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RENORMALIZATIONS IN ABELIAN CHERN-SIMONS FIELD THEORIES WITH MATTER

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Abstract

We consider renormalization in $D = 3$ abelian Chern-Simons field theories with matter. The anomalous dimensions and β functions in a general renormalizable theory with the global $SU(n)$ symmetry are calculated in the leading two-loop approximation. All fixed points are found and their stability properties are examined. Some of them exhibit the $N = 1$ and $N = 2$ supersymmetry. Explicit calculations in terms of the $N = 1$ and $N = 2$ superfields confirm the validity of the component-field results. A possible relation to the anyon superconductivity is briefly discussed.

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1 Introduction

Gauge theories in (2+1) dimensions have several features not seen in their (3+1)-dimensional counterparts. These center on the possible appearance of the Chern-Simons three-form in the action [1] and are connected with parity anomalies [2] and fractional statistics [3]. Original motivation for their study was their resemblance to the high-temperature limit of gauge theories in (3+1) dimensions. More recently, there has been interest in (2+1)-dimensional electrodynamics in models of certain condensed-matter phenomena, particularly high- T_c superconductivity [4] and quantum Hall effect [5].

The Chern-Simons gauge theories, although very similar in some respects to the Yang-Mills theories, exhibit various specific properties. Contrary to the Yang-Mills and Maxwell case, they are intimately connected with a particular dimension and cannot be universally continued to other dimensions. They are also more rigid, in the non-abelian case, due to their topological nature. This property leads, in particular, to the absence of infinite Chern-Simons charge renormalization in non-abelian theories [6]. The same, although not so obvious, is true in the abelian case [7]. The only possibility is a finite shift of the bare coupling, happening in the lowest order [8].

The situation is much more complicated when the matter fields are added. Here it is close to what occurs in the Yang-Mills and Maxwell theories in (3+1) dimensions. Though the Chern-Simons charge is still not renormalized, this is no longer true for the matter couplings. One has non-zero β functions with a complicated renormalization group (RG) flow, possessing plenty of fixed points. Some of these fixed points and fixed lines quite unexpectedly exhibit supersymmetry and extended supersymmetry, being realized as stable or saddle-point solutions of the corresponding RG equations [9].

Below we present the results of our calculations of the renormalization group functions for the abelian $D = 3$ Chern-Simons gauge field theory with scalar and spinor matter fields to the lowest order of perturbation theory. Owing to the odd number of dimensions, there are no genuine ultraviolet (UV) divergences in the one-loop approximation as well as for any odd number of loops. The first nontrivial contribution comes from the two-loop diagrams.

2 UV Divergences in Abelian Chern-Simons Theory with Matter

The Lagrangian for a general renormalizable abelian Chern-Simons gauge field theory with scalar and spinor matter fields in three dimensions can be written as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + |D_\mu\phi_j|^2 + i\bar{\psi}_j\hat{D}\psi_j - m^2\phi_j^*\phi_j - M\bar{\psi}_j\psi_j - q(\phi_j^*\phi_j)^2 \\ & + \alpha\bar{\psi}_j\psi_j\phi_k^*\phi_k + \beta\bar{\psi}_j\psi_k\phi_k^*\phi_j + \frac{\gamma}{4}[\bar{\psi}_j\psi_k^c\phi_j\phi_k + \bar{\psi}_j^c\psi_k\phi_j^*\phi_k^*] - h(\phi_j^*\phi_j)^3, \end{aligned} \quad (1)$$

where the covariant derivative is, as usual, $D_\mu = \partial_\mu - ieA_\mu$, and we have introduced a global $SU(n)$ symmetry, so that the matter fields carry indices of the fundamental representation of the group ($j, k = 1, 2, \dots, n$). The charge conjugate of a spinor ψ is denoted by ψ^c . In the Majorana basis it is obtained by complex conjugation.

Here we see a difference from a four dimensional theory where the terms containing charge-conjugate spinors are absent. As we shall see later, in three dimensions, these terms inevitably appear in the component-field formulation of supersymmetric theories [10, 11]. Physically, they lead to fermion number violation which may be of interest for condensed-matter applications.

The model at hand is renormalizable by the standard power counting procedure, and possesses the usual UV divergences. To deal with them, one has to apply some kind of regularization, the choice of which is not very obvious for the present case. On the one hand, to perform the two-loop calculations analytically, one tries to find the simplest regularization that does not break the gauge invariance. This is apparently dimensional regularization. On the other hand, owing to the ϵ tensor in eq.(1), the Chern-Simons theories are essentially three dimensional and cannot be freely continued to an arbitrary dimension. In this situation the use of a higher-derivative regularization is safe enough but makes calculations very tedious [12]. Therefore, bearing in mind that we need to know only divergences and are not interested in the finite parts of the two-loop diagrams, we apply the so-called regularization by dimensional reduction [13, 14]. This means that all the contractions of ϵ tensors and γ matrices are performed in three dimensions, while the momentum integrals are evaluated afterwards in $3-2\epsilon$ dimensions. Strictly speaking, this procedure is not self-consistent [15], though the only possible ambiguity may appear for the finite parts of the two-loop diagrams and not for the divergences. This is because in three dimensions all the one-loop momentum integrals are finite and the two-loop divergences are just $1/\epsilon$ and not $1/\epsilon^2$ as in four dimensions. Hence, one does not encounter any problem when calculating the leading divergences by means of the dimensional reduction procedure.

