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**INFLATION-DRIVEN STRING INSTABILITIES:
TOWARDS A SYSTEMATIC LARGE- R EXPANSION**

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Abstract

We present the general structure of a large- R expansion for the evolution of classical strings during a phase of fast inflation together with explicit results for several subleading terms. We are thus able to check the consistency of the expansion and to describe the approach to asymptopia both in terms of string shapes and of their equation of state.

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1 OUTLINE

Recently an analytic solution has been found [1] for the late-time (large scale factor R) asymptotic behaviour of highly unstable fundamental strings [2]-[4] during a phase of fast inflation. This paper is devoted to a systematic study of the subleading corrections.

Interest in the structure of a systematic large- R expansion stems from different reasons, among these:

- a) it is important to check that the mutual consistency of equations of motion and constraints keeps working, order by order, in an expansion around the unstable regime;
- b) the magnitude of the non-leading terms reveals the typical scale of time needed in order to reach asymptopia;
- c) the form of the non-leading terms should reveal in which direction asymptopia is approached both in terms of the string's shape and in terms of the equation of state satisfied by an ideal gas of unstable strings. The latter is relevant for checking the viability of models of self-sustained inflation [5].

In Section 2 we shall present our results on the general structure of subleading terms and for the explicit expression of the first few of them. Those expressions only make sense at non-exceptional values of an exponent α , characterizing the inflationary metric.

In Section 3 we discuss the two exceptional cases: $\alpha = 1/2$ and $\alpha = 3/2$.

In Section 4 we describe the approach to asymptopia both for the string shape and for its equation of state.

Section 5 contains our conclusions.

2 SUBLEADING TERMS IN THE GENERIC CASE

We recall here the problem at hand [1]. Let us consider a closed string evolving, at the classical level, in a cosmological metric of the form

$$ds^2 = G_{AB}dX^A dX^B = dt^2 - R^2(t) d\vec{X}^2.$$

Its equations of motion read, in the orthonormal gauge, ($i = 1, 2, \dots, D-1$; $X^0 \equiv t$)

$$\begin{aligned}\ddot{X}^0 - X''^0 &= R \frac{dR}{dt} \sum_i \left[(X'^i)^2 - (\dot{X}^i)^2 \right] \\ \ddot{X}^i - X''^i &= \frac{2}{R} \frac{dR}{dt} (X'^0 X'^i - \dot{X}^0 \dot{X}^i),\end{aligned}\tag{2.1}$$

subject to the constraints

$$\begin{aligned}(\dot{X}^0)^2 + (X'^0)^2 &= R^2 \sum_i \left[(X'^i)^2 + (\dot{X}^i)^2 \right] \\ X'^0 \dot{X}^0 &= R^2 \sum_i \dot{X}^i X'^i.\end{aligned}\tag{2.2}$$

For $\dot{R} = 0$ this is the usual free string while, for a sufficiently large $\dot{R}$ and (positive) $\ddot{R}$ a new kind of asymptotic ($R \rightarrow \infty$) solution was constructed in [1].

In order to describe our systematic expansion of the exact solution in the simplest possible terms, it is convenient to rewrite eqs. (2.1) and (2.2) in terms of conformal (rather than cosmic) time η . We recall that, by definition,

$$R d\eta \equiv dt\tag{2.3}$$

so that, e.g., $\dot{\eta} = R^{-1}\dot{t}$ etc.

Denoting by S, T, C₁, and C₂ the equations of motion for the space and time components and the two constraints (we are dealing here with closed strings) we find:

$$\begin{aligned} S &\equiv \eta (\ddot{X} - X'') + 2\eta R^{-1} dR/d\eta (\dot{\eta}\dot{X} - \eta'X') = 0 \\ T &\equiv \dot{\eta}^2 - \eta'^2 + \dot{X}^2 - X'^2 - R (dR/d\eta)^{-1} (\eta'' - \ddot{\eta}) = 0 \\ C_1 &\equiv \dot{\eta}^2 + \eta'^2 - (X'^2 + \dot{X}^2) = 0 \\ C_2 &\equiv \dot{\eta}\eta' - \dot{X}X' = 0 \end{aligned} \quad (2.4)$$

