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MULTIPHOTON RADIATION IN QUANTUM BEAMSTRAHLUNG

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ABSTRACT

A detailed analysis of the dominant piece of the two-photon yield in electron-positron bunch collisions at high energy is used to obtain a factorization property which gives directly the leading contribution to all orders. This provides a probabilistic approach to the global radiation yield in beamstrahlung.

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1. INTRODUCTION

The special properties of beam radiation in the deep quantum regime have been recently the object of detailed studies [1,2]. This is a regime into which one is driven with linear electron-positron colliders operating in the TeV regime at the appropriate luminosity [3]. Special behaviour results from the fact that the radiative quantum length $L_e = \frac{\hbar\gamma}{mc}$ becomes much larger than the classical coherent radiation length L_c , which is well known in synchrotron radiation [4]. The opposite prevails in the usual, classical regime. Important radiation losses occur and the photon spectrum is hard. With the intense field present inside the colliding bunches, this altogether gives rise to an intense production of pairs [5,6]. This puts serious constraints on machine design.

It has been shown that the dominant contribution to the radiation rate is proportional to the bunch length and can be simply expressed in terms of a new coherence length $\ell_c = (L_c^2 L_e)^{2/3}$, namely [1]

$$I(X) = K \frac{\alpha}{\pi} \frac{L_b}{\ell_c} \left(\frac{1-X}{X} \right)^{2/3} \quad (1.1)$$

Here X is the fraction of the incident electron energy taken by the photon, L_b is the length of the bunch and K is a numerical factor of order one [1,2]. We give the expression of the radiation flux in the Klein-Gordon case which will be used throughout as a paradigm. In the high-energy regime, where helicity flip is negligible, one obtains the realistic Dirac case results by a simple multiplication of the differential yield (1.1) by an extra factor $\frac{1}{2}(1 + (1-X)^2)/(1-X)$ [1]. Using the Dirac equation complicates the algebra whilst the physics issues most simply appear in the Klein-Gordon case.

The important fact is that the radiation rate, calculated to lowest order in the distorted wave approximation which is followed, is proportional to $\frac{\alpha}{\pi} \frac{L_b}{\ell_c}$. The rather large value of $\frac{L_b}{\ell_c}$, which is typically of the order of 50, partly compensates the natural smallness of $\frac{\alpha}{\pi}$. The mean energy loss typically exceeds 10 % [1,2].

The intense hard photon fluxes thus obtained have been first considered a bonus [7]. The abundant pair production which they also create raises, however, a serious problem [5,6]. Since $\frac{\alpha}{\pi} \frac{L_b}{\ell_c}$ is not a very small quantity, the importance of the radiation probability raises the question of multiphoton radiation. Finding a considerable value for the mean radiative loss at lowest order, it is natural to investigate the global photon flux to higher orders. It is important to assess the properties of the full photon spectrum originating from bunch crossing.

Whilst it is not necessary to calculate to all orders in α , one should definitely consider the higher order terms in $(\alpha \frac{L_b}{\ell_c})$, which is the relevant expansion parameter (1.1).

In this paper, we investigate in detail the two-photon process. Calculating the radiation yield to order $(\alpha \frac{L_b}{\ell_c})^2$ we come across general properties which can be readily generalized to all orders. This supports a probabilistic approach which can then be used to derive the dominant properties of the radiation spectrum.

The calculation of the two-photon rate is used as a test case to obtain factorization properties which can be directly used later. Whilst this paper mainly focuses on two-photon production, the direct study of the spectrum, using the results now presented, is deferred to another paper [8].

The method used for the two-photon case is the direct generalization of the one followed earlier in the one-photon case [1]. This has also been investigated recently by Jedrzejczak and Kurek [9]. We reproduce on the way some of their findings but branch off with a different approach to our final results.

There are several different contributions to consider in a second order calculation. However, since we are only after terms of order $(\alpha \frac{L_b}{\ell_c})^2$, it is clear that the diagram of Fig. 1a, together with the one obtained by switching the two photons, contain the dominant contribution. Two photons, taking respectively fraction X and X' of the incident electron energy, are successively emitted. The seagull term, present in the Klein-Gordon case, would give a contribution of order $\alpha^2 \frac{L_b}{\ell_c}$ only, and can be neglected.

