | \gamma_n \rangle_{1 \leq m, n \leq N_j} \| \det \|\langle \beta_p | \beta_q \rangle_{1 \leq p, q \leq N_{1-j}} \|} \right) , \quad (17)$$

where all metrics are associated with

$$ds^2 = \rho |dz + \mu d\bar{z}|^2 , \quad (18)$$

and [10]

$$\ln \det'_\zeta(-\Delta_j) = -\frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} dt \left(\text{Tr } e^{t\Delta_j} - N_j \right) . \quad (19)$$

Since the γ_m 's and β_p 's are assumed to be holomorphic in μ and independent of ρ , $\Gamma(\rho, \mu, \bar{\mu})$ is defined up to addition of a functional of μ alone and its complex conjugate.

³This can be done locally in μ .

3 The construction

We now consider

$$\Gamma = \Gamma(\rho, \mu, \bar{\mu}) - \Gamma(\rho, 0, 0) . \quad (20)$$

Note that in this expression the zero mode subtraction term N_j in Eq.(19) drops out. One can convince oneself that up to the zero mode terms this expression sums up a renormalized perturbative series in $\mu, \bar{\mu}$ based on the action Eq.(11). Before making such an assertion one has of course to define a Feynman propagator. This can be done by introducing a mass term [11]

$$\det'_\zeta(-\Delta_j) \longrightarrow \det_\zeta(-\Delta_j + M^2) . \quad (21)$$

One may then compute the Weyl anomaly and of course one finds a non local contribution produced by the mass term. In the limit $M \rightarrow 0$ the latter degenerates into the contribution to the Weyl anomaly of the zero mode terms included in the definition of Γ whose Weyl anomaly thus becomes local.

Next, one may add to Γ the local counterterm $\Delta\Gamma$ [9] which shifts the Weyl anomaly into the chirally split form of the diffeomorphism anomaly

$$\begin{aligned} \Delta\Gamma(\rho, \mu, \bar{\mu}) = & \frac{C_j}{12\pi} \int_\Sigma \frac{d\bar{z} \wedge dz}{2i} \left(\mu(\mathcal{R} - R) + c.c. - \frac{1}{1 - \mu\bar{\mu}} \left[(\partial + \Gamma)\mu(\bar{\partial} + \bar{\Gamma})\bar{\mu} \right. \right. \\ & \left. \left. - \frac{1}{2}\bar{\mu}((\partial + \Gamma)\mu)^2 - \frac{1}{2}\mu((\bar{\partial} + \bar{\Gamma})\bar{\mu})^2 \right] \right) , \end{aligned} \quad (22)$$

where $\mathcal{R}$ is the coefficient of a projective connection,

$$\begin{aligned} \Gamma &= \partial \ln \rho , \quad \bar{\Gamma} = \bar{\partial} \ln \rho \\ R &= \partial \Gamma - \frac{1}{2}\Gamma^2 \end{aligned} \quad (23)$$

and C_j is the central charge of the problem

$$C_j = 6j(j-1) + 1 . \quad (24)$$

One can check that $\Gamma + \Delta\Gamma$ does not depend on ρ any more, i.e. the Weyl anomaly has indeed been eliminated.

The expectation is then that $\Gamma + \Delta\Gamma$ should split into a part holomorphic in μ and one anti-holomorphic, i.e. ⁴

$$\delta_{\bar{\mu}}\delta_{\mu} [\Gamma + \Delta\Gamma] = 0. \quad (25)$$

This can be proved by direct perturbative computation. It is relatively simple to show - introducing and removing a mass regularization, as explained before- that $\delta_{\bar{\mu}}\delta_{\mu}\Gamma$ is local in $\mu, \bar{\mu}, \delta\mu, \delta\bar{\mu}$. The computation of this local contribution, by heat kernel expansion techniques is however lengthy and the fact that it is of the form

$$\begin{aligned} \delta_{\bar{\mu}}\delta_{\mu}\Gamma &= -\frac{C_j}{12\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \left[\partial + \Gamma - \frac{(\bar{\partial} + \bar{\Gamma})\bar{\mu}}{1 - \mu\bar{\mu}} + \bar{\mu} \frac{(\partial + \Gamma)\mu}{1 - \mu\bar{\mu}} \right] \delta\mu \\ &\quad \times \frac{1}{1 - \mu\bar{\mu}} \left[\bar{\partial} + \bar{\Gamma} - \frac{(\partial + \Gamma)\mu}{1 - \mu\bar{\mu}} + \mu \frac{(\bar{\partial} + \bar{\Gamma})\bar{\mu}}{1 - \mu\bar{\mu}} \right] \delta\bar{\mu} \quad (26) \\ &= -\delta_{\bar{\mu}}\delta_{\mu}\Delta\Gamma, \end{aligned}$$

comes as a surprise [11].