Here X stands for the vector $\vec{X}$ and bilinears in X denote, of course, scalar products with a flat metric. Dots and primes denote, respectively, differentiation with respect to world sheet time τ and space σ (which has, by definition range $0 - 2\pi$). We shall restrict our attention to “power-like” scale factors in η :

$$R(\eta) = \text{const. } (-\eta)^{-\alpha} \quad (\alpha > 0). \quad (2.5)$$

This comprises power inflation ($\alpha > 1$), de Sitter (exponential) inflation ($\alpha = 1$) and the so-called super (or pole) inflation ($\alpha < 1$) i.e., all the range of interesting inflationary backgrounds. Under assumption (2.5) eqs. (2.4) simplify since $\eta R^{-1} dR/d\eta = -\alpha$ and thus eqs. (2.4) all become quadratic. Yet it is quite miraculous that they can be solved analytically in the form of an asymptotic expansion.

As explained in detail in I, highly unstable solutions are characterized by $\dot{X}^2 \ll X'^2$, $\dot{\eta} \gg \eta'$. From eqs.(2.4) this implies immediately that, asymptotically, $\eta = \tau f(\sigma)$, $X = X(\sigma)$, which is precisely the result obtained in I by using cosmic time.

We shall attempt to construct the non leading corrections in the form of a power series in τ (because of the proportionality of η and τ and of eq. (2.5), this is a large- R expansion). We thus write

$$\begin{aligned} \eta &= \sum_{n=0}^{\infty} \eta_n \tau^{2n+1} + \sum_{m=0}^{\infty} \lambda_m \tau^{2\alpha+2m+2} + \sum_{p=0}^{\infty} \zeta_p \tau^{4\alpha+2p+1} \\ X &= \sum_{n=0}^{\infty} A_n \tau^{2n} + \sum_{m=0}^{\infty} B_m \tau^{2\alpha+2m+1} + \sum_{p=0}^{\infty} C_p \tau^{4\alpha+2p+2} + \dots \end{aligned} \quad (2.6)$$

We have seen that this ansatz works perfectly well up to the order investigated and have all reasons to believe that the series with continue forever will generic terms of the form $\tau^{2n_1\alpha+n_2}$ ($n_i = \text{positive integers}$). However, as discussed in I and below, logarithms of τ can appear for special values of α .

The general solution of eqs. (2.4) should depend [1] upon $2(D-2)$ arbitrary functions of σ . These are identified within the set

$$A_0^i, B_0^i, i = 1, 2, \dots, (D-1)$$

subject to the condition

$$B_0 \cdot A_0' = 0. \quad (2.7)$$

The remaining “redundant” arbitrary function can be conveniently eliminated by a σ -gauge choice.

The results of I imply that

$$\eta_0^2 = A'_0 A'_0 \quad (2.8)$$

and also:

$$A_1 = \frac{1}{2(1-2\alpha)} \left(A''_0 - 2\alpha \frac{A'_0 (A'_0 \cdot A'_0)}{(A'_0 \cdot A'_0)} \right). \quad (2.9)$$

It is thus clear that a convenient gauge choice is one in which

$$A'_0 \cdot A'_0 = L^2 = \text{constant}. \quad (2.10)$$

Since $R\sqrt{A'_0 \cdot A'_0} d\sigma$ is the proper string length element, this gauge choice corresponds to $\sigma = \ell/L$ where $R\ell$ is the length measured along the string (from a given point $\sigma = 0$) and $2\pi LR$ is the total length. In this gauge eqs. (2.8) and (2.9) become

$$\eta_0 = L, \quad A_1 = \frac{1}{2(1-2\alpha)} A''_0. \quad (2.11)$$

We are now ready to present the result of our calculations determining completely a number of non-leading coefficients. These are displayed for convenience in Table 1. All coefficients are given explicitly in terms of A_0 and B_0 with the single exception of λ_1 for which the expressions of A_1 and B_2 have still to be inserted.