One should not forget, however, that at order $(\alpha \frac{L_b}{\ell_c})^2$, the one-photon yield is also affected. In this case, one should consider more particularly the diagrams of Fig. 1b, whilst other diagrams with a virtual photon (Fig. 1c) contribute only to order $\alpha^2 \frac{L_b}{\ell_c}$ to the single photon yield. This separation according to cutting rules supports our probabilistic calculation of the full radiation rate (one and two photon) to order $(\alpha \frac{L_b}{\ell_c})^2$, which can be then generalized to higher orders [8]. One finds that the mean global energy loss hardly changes when going from first to second order in $(\alpha \frac{L_b}{\ell_c})$! The sizeable energy flux associated with the two-photon yield is practically compensated by the reduction of the single photon flux when calculated to order $\alpha(\frac{L_b}{\ell_c})^2$.

Our analytic approach corresponds to a small disruption approximation. The key small parameter is [1]

$$\epsilon = \frac{N\alpha}{R^2 k_i} L_b \quad (1.2)$$

Here N is the number of particles in the bunch of radius R and k_i the incident electron

energy. This parameter ϵ is dimensionless, with a typical value of 0.1. In the calculation we keep terms linear in ϵ but drop those of order ϵ^2 . The maximum bending momentum communicated by the bunch is $\frac{2N\alpha}{R}$ [1]. The focal length f is therefore $\frac{R^2 k_i}{2N\alpha}$. The disruption parameter D is defined as $\frac{L_b}{f}$. It is equal to 2ϵ . All analytical calculations done so far are limited to linear terms in D [1,2].

We use a simple cylindrical and uniform bunch to best illustrate the physical issues. As done in the one-photon case [10,11], the approach can be generalized to more realistic bunch shapes. Calculations are done in the rest frame of the bunch. In the low ϵ approximation, electrons in a bunch do not react on one another in an appreciable way as they traverse a target bunch. The incident particles radiate independently as they are bent by the strong field.

This paper is organized as follows. In Section 2 we calculate the electron propagator in the bunch field. This is an essential element of the calculation of the graph of Fig. 1a. Section 3 is a study of the stationary phase conditions which provide some space localization of the radiation. In Section 4 we calculate the production amplitude, borrowing from our previous work [1,10]. The radiation rate associated with this amplitude is discussed in Section 5. This is an introduction to a probabilistic approach valid to higher orders. The calculation technique is sketched in the Appendix. Section 6 gives the conclusions and an example.

2. THE ELECTRON PROPAGATOR IN THE INTENSE BUNCH FIELD

The electron propagator $D(x)$, which also depends on x_0 , satisfies the equation:

$$\left((E - eV)^2 + \vec{\nabla}^2 - m^2 \right) D(x) = \delta^3(x - x_0) \quad (2.1)$$

Here E is the energy of the electron with mass m and V the potential in the bunch [1].

We write the propagator as ($r = |x - x_0|$):

$$D(x) = \frac{1}{4\pi r} e^{i\phi(x)} A(x), \quad (2.2)$$

with

$$\phi(x) = Er - \eta(x) \quad (2.3)$$

neglecting the electron mass in front of E and not mentioning explicitly the x_0 dependence.

Following the optical approximation [1], we write

$$(E - eV)^2 - (\vec{\nabla}\phi)^2 - m^2 = 0 \quad (2.4)$$

Using (2.2) and (2.4) in (2.1), we obtain the equation:

$$(\vec{\nabla}^2\phi - i\vec{\nabla}^2 - \frac{2}{r}\frac{\partial}{\partial r}\phi + 2i\frac{1}{r}\frac{\partial}{\partial r} + 2\vec{\nabla}\phi \cdot \vec{\nabla})A = 0 \quad (2.5)$$

In order to solve it, we write η and A to first order in E^{-1} as:

$$\begin{aligned} \eta &= \eta_0 + \frac{\eta_1}{E} \\ A &= A_0 + \frac{A_1}{E} \end{aligned} \quad (2.6)$$

and derive the equations:

$$\begin{aligned} \frac{\partial\eta_0}{\partial r} &= eV \\ \frac{\partial\eta_1}{\partial r} &= \frac{1}{2}(\vec{\nabla}\eta_0)^2 - \frac{1}{2}e^2V^2 \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} A_0 &= 1 \\ \frac{\partial A_1}{\partial r} &= \frac{1}{2}(\vec{\nabla}^2\eta_0 - \frac{2}{r}\frac{\partial\eta_0}{\partial r}) \end{aligned} \quad (2.8)$$