4 An application of the local index theorem for families

This result can also be reached using the Atiyah-Singer theorem for families of elliptic operators in the local form given by J.-M. Bismut & coworkers [5,12,13,14].

Indeed $\delta_{\bar{\mu}}\delta_{\mu}\Gamma$ represents the first Chern class of the determinant line bundle over $\mathcal{M}$ (the space of μ 's) associated with the μ -dependent $\bar{\partial}$ operator on j -differentials. This representative is computed using Quillen's metric [5,15].

The index theorem for families indicates that $\delta_{\bar{\mu}}\delta_{\mu}\Gamma$ is equal to the integral over Σ of the square of a certain curvature, which becomes a $(1,1)$ -form on $\mathcal{M}$ [4,5,12,13].

The geometrical situation is as follows : One considers the space $\Sigma \times \mathcal{M}$ ⁵ with local coordinates $(z, \bar{z}, \mu, \bar{\mu})$ and the total differential

$$\begin{aligned} D &= d + \Delta \\ d &= dz\partial_z + d\bar{z}\partial_{\bar{z}} = \partial + \bar{\partial} \end{aligned} \quad (27)$$

$$\Delta = \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \left(\delta\mu \frac{\delta}{\delta\mu} + \delta\bar{\mu} \frac{\delta}{\delta\bar{\mu}} \right) (z, \bar{z})$$

⁴Here δ_{μ} and $\delta_{\bar{\mu}}$ are the differentials with respect to μ and $\bar{\mu}$. Eq.(25) implies the holomorphic splitting property locally in μ .

⁵We shall treat the infinite dimensional manifold $\mathcal{M}$ formally ; equivalently all statements which follow hold on finite dimensional analytic subspaces of $\mathcal{M}$.

One considers the complex analytic structure defined by the local coordinates $(Z, \bar{Z}, \mu, \bar{\mu})$ where the Z coordinates are obtained by solving the Beltrami equation (cf Eq.(1))

$$(\bar{\partial} - \mu\partial)Z = 0, \quad (28)$$

and picking out solutions holomorphic in μ . For the sake of definiteness, let us consider an analytic submanifold $\mathcal{M}_t \subset \mathcal{M}$ parametrized by a finite dimensional holomorphic family μ_t , where t denotes a set of local coordinates of a finite dimensional analytic manifold. Z is then chosen to be holomorphic in t .

To this complex analytic structure there corresponds the splitting of the total differential

$$D = \mathcal{D} + \bar{\mathcal{D}} \quad (29)$$

with

$$\mathcal{D} = dZ\partial_Z + dt\partial_{t|Z} \quad (30)$$

where $\partial_{t|Z}$ denotes partial derivatives with respect to the t coordinates at constant $Z, \bar{Z}$.

On the vertical holomorphic tangent bundle to $\Sigma \times \mathcal{M}_t$, one considers the metric

$$\rho_{ZZ} = \frac{\rho_{z\bar{z}}}{\lambda\bar{\lambda}},$$

see Eq.(8).

The corresponding curvature form (on $\Sigma \times \mathcal{M}_t$) is given by ⁶

$$\begin{aligned} \mathcal{K} &= \bar{\mathcal{D}}\mathcal{D}\ln\rho_{ZZ} \\ &= (D\bar{Z}\partial_Z + D\bar{t}\partial_{\bar{t}|Z})(DZ\partial_Z + Dt\partial_{t|Z})\ln\rho_{ZZ} \\ &= (D\bar{Z}DZ\partial_Z\partial_Z + D\bar{t}DZ\partial_{\bar{t}|Z}\partial_Z + D\bar{Z}Dt\partial_Z\partial_{\bar{t}|Z} + D\bar{t}Dt\partial_{\bar{t}|Z}\partial_{t|Z})\ln\rho_{ZZ}. \end{aligned} \quad (31)$$