3 EXCEPTIONAL VALUES OF α

As already pointed out in I – and as evident from Table 1 – singularities occur in the expression of non-leading terms (such as A_1) for special values of α . Keeping in mind that $\alpha > 0$, we see that the singularities occur, for the terms we have computed, at $\alpha = 1/2$ and $3/2$. Eq. (2.6) shows that, at this values, two or more of the non-leading terms multiply the same power of τ . This is a typical situation in which two distinct power solutions coalesce into a single power and a power times logarithms. A convenient way to proceed in order to construct the explicit solutions for these cases is to play with the arbitrariness of A_0^i, B_0^i [subject to the restrictions (2.7) and (2.10)] in order to cancel the singularities at $\alpha = 1/2, 3/2$ and to arrive at a finite answer.

Let us illustrate this procedure in order to recover a result of I for $\alpha = 1/2$. Since, for $\alpha \rightarrow 1/2$,

$$\begin{aligned} X &= A_0 + \tau^2(A_1 + B_0\tau^{2\alpha-1}) + \dots \\ &= A_0 + \tau^2 \left(\frac{A''_0}{2(1-2\alpha)} + B_0\tau^{2\alpha-1} \right) + \dots \end{aligned} \quad (3.1)$$

we take

$$B_0 = \frac{A''_0}{2(2\alpha-1)} + \tilde{B}_0, \quad \tilde{B}_0 A'_0 = 0, \quad \tilde{B}_0 \text{ finite at } \alpha = 1/2. \quad (3.2)$$

The $\alpha \rightarrow 1/2$ limit can be now taken safely yielding:

$$X = A_0 + \tau^2 \tilde{B}_0 + \frac{\tau^2 A''_0}{2} \ln(-\tau) + \dots \quad (3.3)$$

which is the result of I in the “gauge” (2.8).

Let us see how this procedure extends to higher orders. If we look at η we see that double poles at $\alpha = 1/2$ appear in η_1, λ_0 and ζ_0 (the latter two are induced by the previous

requirement (3.2)). Fortunately, all of them multiply the same power of τ at $\alpha = 1/2$. Indeed,

$$\eta = \eta_0 \tau + \tau^3 \left[\eta_1 + \lambda_0 \tau^{2\alpha-1} + \zeta_0 \tau^{4\alpha-2} \right] + \dots \quad (3.4)$$

Inserting the expressions of $\eta_1, \lambda_0, \zeta_0$ from Table 1 and the ansatz (3.2) we find that double and simple poles cancel in (3.4). The limit $\alpha \rightarrow 1/2$ yields

$$\eta = L\tau + \frac{(A_0'' A_0'')}{L} \tau^3 \left[\frac{17}{216} - \frac{2}{9} \ln(-\tau) + 1/6 \ln^2(-\tau) \right]. \quad (3.5)$$

A similar situation seems to appear in X where single, double and triple poles at $\alpha = 1/2$ appear in A_2, B_1 and C_0 (all multiplying τ^4 for $\alpha = 1/2$). To our initial surprise we have found that the cancellation does not work here. At the triple pole level we find, for $\alpha \rightarrow 1/2$

$$A_2 + B_1 + C_0 \longrightarrow A_2 \rightarrow -\frac{1}{24L^2} \frac{1}{(2\alpha-1)^3} (A_0'' A_0'') A_0''. \quad (3.6)$$

Unless there is an algebraic error in our Table 1 (which has passed a number of severe tests), the absence of cancellation calls for extra non-leading terms in X which also “coalesce” with the above three at $\alpha = 1/2$, e.g., a term¹⁾ like $D_0 \tau^{6\alpha+1}$ with a triple pole at $\alpha = 1/2$ in D_0 .

The situation can be settled by an explicit computation of this level of subleading terms, but we have not attempted to do so yet.