To perform the calculations in gauge theories, one has to fix a gauge. We choose it in a way to simplify the calculations, namely we add to the Lagrangian (1) a gauge fixing term

$$\frac{1}{2\xi}(\partial_\mu A_\mu)^2, \quad (2)$$

which in the Landau gauge ($\xi \rightarrow 0$) gives us a transverse propagator

$$\langle A_\mu(-p)A_\nu(p) \rangle = \frac{\epsilon^{\mu\nu\lambda}p_\lambda}{p^2 + i0}.$$

The complete set of Feynman rules for the model (1) is given in Appendix A.

In what follows we perform renormalizations using the minimal subtraction (MS) scheme. In this case, the β functions and anomalous dimensions depend only on dimensionless couplings e , α , β , γ , and h . The dependence on the parameters of dimension of a mass (m , M , and q) is absent.

Now we present the results of our calculations of anomalous dimensions of the fields as well as of the β functions of the theory. They are related to the anomalous dimensions of the vertex and propagator-type diagrams according to the following formulae:

$$\begin{aligned} \beta_{e^2} &= e^2\gamma_A, & \beta_\alpha &= \alpha(\gamma_\alpha + \gamma_\psi + \gamma_\phi), & \beta_\beta &= \beta(\gamma_\beta + \gamma_\psi + \gamma_\phi), \\ \beta_\gamma &= \gamma(\gamma_\gamma + \gamma_\psi + \gamma_\phi), & \beta_h &= h(\gamma_h + 3\gamma_\phi), \end{aligned}$$

where γ_j are the anomalous dimensions of the corresponding vertices and propagators.

The results are:

$$\begin{aligned}
\gamma_A &= 0, \\
\gamma_\phi &= \frac{1}{3}(n\alpha^2 + 2\alpha\beta + n\beta^2) + \frac{1}{12}(n+1)\gamma^2 - \frac{8n+10}{3}e^4, \\
\gamma_\psi &= \frac{1}{3}(n\alpha^2 + 2\alpha\beta + n\beta^2) + \frac{1}{12}(n+1)\gamma^2 - \frac{2n+1}{3}e^4, \\
\beta_{e^2} &= 0, \\
\beta_\alpha &= \left(\frac{8}{3}n+2\right)\alpha^3 + \frac{16}{3}\alpha^2\beta + \left(\frac{8}{3}n+3\right)\alpha\beta^2 + (n+2)\beta^3 + \frac{3}{4}(n+2)\gamma^2(\alpha+\beta) \\
&\quad - \frac{n+1}{12}\gamma^2\alpha + 3\beta^2e^2 + \frac{\gamma^2}{4}e^2 - \left(\frac{40}{3}n + \frac{62}{3}\right)\alpha e^4 - 8\beta e^4 - 8(n+2)e^6, \\
\beta_\beta &= \left(\frac{8}{3}n+6\right)\alpha^2\beta + \left(3n + \frac{16}{3}\right)\alpha\beta^2 + \left(1 + \frac{2n}{3}\right)\beta^3 + \frac{3}{4}(n+2)\gamma^2(\alpha+\beta) \\
&\quad - \frac{n+1}{12}\gamma^2\beta - 3n\beta^2e^2 + \frac{n+2}{4}\gamma^2e^2 - \left(\frac{16}{3}n + \frac{62}{3}\right)\beta e^4, \\
\beta_\gamma &= \left(\frac{8}{3}n+6\right)\alpha^2\gamma + \left(6n + \frac{34}{3}\right)\alpha\beta\gamma + \left(\frac{8}{3}n+6\right)\beta^2\gamma + \frac{n+1}{6}\gamma^3 \\
&\quad + 4\alpha\gamma e^2 + 2(n+1)\beta\gamma e^2 - \frac{4n-10}{3}\gamma e^4, \\
\beta_h &= 12(3n+11)h^2 + 4h[4n\alpha^2 + 8\alpha\beta + (n+3)\beta^2] + (n+4)h\gamma^2 - 4(5n+16)he^4 \\
&\quad - 4[n\alpha^4 + 4\alpha^3\beta + (n+5)\alpha^2\beta^2 + (n+3)\alpha\beta^3 + \frac{n+3}{4}\beta^4] - \frac{n+3}{16}\gamma^4 \\
&\quad - \gamma^2[(n+6)\alpha^2 + (3n+11)\alpha\beta + (n+6)\beta^2] - 2(\alpha+\beta)\gamma^2e^2 \\
&\quad + 4(n\alpha^2 + 2\alpha\beta + \beta^2)e^4 - \gamma^2e^4 + 8(n\alpha + \beta)e^6 + (8n+28)e^8.
\end{aligned} \tag{3}$$

Hereafter n refers to the global $SU(n)$ group and we omit the obvious factors of $(64\pi^2)^{-1}$. The β functions are gauge invariant, while the anomalous dimensions of the fields are given in the Landau gauge.

Note that the $n = 1$ case is degenerate because α and β couplings are indistinguishable. To obtain the correct result for $n = 1$, one should sum up the expressions for β_α and β_β .