The “restriction” of $\mathcal{K}$ to a fiber Σ_t is given by

$$\begin{aligned} \mathcal{K}_{|\Sigma_t} &= [(d\bar{Z} + d\bar{t}\partial_{\bar{t}}\bar{Z})(dZ + dt\partial_t Z)\partial_Z\partial_Z + dt(dZ + dt\partial_t Z)\partial_{\bar{t}|Z}\partial_Z \\ &\quad + (d\bar{Z} + d\bar{t}\partial_{\bar{t}}\bar{Z})dt\partial_{t|Z}\partial_Z + d\bar{t}dt\partial_{\bar{t}|Z}\partial_{t|Z}] \ln\rho_{ZZ} \end{aligned} \quad (32)$$

⁶In the following, the product of differentials is to be understood as the usual wedge product.

where now we have returned to the $(z, \bar{z}, t, \bar{t})$ coordinates through the expression of dZ , cf Eq.(1). Using now

$$\partial_t = \partial_{t|Z} + (\partial_t Z) \partial_Z \quad (33)$$

where

$$\partial_t = \partial_{t|z} , \quad (34)$$

we get

$$\mathcal{K}_{|\Sigma_t} = \left[d\bar{Z} dZ \partial_Z \partial_Z + d\bar{Z} dt \partial_t \partial_Z + d\bar{t} dZ \partial_t \partial_Z \right] \ln \rho_{ZZ} \quad (35)$$

and

$$-\frac{1}{2} \left(\mathcal{K}_{|\Sigma_t} \right)_{2,2}^2 = d\bar{Z} dZ d\bar{t} dt |\partial_t \partial_Z \ln \rho_{ZZ}|^2 \quad (36)$$

where (2, 2) refers to the total degrees on Σ_t and on $\mathcal{M}_t$. Using repeatedly Eqs.(3,7,8,28) one finds

$$\partial_t \partial_Z \ln \rho_{ZZ} = - \frac{1}{\bar{\lambda}(1 - \mu\bar{\mu})} \left[\partial + \Gamma - \frac{(\bar{\partial} + \bar{\Gamma})\bar{\mu}}{1 - \mu\bar{\mu}} + \bar{\mu} \frac{(\partial + \Gamma)\mu}{1 - \mu\bar{\mu}} \right] \partial_t \mu \quad (37)$$

hence

$$\begin{aligned} -\frac{1}{2} \left(\mathcal{K}_{|\Sigma_t} \right)_{2,2}^2 &= d\bar{z} \wedge dz \left[\bar{\partial} + \bar{\Gamma} - \frac{(\partial + \Gamma)\mu}{1 - \mu\bar{\mu}} + \mu \frac{(\bar{\partial} + \bar{\Gamma})\bar{\mu}}{1 - \mu\bar{\mu}} \right] d_t \bar{\mu} \\ &\quad \times \frac{1}{1 - \mu\bar{\mu}} \left[\partial + \Gamma - \frac{(\bar{\partial} + \bar{\Gamma})\bar{\mu}}{1 - \mu\bar{\mu}} + \bar{\mu} \frac{(\partial + \Gamma)\mu}{1 - \mu\bar{\mu}} \right] d_t \mu . \end{aligned} \quad (38)$$

Now, the local form of the index theorem for families reads

$$d_{\bar{t}} d_t \Gamma = -\frac{C_i}{48\pi i} \int_{\Sigma_t} \left(\mathcal{K}_{|\Sigma_t} \right)_{2,2}^2 . \quad (39)$$

It is remarkable that

$$\frac{C_i}{48\pi i} \int_{\Sigma_t} \left(\mathcal{K}_{|\Sigma_t} \right)_{2,2}^2 = d_{\bar{t}} d_t \Delta \Gamma , \quad (40)$$

which is the conjectured result.

The numerical coefficient C_j comes from a classical analysis of the spin j family [4,5]. We have not found so far an a priori geometrical reason why

$$\int_{\Sigma_t} \left(\mathcal{K}_{|\Sigma_t} \right)_{2,2}^2 = d_t d_t (\text{local expression})$$

which looks like a generalization of a Bott-Chern formula [5,12].

5 Conclusion

The derivation of the holomorphic factorization property exhibited here provides a bridge between field theory à la Feynman, based on locality, and one of the main properties of conformal field theories in their holomorphic versions [16].

The result of the foregoing analysis is that the holomorphic factorization property, at least in the free field case, requires the elimination of the usual Weyl anomaly to the benefit of a chirally split diffeomorphism anomaly.

Although this conclusion was reached only for free fields, the topological smell -universality- of the result helps one conjecture that it should survive *perturbative* renormalization for non free models [17].

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