Let us now turn to the study of the other exceptional case $\alpha = 3/2$. Such a singularity appears in X through the term $A_2 \tau^4$. The only non-leading term coalescing with the above one at $\alpha = 3/2$ appears to be $B_0 \tau^{2\alpha+1}$. We have already committed ourselves to eq. (3.2) for B_0 but we still have the possibility of choosing $\tilde{B}_0$ to be singular at $\alpha = 3/2$. Since, for $\alpha \rightarrow 3/2$

$$A_2 \sim \frac{1}{(2\alpha-3)} \frac{1}{16} \left[A_0'''' + \frac{3}{L^2} \left((A_0'' A_0''') A_0' + \frac{1}{2} (A_0'' A_0'') A_0'' \right) \right] \quad (3.7)$$

we take

$$\lim_{\alpha \rightarrow 3/2} (\alpha - 3/2) \tilde{B}_0 = - \lim_{\alpha \rightarrow 3/2} (\alpha - 3/2) A_2,$$

One could worry (and rightly so) that the transversality condition $\tilde{B}_0 A_0' = 0$ should be fulfilled. This is (fortunately?) the case since, as $\alpha \rightarrow 3/2$

$$A_2 A_0' \sim \frac{1}{(2\alpha-3)} \frac{1}{16} [A_0'''' A_0' + 3 (A_0'' A_0''')] = 0 \quad (3.8)$$

after repeated use of eq. (2.10) and of its σ -derivatives. With this ansatz for $\tilde{B}_0$ on finally gets at $\alpha = 3/2$

$$X = A_0 + \dots + \tau^4 \tilde{B}_0 - \frac{\tau^4 \ln(-\tau)}{16} \left[A_0'''' + \frac{3}{L^2} \left((A_0'' A_0''') A_0' + \frac{1}{2} (A_0'' A_0'') A_0'' \right) \right] \quad (3.9)$$

Having fixed our only adjustable function $\tilde{B}_0$ the cancellation of $\alpha = 3/2$ singularities should automatically occur in other places, e.g., in η .

1) We have independent arguments for the presence of such terms.

From Table 1 we see that λ_0 , λ_1 , ζ_0 have singularities at $\alpha = 3/2$. The first multiplies τ^5 while the latter two multiply τ^7 . Cancellation of the λ_0 singularity should thus be provided by $\eta_2\tau^5$ (which, unfortunately, we have not computed) while the singularities in λ_1 , ζ_0 should cancel against $\eta_3\tau^7$ (not computed either). Although it would be nice to check explicitly that all these cancellations do occur, we have little doubt that this will turn out to be the case.

4 APPROACH TO ASYMPTOPIA

In order to discuss the physical significance of the non-leading terms obtained in Section 2 and, in particular, the approach to asymptopia it is useful to express the space coordinates $\vec{X}$ directly in terms of “time” η . This involves inverting the relation between η and τ and inserting the result in the expression of $\vec{X}$, a straight forward though tedious calculation.

Before presenting the final outcome, we shall also replace the world sheet spatial variable σ by a more physical quantity, the physical length $\ell \equiv L\sigma$ measured by a comoving observer along the string (recall that $2\pi LR$ is the total proper length, $2\pi L$ the total comoving length). Denoting from now on by primes derivatives with respect to ℓ (rather than σ) we arrive at the final result

$$\begin{aligned}
X = & A_0 + \frac{A_0''}{2(1-2\alpha)}\eta^2 \\
& + \left[A_2 - \frac{\alpha}{3(1-2\alpha)^3} (A_0'' A_0'') A_0'' \right] \eta^4 \\
& + B_0(\eta/L)^{2\alpha+1} + \left(B_1 + \frac{\alpha}{(\alpha+1)(2\alpha-1)^2} (B_0' A_0') A_0'' - \frac{\alpha(2\alpha+1)}{3(1-2\alpha)^2} \right. \\
& \quad \left. \cdot B_0(A_0'' A_0'') \right) (\eta/L)^{2\alpha+1} \eta^2 \\
& + \left(C_0 - \frac{2\alpha(2\alpha+1)}{(\alpha+1)(2\alpha-1)} B_0(B_0' A_0') \right) (\eta/L)^{4\alpha+2} \\
& + \dots
\end{aligned} \tag{4.1}$$