We have calculated the anomalous dimensions of the dimensional parameters in our model as well. In the MS scheme they are also gauge invariant and happen to be equal to each other

$$\gamma_m = \gamma_M = \gamma_q = \frac{4}{3}[n\alpha^2 + 2\alpha\beta + n\beta^2 + \frac{1}{4}(n+1)\gamma^2 - (5n+4)e^4]. \tag{4}$$

For a particular choice of the couplings, the Lagrangian (1) possesses the $N = 1$ supersymmetry, which is preserved by renormalization. This choice corresponds to

$$\begin{aligned}
\alpha &= \lambda, \quad \beta = \lambda + e^2, \quad \gamma = 2(\lambda - e^2), \quad h = \lambda^2, \\
M &= m, \quad q = 2\lambda m,
\end{aligned} \tag{5}$$

where λ is the only remaining dimensionless non-gauge coupling constant. Eq.(5) is consistent with eq.(3) for the gauge-invariant quantities. One has

$$\beta_\lambda = 4(\lambda - e^2)[3(n+2)\lambda^2 + (3n+7)\lambda e^2 + (n+3)e^4] \tag{6}$$

and

$$\gamma_m = 4(n+1)(\lambda^2 - e^4). \quad (7)$$

We consider the superfield formulation of the model in the next section and check the above expressions by an explicit superfield calculation.

Further restriction of the model enables us to get an extended $N = 2$ supersymmetry, which, contrary to the Yang-Mills case, can be reached for an arbitrary value of n and corresponds to the value of λ equal to e^2 . It follows from eqs.(6,7) that both the β function and the anomalous dimension of mass vanish for the $N = 2$ supersymmetry.

3 Supersymmetric Abelian Chern-Simons Theory

In three dimensions the $N = 1$ supersymmetric action for a general renormalizable abelian Chern-Simons theory has the form [10, 11]

$$S = \int d^3x d^2\theta \left\{ -\frac{1}{4}(D^\alpha \Gamma^\beta) D_\beta \Gamma_\alpha - \frac{1}{2}[(D^\alpha + ig\Gamma^\alpha)\Phi^\dagger](D_\alpha - ig\Gamma_\alpha)\Phi - m\Phi^\dagger\Phi + \frac{\eta}{4}(\Phi^\dagger\Phi)^2 \right\}, \quad (8)$$

where Γ_α is a real (Majorana) gauge superfield and Φ is a complex matter supermultiplet belonging to the fundamental representation of the global $SU(n)$. The first term gives the supersymmetric generalization of the Chern-Simons term in eq.(1). The coupling constants are related to those in the component-field Lagrangian (1),(5) as $g = e\sqrt{2}$, $\eta = 2\lambda^2$. The super-covariant derivatives satisfy the anticommutation relation

$$\{D_\alpha, D^\beta\} = 2i\partial_\alpha^\beta. \quad (9)$$

Some useful formulae and our conventions about the θ algebra are specified in Appendix B.

Expansion of superfields in θ , or alternatively, the values of superfields and their spinor derivatives at $\theta = 0$ determine the component-field contents of superfields

$$\begin{aligned} \Gamma_\alpha|_{\theta=0} &= \chi_\alpha, \quad \frac{1}{2}D^\alpha \Gamma_\alpha| = B, \quad D_\alpha \Gamma^\beta| = \frac{i}{\sqrt{2}}(\gamma_\mu)_\alpha^\beta A_\mu - \delta_\alpha^\beta B, \quad \frac{1}{2}D^\beta D_\alpha \Gamma_\beta| = \lambda_\alpha; \\ \Phi| &= \phi, \quad D_\alpha \Phi| = \psi_\alpha, \quad D^2 \Phi| = F. \end{aligned} \quad (10)$$

To express the superfield action (8) in terms of component fields (10), we replace the integration over θ by differentiation, eq.(B.11), apply the derivatives according to the Leibnitz rule, and perform the identifications (10). The fields χ_α and B can be gauged away via a specific superfield gauge transformation (fixing the Wess-Zumino gauge). The auxiliary fields λ_α , F and $F^\dagger$ have no derivatives in kinetic terms and can be eliminated by using their equations of motion, which modifies the interactions of the physical component fields. In this way, the component-field Lagrangian (1) is reproduced from eq.(8) with the particular correlation between the couplings given by eq.(5). Note that contrary to the Maxwell case, for a Chern-Simons supersymmetric theory the spinor superpartner λ_α of the vector field proves to be an auxiliary field and does not appear in the on-shell component-field Lagrangian. Nevertheless, irrespective of n , the equality of the number

of bosonic and fermionic degrees of freedom is retained on shell since the Chern-Simons field has no physical degrees of freedom.

To preserve the linear realization of the manifest supersymmetry in our calculations, we must choose a supersymmetric rather than the Wess-Zumino gauge. This can be attained by writing the gauge-fixing term like

$$\int d^3x d^2\theta \frac{1}{4\xi} (D^\alpha \Gamma_\alpha) D^\beta \Gamma_\beta, \quad (11)$$

which leads to the following propagator for the gauge superfield

$$\langle \Gamma_\alpha(-p, \theta_1) \Gamma^\beta(p, \theta_2) \rangle = -\frac{i}{p^2 + i0} \left[\frac{1+\xi}{2} p_\alpha^\beta + \frac{1-\xi}{2} \delta_\alpha^\beta D^2 \right] \delta^2(\theta_1 - \theta_2).$$

The simplest case is the supersymmetric Feynman gauge $\xi = 1$ which we used in our calculations. The Feynman rules are presented in Appendix B.

The recipe of the regularization by dimensional reduction requires us to perform all the super-algebra first in three dimensions and then take the resulting momentum integrals in $3-2\epsilon$ dimensions. The D algebra in supergraphs is performed as usual [14] by integrating the spinor derivatives by parts (see eq.(B.12)) from a line of a graph to other lines and by reducing their number on any line to at most two with the aid of eqs.(B.6-B.10). This enables us to remove all the derivatives from some lines, leaving the pure δ functions, so that the θ integration is easily performed according to eqs.(B.13-B.15). The higher powers of the derivatives generate additional momentum factors in numerators or cancel some denominators.