For cylindrical, uniform bunches [1], these equations can be integrated explicitly to give:

$$\eta_0 = \frac{1}{3} \frac{N\alpha}{L_b R^2} r ((x_\perp - x_{0\perp})^2 + 3x_\perp \cdot x_{0\perp}) \quad (2.9)$$

$$\eta_1 = \frac{1}{2} \left(\frac{1}{3} \frac{N\alpha}{L_b R^2} \right)^2 r \left\{ 3(x_{\perp 0} \wedge x_\perp)^2 + 3(z - z_0)^2 (x_{0\perp} \cdot x_\perp + \frac{4}{15}(x_\perp - x_{0\perp})^2) \right\} \quad (2.10)$$

Here $x_\perp$ corresponds to the co-ordinate orthogonal to the bunch axis, which defines the z axis.

From (2.8) and (2.9), one then obtains

$$A_1 = \frac{N\alpha}{3L_b R^2} r^2 \quad (2.11)$$

The quantity $\frac{A_1}{E}$ in (2.6) is only of order ϵ , and therefore neglected in front of A_0 . For the same low bending reason, we can approximate η_0 and $\frac{\eta_1}{E}$ by, for $z > z_0$,

$$\eta_0 = \frac{1}{3} \frac{N\alpha}{L_b R^2} (z - z_0) ((x_\perp - x_{0\perp})^2 + 3x_\perp \cdot x_{0\perp}) \quad (2.12)$$

and

$$\frac{\eta_1}{E} = \frac{1}{6} \frac{N\alpha\epsilon}{1-X} (z - z_0)^3 \frac{1}{L_b^3 R^2} (x_{0\perp} \cdot x_\perp) \quad (2.13)$$

As in the one-photon case [1], we have to keep terms of order ϵ since the leading terms will cancel against one another in the expression for the phase of the amplitude.

We can now collect all terms and write the electron propagator between points x and x' . It reads ($r = |x' - x|$):

$$D = \frac{1}{4\pi r} \exp i \left\{ Er - \frac{N\alpha}{6} (\tilde{z}' - \tilde{z}) ((\tilde{x}'_\perp - \tilde{x}_\perp)^2 + 3\tilde{x}'_\perp \cdot \tilde{x}_\perp) - \frac{1}{48} \frac{N\alpha\epsilon}{1-X} (\tilde{z}' - \tilde{z})^3 (\tilde{x}_\perp \cdot \tilde{x}'_\perp) \right\} \quad (2.14)$$

We have introduced scaled quantities which are convenient to use, namely:

$$z = \frac{L_b}{2} \tilde{z} \quad (-1 \leq \tilde{z} \leq 1) \quad (2.15)$$

and

$$x_\perp = R\tilde{x}_\perp \quad (-1 \leq |\tilde{x}_\perp| \leq 1) \quad (2.16)$$

3. THE STATIONARY PHASE CONDITIONS

It is clear that the amplitude, which is an integral over x and x' , will collect most of its value from the zones where the phase is stationary. We have therefore to first concentrate on the phase. Denoting by ϕ_0 the phase of the propagator, the phase of the amplitude is:

$$\phi = \phi_i + \phi_0 - \phi_f - k \cdot x - k' \cdot x' \quad (3.1)$$

where ϕ_i is the phase of the incoming electron wave function at point x and ϕ_f is the phase of the outgoing electron wave function at point x' .

This assumes an ordered emission where the photon with momentum k is emitted first whilst the photon with momentum k' is emitted later, after some bending of the electron trajectory. We shall later integrate over $\tilde{z}$ and $\tilde{z}'$ from -1 to +1 but keeping this $\tilde{z}' > \tilde{z}$ ordering in mind when computing the rate since this condition puts limits on the k and k' integrations. Interferences are negligible to order $(\alpha \frac{L_b}{\ell_c})^2$, since they contribute to order $\frac{\alpha^2 L_b}{\ell_c}$ only.