We are now in a position to discuss various aspects of the approach to asymptopia:

a) Rate of the approach

This is given by the magnitude of the non-leading terms relative to the leading one. Suppose that $A_0(\sigma)$ is a given (closed) string (with $\int d\sigma A = 0$), the next to leading term will become relevant for $\eta^2 \approx 0(A_0/A_0'')$. If A_0 has a smooth shape, say a slightly deformed circle of comoving radius r , $A_0/A_0'' \approx r^2$ and we see that the asymptotic regime is achieved at conformal times $\leq r$. However, if A_0 is a highly convoluted string of size r one may have $A_0/A_0'' \ll r^2$ and asymptopia is reached much later. This is our counterpart to the result of Ref. 3 that the higher the frequency the later its amplitude grows. In any case, it is not possible to trust the leading order approximation at conformal times η smaller than the comoving size of the string, a result which has been recently used [6] to constrain a model of self-sustained inflation and simultaneous dimensional reduction.

b) Shape changes

After taking out the scale factor R the size of unstable strings freezes as $\eta \rightarrow 0^-$.

We may ask: what was the string's shape just before it froze? The answer depends on α and thus we distinguish three cases:

i) $\alpha > 1/2$. In this case

$$X = A_0 - C(\alpha) A_0'' \eta^2; \quad C(\alpha) > 0. \quad (4.2)$$

As we move a bit backward in time from $\eta = 0$ we see that points on the string move opposite to A_0'' i.e. inward where the string is concave and outward where it is convex. Playing the "movie" forward in time, one sees that the string "rounds up" as time goes on, becoming more and more regular. In fact a circle remains a circle while an ellipse tends to become more and more circular. We also see that, for a circle, the comoving radius shrinks a little (owing to the η^2 correction) so that the proper size grows a little slower than R subasymptotically.

ii) $\alpha = 1/2$. Using eq. (3.3) we see that, for sufficiently small η , the logarithmically enhanced correction wins and is again opposite to A_0'' . The situation is thus similar to the one for $\alpha > 1/2$.

iii) $\alpha < 1/2$. In this case the asymptotic expansion of X reads

$$X^i = A_0^i + B_0^i (\eta/L)^{2\alpha+1} + \dots \quad (4.3)$$

The correction is of course orthogonal to the string but not necessarily in the direction of $-\vec{A}_0''$. If, for instance, $\vec{B}_0$ is parallel to $\vec{A}_0''$ the strings proper size grows faster than R while its shape becomes less and less round as it expands. We have to remember that for $\alpha < 1/2$ the expansion is extremely fast (for $\alpha < 1$ $R = \infty$ is reached in a finite cosmic time) and the horizon shrinks like η .

One might guess that, in this case, subasymptotic corrections to the equation of state could go in the direction of making the pressure more negative than its asymptotic value $p(\eta = 0) = -\rho/(D-1)$. This could allow a period of self-sustained inflation in $D = 4$. Unfortunately, as we shall see, this guess turns out to be incorrect.

c) Corrections to the equation of state

As shown in I the equation of state of an ideal gas of unstable strings can be obtained by combining the general form of the energy-momentum tensor

$$T^{AB}(x) \propto \int d\sigma d\tau (\dot{X}^A \dot{X}^B - X'^A X'^B) \delta(X - x) \quad (4.4)$$

with the constraints (2.2) and with the asymptotic behaviours of X^0 and X^i . For unstable strings it is most useful to consider the combination

$$T_0^0 - T_i^i = \rho + (D-1)p = \int \left[(X'^0)^2 - R^2 \dot{X}^i \dot{X}^i \right] = -R^2 \int [\eta'^2 - \dot{X}^2] \quad (4.5)$$

where the constraint (2.2) has been used.

In order to have $p < -\rho/(D-1)$ the right-hand side of eq. (4.5) should be negative. Unfortunately, using eq. (2.6), we find

$$\eta' = 0(\tau^3) + (\tau^{2\alpha+2}), \quad \dot{X} = 0(\tau) + 0(\tau^{2\alpha}) \quad (4.6)$$

and therefore the right-hand side of eq. (4.5) is positive for $\tau \rightarrow 0$.