Supersymmetry reduces the maximum degree of divergence of the supergraphs, so that some of the diagrams become trivially equal to zero or finite. However, UV divergences are still present: linear in propagators and logarithmic in vertices. Here, we write down the results of our superfield calculations of the anomalous dimensions and the β functions

$$\gamma_\Gamma = 0, \quad (12)$$

$$\gamma_\Phi = \frac{n+1}{4} \eta^2 - \frac{2n+3}{4} g^4, \quad (13)$$

$$\gamma_m = (n+1)(\eta^2 - g^4), \quad (14)$$

$$\beta_{g^2} = 0, \quad (15)$$

$$\beta_\eta = 3(n+2)\eta^3 + \eta^2 g^2 - 2(n+2)\eta g^4 - (n+3)g^6, \quad (16)$$

which confirm the validity of the component-field calculations of the previous section. Note that in the supersymmetric gauge (11) both γ_ϕ and γ_ψ are given by γ_Φ (13), while in the Wess-Zumino gauge they are different (see eq.(3)). The gauge-invariant quantities of eqs.(14,16) and (7,6) coincide as expected.

At $\eta = g^2$, that is $\lambda = e^2$, the β function and the anomalous dimension of the mass become equal to zero. This fixed point, where the UV divergences cancel, corresponds to the extended $N = 2$ supersymmetry of the model. A manifestly $N = 2$ covariant formalism can be applied [16] (see Appendix C for the notation). The action is written as

$$S = \int d^3x d^4\theta \left[\frac{1}{2} V \bar{D}^\alpha D_\alpha V + \kappa V + \Psi^\dagger \exp(-gV) \Psi \right], \quad (17)$$

where V is a general real gauge superfield and Ψ is a chiral

$$\bar{D}_\alpha \Psi = 0, \quad D_\alpha \Psi^\dagger = 0$$

complex supermultiplet in the fundamental representation of the $SU(n)$. The introduction of the global $SU(n)$ group is possible because the gauge and matter fields form independent $N = 2$ supermultiplets, as opposed to the Yang-Mills case where they are combined into a single supermultiplet.

The D algebra is now defined by

$$\{D_\alpha, D_\beta\} = 0, \quad \{\bar{D}_\alpha, \bar{D}_\beta\} = 0, \quad \{D_\alpha, \bar{D}^\beta\} = i\partial_\alpha{}^\beta. \quad (18)$$

To perform the reduction to the $N = 1$ case, the algebra can be realized as

$$D_\alpha = \frac{1}{2}(D_{1\alpha} + iD_{2\alpha}), \quad \bar{D}_\alpha = \frac{1}{2}(D_{1\alpha} - iD_{2\alpha})$$

with

$$\{D_{j\alpha}, D_k^\beta\} = 2i \delta_{jk} \partial_\alpha{}^\beta, \quad (j, k = 1, 2).$$

Then $\int d^3x d^4\theta = -\int d^3x d^2\theta_1 D_2^2$ at $\theta_2 = 0$. For chiral superfields we have

$$D_{2\alpha} \Psi = -iD_{1\alpha} \Psi, \quad D_{2\alpha} \bar{\Psi} = iD_{1\alpha} \bar{\Psi}.$$

Now, in complete analogy with eqs.(10), the $N = 1$ component superfields are given by the $N = 2$ superfields and their D_2 derivatives at $\theta_2 = 0$. By a gauge transformation we can leave out $V|_{\theta_2=0}$, and there remain the following component superfields

$$D_{2\alpha} V|_{\theta_2=0} = \Gamma_\alpha(x, \theta_1), \quad D_2^2 V| = R(x, \theta_1), \quad \Psi| = \frac{1}{\sqrt{2}}\Phi(x, \theta_1), \quad \Psi^\dagger| = \frac{1}{\sqrt{2}}\Phi^\dagger(x, \theta_1).$$

The auxiliary superfield R is eliminated via its equation of motion. This procedure generates the mass term and the quartic interaction term of eq.(8) with $\eta = g^2$ and $m = g\kappa$.

The $N = 2$ supersymmetric gauge-fixing term can be chosen in the form

$$\int d^3x d^4\theta \frac{1}{2} V \left[\frac{1}{a}(D^2 + \bar{D}^2) + \frac{1}{ib}(D^2 - \bar{D}^2) \right] V, \quad (19)$$

which results in the following gauge-superfield propagator:

$$\begin{aligned} &< V(-p, \theta) V(p, \theta') > \\ &= \frac{i}{p^2 + i0} \left\{ \bar{D}^\alpha D_\alpha + \frac{a^2 b^2}{a^2 + b^2} \left[\frac{1}{a}(D^2 + \bar{D}^2) + \frac{1}{ib}(D^2 - \bar{D}^2) \right] \right\} \delta^4(\theta - \theta') \end{aligned}$$

We used an analogue of the Landau gauge $a, b \rightarrow 0$.

The results of our two-loop calculations are

$$\gamma_v = 0, \quad (20)$$

$$\gamma_\psi = (n+1)g^4, \quad (21)$$

$$\gamma_\kappa = 0, \quad (22)$$

$$\beta_{g^2} = 0. \quad (23)$$

Table 1: RG fixed points for $n = 1$

i	$\alpha_i + \beta_i$	γ_i	h_i	Comment
1	3	0	1	stable
2	3	0	-19/14	saddle point
3	-2.2319	0	0.6164	stable
4	-2.2319	0	-0.5909	saddle point

In fact, eq.(22) is a direct consequence of eq.(20) because in a massless renormalization scheme there cannot be any independent linearly divergent counterterm of the κV -type generated by the gauge interaction, except for the one coming from the field renormalization. Our direct calculations show that the latter is absent too, eq.(20). The vanishing of the renormalizations eqs.(22,23) should extend to any order of perturbation theory [7].