The values of ϕ_i and ϕ_f are taken from Ref. [1].

$$\begin{aligned} \phi_i &= k_i z - \frac{N\alpha}{2} \tilde{x}_\perp^2 (\tilde{z} + 1) \left(1 + \frac{1}{6} \epsilon (\tilde{z} + 1)^2 \right) \\ \phi_f &= k_f \cdot x' + \frac{N\alpha}{2} \tilde{x}_\perp'^2 (1 - \tilde{z}') + \frac{k_{f\perp} \cdot x'_\perp}{4k_f} (k_i \epsilon) (1 - \tilde{z}')^2 \\ &\quad + \frac{1}{12k_f} \frac{x_\perp'^2}{L_b} (k_i \epsilon)^2 (1 - \tilde{z}')^3 \end{aligned} \quad (3.2)$$

Since the maximum bending momentum is $\frac{2N\alpha}{R}$, we scale all transverse momenta accordingly, namely

$$k_\perp = \tilde{k}_\perp \frac{2N\alpha}{R} \quad (3.3)$$

and rewrite ϕ_f (3.2) as:

$$\begin{aligned} \phi_f &= k_f \cdot x' + \frac{N\alpha}{2} \tilde{x}_\perp'^2 (1 - \tilde{z}') + \frac{N\alpha\epsilon}{2(1 - X - X')} \tilde{k}_{f\perp} \cdot \tilde{x}'_\perp (1 - \tilde{z}')^2 \\ &\quad + \frac{N\alpha\epsilon}{12(1 - X - X')} \tilde{x}_\perp'^2 (1 - \tilde{z}')^3 \end{aligned} \quad (3.4)$$

As previously said, we keep terms of order ϵ , since all leading terms will cancel against one another in the overall phase (3.1).

The electron and the photons come out with relatively low transverse momenta as compared to their energy. We thus write, neglecting the electron mass

$$\begin{aligned} k_{fz} &= k_f \left(1 - \frac{k_{f\perp}^2}{2k_f^2} \right) \\ k_z &= k \left(1 - \frac{k_\perp^2}{2k^2} \right) \\ k'_z &= k' \left(1 - \frac{k'^2_\perp}{2k'^2} \right) \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} k &= X k_i \\ k' &= X' k_i \\ k_f &= (1 - X - X') k_i \end{aligned} \tag{3.6}$$

We can collect first all terms of order k_i in (3.1). They cancel. This justifies the ordering imposed when writing the amplitude with the expression obtained from the propagator. The other choice leads to a strong oscillation of the amplitude and a vanishing result upon integration.

We now need to write the remaining expression for the phase. Since all quantities are now scaled ((2.15),(2.16),(3.3)), we simplify the notations not indicating scaling through a twiddle sign over the variable, as done so far. Collecting all terms, we have:

$$\begin{aligned} \phi &= \frac{k_f^2}{1 - X - X'} N \alpha \epsilon z' - 2N \alpha (k_{f\perp} x'_\perp + k_\perp x_\perp + k'_\perp x'_\perp) \\ &+ (1 - X) \frac{N \alpha}{\epsilon} \frac{(x'_\perp - x_\perp)^2}{z' - z} + \frac{N \alpha \epsilon}{X} k_\perp^2 z + \frac{N \alpha \epsilon}{X'} k'^2_\perp z' \\ &+ \frac{N \alpha \epsilon}{1 - X - X'} k_{f\perp} x'_\perp (1 - z')^2 - \frac{N \alpha}{2} (x_\perp^2 (1 + z) + x'^2_\perp (1 - z')) \\ &- \frac{N \alpha \epsilon}{12} \left(x_\perp^2 (1 + z)^3 + \frac{1}{1 - X - X'} x'^2_\perp (1 - z')^3 \right) \\ &- \frac{N \alpha}{2} (z' - z) (x'_\perp x_\perp) \\ &- \frac{1}{48} \frac{N \alpha \epsilon}{1 - X} (x_\perp \cdot x'_\perp) (z' - z)^3 \end{aligned} \tag{3.7}$$

We remark that $|x'_\perp - x_\perp|/|z' - z|$ is of order ϵ . We thus have only terms of orders $N\alpha$ and $N\alpha\epsilon$.