Working out the answer full we find

$$(D-1)p = -\rho \left(1 - 2\eta^2 \left((1-2\alpha)^{-1} A_0'' + \frac{(1+2\alpha)}{L^2} (\eta/L)^{2\alpha-1} B_0 \right)^2 \right) > -\rho. \quad (4.7)$$

Thus the pressure approaches the asymptotic value $-\rho/(D-1)$ from above within a conformal time of order L .

5 CONCLUSION

By studying the systematic form of the large- R expansion and by constructing its first few terms we have been able to check the consistency of the method presented in I. We have also been able to discuss the rate of approach to asymptopia and the direction from which the asymptotic solution is approached in what concerns both the string shape and its equation of state.

Besides providing useful information in view of physical applications (see e.g. ref. 6), we believe our results to become relevant once the problem of quantization will be tackled.

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REFERENCES

- [1] M. Gasperini, N. Sanchez and G. Veneziano, Highly Unstable Fundamental Strings in Inflationary Cosmologies, CERN preprint TH. 5893 (1990), here after referred to as I.
- [2] N. Turok and P. Bhattacharjee, Phys. Rev. **D29** (1984) 1557.
- [3] H. J. De Vega and N. Sanchez, Phys. Lett. **B197** (1987) 322.
N. Sanchez and G. Veneziano, Nucl. Phys. **B333** (1990) 253.
- [4] M. Gasperini, Phys. Lett. **B258** (1991) 70
- [5] Y. Aharonov, F. Englert and J. Orloff, Phys. Lett. **B199** (1987) 366.
N. Turok, Phys. Rev. Lett. **60** (1988) 543.
- [6] M. Gasperini, N. Sanchez and G. Veneziano, Self Sustained Inflation and Dimensional Reduction from Fundamental Strings, CERN preprint TH. 6010 (1991).

TABLE 1
Explicit results for the first few coefficients of the expansion (2.6)

A_0, B_0 arbitrary subject to $A'_0 A''_0 = A'_0 B_0 = 0$

$$\eta_0 = L \equiv \sqrt{A'_0 A'_0}$$

$$\eta_1 = \frac{\alpha}{3L} \frac{(A''_0 A''_0)}{(1 - 2\alpha)^2}$$

$$\lambda_0 = \frac{2\alpha}{L} \frac{(A'_0 B'_0)}{(\alpha + 1)(2\alpha - 1)}$$

$$\lambda_1 = \frac{1}{2L(\alpha + 2)} [(A'_0 B'_1 + A'_1 B'_0) + 2(2\alpha + 3)(A_1 B_1) + 4(2\alpha + 1)(A_2 B_0) - 6(\alpha + 1)\eta_1 \lambda_0]$$

$$\zeta_0 = \frac{(2\alpha + 1)^2}{2(4\alpha + 1)} \frac{(B_0 B_0)}{L}$$

$$A_1 = \frac{A''_0}{2(1 - 2\alpha)}$$

$$A_2 = \frac{1}{(3 - 2\alpha)(1 - 2\alpha)} \left[\frac{A''_0}{8} - \frac{\alpha^2}{3(1 - 2\alpha)L^2} \left[(A''_0 A''_0) A'_0 - \frac{(A''_0 A''_0) A''_0}{(1 - 2\alpha)} \right] \right]$$

$$B_1 = \frac{1}{(2\alpha + 3)L^2} \left[\frac{2\alpha^2(2\alpha + 1)}{(2\alpha - 1)^2} \left(\frac{1}{3} (A''_0 A''_0) B_0 - \frac{(A'_0 B'_0) A''_0}{(\alpha + 1)} \right) - \frac{2\alpha^2}{(\alpha + 1)(2\alpha - 1)} (A'_0 B'_0)' A'_0 + \frac{1}{2} L^2 B''_0 \right]$$

$$C_0 = \frac{\alpha}{L^2} \left[\frac{2\alpha}{(\alpha + 1)(2\alpha - 1)} (A'_0 B'_0) B_0 + \frac{2\alpha}{(4\alpha + 1)(1 - 2\alpha)} (B_0 B_0) A''_0 - \frac{1}{(4\alpha + 1)} (B_0 B'_0) A'_0 \right]$$