The difference in the matter-field renormalization between eqs.(21) and(13) is due to the fact that the gauges (19) and (11) are essentially different.

4 Renormalization Group Fixed Points

Now we are in a position to calculate the fixed points of the RG equations (3). They are determined by zeros of the β functions. As the Chern-Simons coupling e is not renormalized, i.e. $\beta_e \equiv 0$ to all orders of perturbation theory, we consider e as a fixed parameter in the remaining equations for α , β , γ , and h . Then the fixed points are of the form

$$\alpha^* = \alpha_i e^2, \quad \beta^* = \beta_i e^2, \quad \gamma^* = \gamma_i e^2, \quad h^* = h_i e^4.$$

The results for various values of n are presented in Tables 1-3. The stability properties refer to the infrared (IR), or long-distance, behaviour.

Though the values of the couplings for various values of n are different, there are some common features. One of them is the existence of a fixed point which corresponds to the $N = 2$ supersymmetry. For $n = 1$ this is the stable fixed point # 1, while for $n > 1$ these are the saddle points # 7. Thus, the $N = 2$ extended supersymmetry exists for any global $SU(n)$ group, and it is realized as a fixed-point solution of the RG equations. The $N = 1$ supersymmetry is also consistent with eq.(3) and manifests itself as a fixed line corresponding to eq.(5). This is valid for any value of n , too. The only fixed point on this line is the $N = 2$ supersymmetric one. It is IR stable along the line, i.e. when the $N = 1$ supersymmetry is imposed.

We would like to stress that the fixed points obtained here to the leading order of perturbation theory are steady with respect to the higher-order corrections. This means that neither will any new fixed points appear nor will any of them disappear in higher orders. Their stability properties will not be subject to a change either. The fixed points will only get corrections in the form of a power series in e^2 . Moreover, the $N = 2$ supersymmetric fixed point is exact to all orders of perturbation theory if the regularization applied preserves supersymmetry.

Table 2: RG fixed points for $n = 2$

i	α_i	β_i	γ_i	h_i	Comment
1	2.8280	0	0	1.0045	stable
2	2.8280	0	0	-1.7492	saddle point
3	-2.0896	0	0	0.6420	stable
4	-2.0896	0	0	-0.8171	saddle point
5	2.6192	-2.6533	0	0.7586	saddle point
6	2.6192	-2.6533	0	-0.9250	saddle point
7	1	2	0	1	saddle point
8	1	2	0	-23/17	saddle point
9	-0.9733	-1.0565	0	0.4975	stable
10	-0.9733	-1.0565	0	-0.4070	saddle point
11	-2.2121	0.8279	0	0.5953	saddle point
12	-2.2121	0.8279	0	-0.6330	saddle point
13	3.0113	-3.6686	± 4.8514	0.7379	stable
14	3.0113	-3.6686	± 4.8514	-1.9293	saddle point
15	3	-4	± 4	0.7789	saddle point
16	3	-4	± 4	-1.8377	saddle point
17	2.6690	-2.1728	± 2.3536	0.7140	saddle point
18	2.6690	-2.1728	± 2.3536	-1.0377	saddle point

Table 3: RG fixed points for $n \rightarrow \infty$

i	α_i	β_i	γ_i	h_i	Comment
1	2.4909	0	0	0.9057	stable
2	2.4909	0	0	-3.1070	saddle point
3	-1.8342	0	0	0.6659	stable
4	-1.8342	0	0	-1.6057	saddle point
5	23.7209	-70.3767	0	117.2012	saddle point
6	23.7209	-70.3767	0	-917.0468	saddle point
7	1	2	0	1	saddle point
8	1	2	0	-4/3	saddle point
9	-1.7829	0.3885	0	0.6478	saddle point
10	-1.7829	0.3885	0	-1.5218	saddle point
11	2.5905	-3.0251	± 8.5221	-0.3168	stable
12	2.5905	-3.0251	± 8.5221	-5.1444	saddle point
13	2.2411	-0.9937	± 1.9829	0.6985	saddle point
14	2.2411	-0.9937	± 1.9829	-2.5942	saddle point

The picture of fixed points changes relatively smoothly with n . Points # 9,10 of Table 2 disappear at $n = 43$. Points # 15,16 flow together with # 5,6 at $n = 6$ and disappear at $n = 9$ (h becomes imaginary). Instead of these, the new far-distant points # 5,6 appear in Table 3. Points # 17,18 of Table 2 correspond to # 13,14 of Table 3.

5 Discussion

Now we discuss some consequences and physical meaning of the fixed points obtained above. In the condensed-matter physics applications the RG fixed points are associated with phase transitions and characterize the critical behaviour of a system. According to usual belief [17], the IR stable fixed points correspond to the second-order phase transitions. In the vicinity of a stable fixed point the Green functions exhibit a power-like behaviour with the powers equal to the proper anomalous dimensions. The same kind of behaviour is typical of the second-order phase transition where the power-like scaling behaviour is governed by the critical exponents. Thus, making the identification of the stable fixed points with the second-order phase transitions, one can calculate the critical exponents in the form of a power series in e^2 .