We first impose stationary conditions with respect to the radial variables.

$$\frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial\phi}{\partial y} = 0 \quad (3.8)$$

$$N\alpha \left\{ -(1-X) \frac{2}{\epsilon} \frac{x-x'}{z'-z} - 2k_x - x(1+z) - \frac{1}{6}\epsilon x(1+z)^3 - \frac{1}{2}(z'-z)x' - \frac{\epsilon}{48} \frac{1}{1-X} (z'-z)^3 x' \right\} = 0 \quad (3.9)$$

with an identical relation for y . We also impose

$$\frac{\partial\phi}{\partial x'} = 0 \quad \frac{\partial\phi}{\partial y'} = 0 \quad (3.10)$$

$$N\alpha \left\{ (1-X) \frac{2}{\epsilon} \frac{x-x'}{z-z'} - 2k_{fx} - 2k'_x - x'(1-z') - \frac{\epsilon}{1-X-X'} k_{fx}(1-z')^2 - \frac{\epsilon x'}{6(1-X-X')} (1-z')^3 - \frac{1}{2}(z'-z)x - \frac{\epsilon}{48} \frac{1}{1-X} (z'-z)^3 x \right\} = 0 \quad (3.11)$$

The values of the final momenta define an impact parameter $x_{0\perp}$ and the actual radiation zones are at a transverse distance from it, which differ only by a quantity of order ϵ . We can combine (3.9) and (3.11) and drop terms of order ϵ to calculate x_0 and y_0 . This gives:

$$x_0 = \Delta_x \quad y_0 = \Delta_y \quad (3.12)$$

$$\Delta = -(k_{f\perp} + k_\perp + k'_\perp) \quad (3.13)$$

where Δ is the momentum transferred to the bunch. Δ is simply $r_\perp/R$ where $r_\perp$ is the classical impact parameter.

We then write

$$\begin{aligned} x_\perp &= x_{0\perp} + \epsilon\delta_\perp \\ x'_\perp &= x_{0\perp} + \epsilon\delta'_\perp \end{aligned} \quad (3.14)$$

We can now substitute (3.14) in (3.7) and use (3.12). All z or z' dependent terms which are of order $N\alpha$ cancel. We are left with leading terms which are of order $N\alpha\epsilon$. They imply $\delta_\perp$ and $\delta'_\perp$ only in the combination $(\delta'_\perp - \delta_\perp)$. Terms including $(\delta'_\perp + \delta_\perp)$ carry an extra ϵ factor and can be dropped.

We therefore need to calculate only $\delta' - \delta$. This is easily done from (3.9) and (3.12), with the result:

$$(\delta'_\perp - \delta_\perp) = -\frac{\Delta(1 + \frac{z+z'}{2}) + 2k_\perp}{2(1-X)}(z' - z) \quad (3.15)$$

The phase of the amplitude can be written as:

$$\phi = N\alpha\Delta^2 + \varphi \quad (3.16)$$

The expression for φ has terms which depend separately on z and z' but also terms which depend on zz' . It turns out that, using (3.15), the latter ones all cancel and the phase depends separately on z and z' . It reads:

$$\begin{aligned} \phi = N\alpha \Bigg\{ & \Delta^2 + \epsilon \left(-\frac{(1+z)^3}{12}\Delta^2 + \frac{k_\perp^2}{X}z \right. \\ & + \frac{1}{1-X}(k_\perp^2 z + \Delta k_\perp \left(z + \frac{z^2}{2} \right) + \frac{\Delta^2}{4}(z + z^2 + \frac{z^3}{3})) \\ & + \frac{k_{f\perp}^2}{1-X-X'}z' + \frac{k_\perp'^2}{X'}z' - \frac{k_{f\perp}\Delta}{2(1-X-X')}(1-z')^2 - \frac{1}{12}\Delta^2 \frac{(1-z')^3}{1-X-X'} \\ & \left. - \frac{1}{1-X}(k_\perp^2 z' + \Delta k_\perp (z' + \frac{z'^2}{2}) + \frac{\Delta^2}{4}(z' + z'^2 + \frac{1}{3}z'^3)) \right\} \end{aligned} \quad (3.17)$$

This separation of the z and z' dependence translates the presence of two successive radiation processes which separately take place along the electron trajectory. It corresponds, of course, to our calculating only terms of order $(\alpha \frac{L_b}{L_c})^2$. This important separation property was found by Jedrzejczak and Kurek [9]. It becomes intuitive only after the hard calculation is done.