As have been mentioned earlier, the Chern-Simons field theories are considered as an effective theory of anyon superconductivity. There the Chern-Simons field arises from the Hopf term of non-linear σ - model which describes the spin excitations in doped antiferromagnets [18]. It has been argued [4, 18] that in this model a phase transition could occur which leads to the superfluidity or, when the Maxwell electromagnetic field is introduced, to superconductivity. We analyse this possibility from the point of view of a field-theoretical description of phase transitions based on the renormalization group fixed points.

Coming back to the Tables, we see that there are several stable fixed points. The values of the anomalous dimensions (4), which govern the behaviour of the correlation length and define the effective potential, can be divided into three classes corresponding to $\gamma_m = 0$, $\gamma_m > 0$, and $\gamma_m < 0$, respectively.

The first class refers to the $N = 2$ supersymmetry. In this case the potential is completely determined by the supersymmetry. The phase transition takes place with the gaussian critical exponents. Presumably, it is a transition from a supersymmetric phase to a non-supersymmetric one, between superconductors of the first and second kind [20]. In the supersymmetric phase vortex solutions exist [10] which satisfy the self-duality equations [19] and are stable. At the transition point, the force between two single vortices changes its sign. Therefore, this transition separates the phase where vortices are stable from the one where they are unstable [20].

When $\gamma_m > 0$, all the dimensional parameters vanish in the IR limit. Thus, the effective potential is controlled by dimensionless couplings and radiative corrections. The fixed point with the coupling $\gamma \neq 0$ (# 13, Table 2) then opens the possibility of getting anomalous charge expectation values of the types $\langle \phi\phi \rangle$ and $\langle \psi\psi \rangle$, violating the fermion number. This may lead to the existence of something like a condensate of Cooper pairs. It is interesting to note that nontrivial fixed points with $\gamma \neq 0$ arise only if we impose the global $SU(n)$ group ($n \geq 2$), which may be related to the introduction of the

anyon flavour in some high- T_c superconductivity models [21]. The phases with nonzero neutral expectation values $\langle \bar{\psi}\psi \rangle$ and $\langle \phi^*\phi \rangle$ are also possible. The phase transition separates the phases with different values of the expectation values.

When $\gamma_m < 0$, the parameters with dimension of mass tend to ∞ in the IR limit. This means that the terms with dimensional couplings become essential and determine the potential and possible phases in the model. Since for these cases the coupling $\gamma = 0$, anomalous expectation values are excluded.

The remaining fixed points are saddle points. According to the same usual belief [17], they are identified with the three-critical points, though here the interpretation is not so clear.

Appendix A: Feynman Rules for the Abelian Chern-Simons Theory with Matter

In this appendix we list the Feynman rules corresponding to the Lagrangian (1).

$\Rightarrow$	$\frac{\epsilon_{\mu\nu\lambda}p^\lambda + i\xi p_\mu p_\nu / p^2}{p^2 + i0},$
$\Rightarrow$	$i \frac{\hat{p} + M}{p^2 - M^2 + i0} \delta_{jk},$
$\Rightarrow$	$\frac{i}{p^2 - m^2 + i0} \delta_{jk},$
$\Rightarrow$	$ie(p + q)_\mu,$
$\Rightarrow$	$ie^2 \delta_{\mu\nu},$
$\Rightarrow$	$ie\gamma^\mu,$
$\Rightarrow$	$i\alpha,$
$\Rightarrow$	$i\beta,$
$\Rightarrow$	$i\frac{\gamma}{2},$
$\Rightarrow$	$i\frac{\gamma}{2},$
$\Rightarrow$	$-6ih.$

In our calculations we use the Landau gauge $\xi = 0$.

Appendix B: The N=1 supersymmetry algebra in three dimensions

We use the Majorana spinor basis. Our conventions follow ref.[14]. Spinor indices denoted by $\alpha, \beta, \dots$ take on two values. They can be raised and lowered

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta, \quad \psi^\beta C_{\beta\alpha} = \psi_\alpha$$

with the antisymmetric tensor

$$C_{\alpha\beta} = -C_{\beta\alpha}, \quad C^{\alpha\beta} = -C^{\beta\alpha}, \quad C^{\alpha\beta}C_{\beta\gamma} = -\delta_\gamma^\alpha.$$

In contrast, the Dirac γ matrices

$$(\gamma_\mu)_\alpha^\beta (\gamma_\nu)_\beta^\gamma + (\gamma_\nu)_\alpha^\beta (\gamma_\mu)_\beta^\gamma = 2g_{\mu\nu}\delta_\alpha^\gamma \quad (\text{B.1})$$

are symmetric in their spinor indices of the same position

$$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_{\beta\alpha}, \quad (\gamma_\mu)^{\alpha\beta} = (\gamma_\mu)^{\beta\alpha}.$$

The standard direction of contracting the spinor indices is chosen to be from left up to right down ($\searrow$), while for the opposite order the sign changes

$$\psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha.$$

Thus, for the anticommuting spinors the scalar combination is symmetric

$$\{\psi_\alpha, \chi_\beta\} = 0 \quad \Rightarrow \quad \psi^\alpha \chi_\alpha = \chi^\alpha \psi_\alpha.$$

By definition, the square of a spinor involves a 1/2 factor

$$\psi^2 \equiv \frac{1}{2}\psi^\alpha \psi_\alpha,$$

which suppresses abundant powers of 2 throughout the calculations.

Since there are just two values for the indices, the following identities hold for any ψ , ϕ , and anticommuting θ

$$\psi^\alpha \chi_\beta - \psi_\beta \chi^\alpha = \delta_\beta^\alpha \psi^\gamma \chi_\gamma, \quad \psi_\alpha \chi_\beta - \psi_\beta \chi_\alpha = C_{\beta\alpha} \psi^\gamma \chi_\gamma, \quad (\text{B.2})$$

$$\{\theta_\alpha, \theta_\beta\} = 0 \quad \Rightarrow \quad \theta^\alpha \theta_\beta = \delta_\beta^\alpha \theta^2, \quad \theta^\alpha \theta^\beta = C^{\beta\alpha} \theta^2, \quad \theta_\alpha \theta_\beta = C_{\beta\alpha} \theta^2. \quad (\text{B.3})$$

A spinor with a lower index is the usual ψ column, while a complex conjugate spinor with an upper index represents the $\bar{\psi}$ row. The complex conjugation of these leads to ψ^c and $\bar{\psi}^c$.

The traces of the γ matrices (with the spinor indices in a product contracted according to the $\searrow$ convention) are derived from eq.(B.1) and from

$$\text{tr } \mathbf{1} = 2, \quad \text{tr } \gamma_\mu = 0, \quad \text{tr}(\gamma_\mu \gamma_\nu \gamma_\lambda) = -2i\epsilon_{\mu\nu\lambda}. \quad (\text{B.4})$$

The ϵ tensor in three dimensions is defined by

$$\epsilon_{\mu\nu\lambda}\epsilon^{\mu'\nu'\lambda'} = g_{\mu}^{[\mu'} g_{\nu}^{\nu'} g_{\lambda}^{\lambda']}, \quad (\text{B.5})$$

where the brackets denote a totally antisymmetrized sum (for the Lorentz indices we do not distinguish the upper and lower positions; covariant contractions are always implied through the metric tensor $g_{\mu\nu}$).

The spinor co-ordinates of superspace θ_α are real (Majorana) in the $N = 1$ case. Here are some useful relations for the super-covariant derivatives (9), written in the momentum space ($\partial_\mu \rightarrow ip_\mu$),

$$D^\alpha D_\beta = -p_\beta^\alpha + \delta_\beta^\alpha D^2, \quad D_\alpha D^\beta = -p_\alpha^\beta - \delta_\alpha^\beta D^2, \quad (\text{B.6})$$

$$D^\alpha D_\beta D_\alpha = 0, \quad (\text{B.7})$$

$$D^2 D_\alpha = p_\alpha^\beta D_\beta, \quad D_\alpha D^2 = -p_\alpha^\beta D_\beta, \quad (\text{B.8})$$

$$D^\alpha D^2 = D^\beta p_\beta^\alpha, \quad D^2 D^\alpha = -D^\beta p_\beta^\alpha, \quad (\text{B.9})$$

$$D^2 D^2 = p^2, \quad D^\alpha D^2 D_\alpha = -2p^2, \quad D^2 D_\alpha D^2 = -p^2 D_\alpha. \quad (\text{B.10})$$

The integration over θ can be defined as

$$\int d^3x d^2\theta F(x, \theta) = \int d^3x D^2 F(x, \theta). \quad (\text{B.11})$$

An obvious consequence of the definition and the properties of the derivatives is

$$\int d^3x d^2\theta D_\alpha F(x, \theta) = 0, \quad (\text{B.12})$$

which allows one to integrate D_α by parts.

The δ function for the θ integration is introduced by

$$\int d^2\theta \delta^2(\theta - \phi) F(\theta) = F(\phi). \quad (\text{B.13})$$

In accordance with eq.(B.11), it satisfies

$$D^2 \delta^2(\theta)|_{\theta=0} = 1, \quad (\text{B.14})$$

while for lower powers of the derivatives the result is zero

$$D_\alpha \delta^2(\theta)|_{\theta=0} = 0, \quad \delta^2(\theta)|_{\theta=0} = 0. \quad (\text{B.15})$$

The Feynman rules for the model (8) in the supersymmetric Feynman gauge are as follows:

$$\begin{aligned}
\Gamma_\alpha(\theta_1) \overset{p}{\rightsquigarrow} \Gamma^\beta(\theta_2) &\Rightarrow \frac{-ip_\alpha^\beta}{p^2 + i0} \delta^2(\theta_2 - \theta_1), \\
\Phi_j(\theta_1) \overset{p}{\longrightarrow} \Phi_k^*(\theta_2) &\Rightarrow i\delta_{jk} \frac{D^2 + m}{p^2 - m^2 + i0} \delta^2(\theta_2 - \theta_1), \\
\begin{array}{c} \Gamma_\alpha \\ \vdots \\ \Phi_j^* \xleftarrow{p} \Phi_k \end{array} &\Rightarrow \frac{1}{2} g \delta_{jk} [D^\alpha(-q) - D^\alpha(p)], \\
\begin{array}{c} \Gamma^\alpha \quad \Gamma^\beta \\ \diagdown \quad \diagup \\ \Phi_j^* \quad \Phi_k \end{array} &\Rightarrow -ig^2 \delta_{jk} \delta_\alpha^\beta, \\
\begin{array}{c} \Phi_j^* \quad \Phi_i \\ \diagdown \quad \diagup \\ \Phi_{k'}^* \quad \Phi_k \end{array} &\Rightarrow \frac{1}{2} i\eta (\delta_{j'i} \delta_{k'k} + \delta_{j'k} \delta_{k'i}).
\end{aligned}$$

Appendix C: The N=2 supersymmetry algebra in three dimensions

In the $N = 2$ case the spinor co-ordinates of superspace are complex. The commutation relations (18) for the super-covariant derivatives lead to the following identities in momentum representation:

$$\begin{aligned}
D^\alpha D_\beta &= \delta_\beta^\alpha D^2, \quad D_\alpha D^\beta = -\delta_\alpha^\beta D^2, \quad D_\alpha D_\beta D_\gamma = 0, \\
\bar{D}^\alpha \bar{D}_\beta &= \delta_\beta^\alpha \bar{D}^2, \quad \bar{D}_\alpha \bar{D}^\beta = -\delta_\alpha^\beta \bar{D}^2, \quad \bar{D}_\alpha \bar{D}_\beta \bar{D}_\gamma = 0, \\
\bar{D}^\alpha D_\alpha &= D^\alpha \bar{D}_\alpha, \quad D^\alpha p_\alpha^\beta \bar{D}_\beta = -2p^2 - \bar{D}^\alpha p_\alpha^\beta D_\beta, \\
[D_\alpha, \bar{D}^2] &= -p_\alpha^\beta \bar{D}_\beta, \quad [D^\alpha, \bar{D}^2] = \bar{D}^\beta p_\beta^\alpha, \\
[D^2, \bar{D}^\alpha] &= -D^\beta p_\beta^\alpha, \quad [D^2, \bar{D}_\alpha] = p_\alpha^\beta D_\beta, \\
D^\alpha \bar{D}^\beta D_\beta &= -\bar{D}^\alpha D^2 + D^\beta p_\beta^\alpha, \quad D_\alpha \bar{D}^\beta D_\beta = -\bar{D}_\alpha D^2 - p_\alpha^\beta D^\beta, \\
[D^2, \bar{D}^2] &= p^2 + \bar{D}^\alpha p_\alpha^\beta D_\beta, \quad D^2 \cdot \bar{D}^\alpha D_\alpha = 0, \quad \bar{D}^\alpha D_\alpha \cdot \bar{D}^2 = 0, \\
D^\alpha \bar{D}^2 D_\alpha &= \bar{D}^\alpha p_\alpha^\beta D_\beta - 2\bar{D}^2 D^2, \quad \bar{D}^\alpha D_\alpha \cdot \bar{D}^\beta D_\beta = -\bar{D}^\alpha p_\alpha^\beta D_\beta - 2\bar{D}^2 D^2, \\
\bar{D}^\alpha p_\alpha^\beta D_\beta \cdot \bar{D}^\gamma D_\gamma &= \bar{D}^\gamma D_\gamma \cdot \bar{D}^\alpha p_\alpha^\beta D_\beta = -p^2 \bar{D}^\alpha D_\alpha, \\
D_\alpha \bar{D}^2 D^2 &= -p_\alpha^\beta \bar{D}_\beta D^2, \quad D^\alpha \bar{D}^2 D^2 = \bar{D}^\beta D^2 p_\beta^\alpha, \\
\bar{D}^2 D^2 \bar{D}_\alpha &= p_\alpha^\beta \bar{D}_\beta D^2, \quad \bar{D}^2 D^2 \bar{D}^\alpha = -\bar{D}^2 D^\beta D^2 p_\beta^\alpha, \\
\bar{D}^\alpha D_\alpha \cdot \bar{D}^\beta D_\beta \cdot \bar{D}^\gamma D_\gamma &= p^2 \bar{D}^\alpha D_\alpha, \quad D^2 \bar{D}^2 D^2 = p^2 D^2, \quad \bar{D}^2 D^2 \bar{D}^2 = p^2 \bar{D}^2.
\end{aligned}$$

Integration over chiral subspaces and over the full superspace is defined as

$$\int d^3x d^2\theta (\dots) = \int d^3x D^2 (\dots), \quad \int d^3x d^2\bar{\theta} (\dots) = \int d^3x \bar{D}^2 (\dots),$$

$$\int d^3x d^4\theta (\dots) = \int d^3x d^2\theta d^2\bar{\theta} (\dots) = \int d^3x D^2 \bar{D}^2 (\dots),$$

with the δ function in the spinor variables given by

$$\int d^4\theta \delta^4(\theta - \phi) F(\theta, \bar{\theta}) = F(\phi, \bar{\phi}).$$

Integration by parts is possible for D_α and $\bar{D}_\alpha$.

In order to have for the chiral superfields ($\bar{D}_\alpha \Phi = 0$, $\bar{D}_\alpha J = 0$) a natural variational identity

$$\frac{\delta}{\delta J(x, \theta)} \int d^3y d^2\phi J(y, \phi) \Phi(y, \phi) = \Phi(x, \theta),$$

one defines

$$\frac{\delta J(y, \phi)}{\delta J(x, \theta)} = \bar{D}^2 \delta^3(x - y) \delta^4(\theta - \phi).$$

To convert from integrals over the full superspace to integrals over its chiral subspace, the following transformations may be useful:

$$\begin{aligned} \int d^3x d^4\theta F D^2 G &= \int d^3x \bar{D}^2 D^2 (F D^2 G) = \int d^3x \bar{D}^2 [(D^2 F) D^2 G] \\ &= \int d^3x d^2\bar{\theta} (D^2 F) D^2 G, \\ \int d^3x d^2\theta F &= \int d^3x D^2 F = \int d^3x D^2 \bar{D}^2 D^2 \square^{-1} F \\ &= \int d^3x d^2\bar{\theta} D^2 \square^{-1} F. \end{aligned}$$

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