The next step is to write the other stationary phase conditions, one for z and one for z' . One finds for z :

$$\frac{\partial \phi}{\partial z} = \frac{N\alpha\epsilon}{4X(1-X)}(\Delta X(1+z) + 2k_\perp)^2 \quad (3.18)$$

The condition for z' is simplified when one introduces the variables X'' and $q_\perp$, with

$$X'' = \frac{X'}{1-X} \quad (3.19)$$

$$q_{\perp} = (k_{\perp}X'' + k'_{\perp}) \quad (3.20)$$

They correspond respectively to the fraction of the remaining electron energy taken by the second photon and to our considering its transverse momentum with respect to the electron classical bending plane. It indeed “freely” collects a transverse momentum $-k_{\perp}X''$ from the first electron recoil. One then finds:

$$\frac{\partial \phi}{\partial z'} = \frac{N\alpha\epsilon}{1-X} \frac{1}{4X''(1-X'')} (\Delta X''(1+z') + 2q_{\perp})^2 \quad (3.21)$$

The two conditions are identical, as indeed expected when only taking into account the past history of the electron, with its radiation of a first photon, resulting in a loss of energy and a transverse kick.

Each condition is also identical to the one obtained in the single photon case [1]. This separation of the stationarity conditions and their identical form when taking into account the history of the process clearly extends to higher orders in $(\alpha \frac{L_b}{\ell_c})$. As discussed in Ref. [1] the stationary conditions cannot be fully met. One finds that one has to approximate ϕ in the neighbourhood of the point where the second derivative vanishes

$$Z = -1 - \frac{2k_{\perp} \cdot \Delta}{X\Delta^2} \quad (3.22)$$

There is only one value of z satisfying (3.22), when we would need two separate ones, to cancel both terms in (3.18) which is the square of a vector. The integration of (3.18) therefore gives an odd cubic in $(z - Z)$, namely

$$\varphi = \frac{N\alpha\epsilon}{4X(1-X)} \left\{ \frac{\Delta^2 X^2}{3} (z - Z)^3 + \frac{4}{\Delta^2} (k_{\perp}^2 \Delta^2 - (k_{\perp} \cdot \Delta)^2) (z - Z) \right\} \quad (3.23)$$

up to an irrelevant constant term. We can write it as:

$$\varphi = c_3(z - Z)^3 + c_1(z - Z)$$

with

$$\begin{aligned} c_3 &= \frac{N\alpha\epsilon X}{12(1-X)} \Delta^2 \\ c_1 &= \frac{N\alpha\epsilon}{X(1-X)\Delta^2} (k_{\perp}^2 \Delta^2 - (k_{\perp} \cdot \Delta)^2) \end{aligned} \quad (3.24)$$

This takes care of the z dependence of the phase near the stationary point Z . We have a similar behaviour near the stationary point Z' , substituting X'' for X , $\epsilon/(1-X)$ for ϵ and $q_{\perp}$ for $k_{\perp}$ in (3.24).

The amplitude receives the main contribution originating from the neighbourhood of Z and Z' respectively. Each relevant part of the integral is practically limited to a range corresponding to the inverse of the third root of the coefficient of the cubic term. In the first case, we get the coherent length ℓ_c , as previously defined in the one-photon case [1]. For the second radiation, the relevant coherent length is $\ell_c(1 - X)^{1/3}$. We have indeed $\ell_c \sim k_i^{+1/3}$ [1]. The radiation rate per unit length is accordingly increased. Going to higher orders without special care we are driven out of the low disruption regime in which calculations are done.

4. THE PRODUCTION AMPLITUDE

We now calculate the matrix elements for the radiation. From our previous discussion we can calculate them separately in the neighbourhood of Z and Z' and take the product. We consider photon polarizations in the x and y planes, respectively.

The orthogonal polarization vectors are written as:

$$\begin{aligned}\epsilon_x &= (1, 0, -\frac{k_x}{Xk_i}) \\ \epsilon_y &= (0, 1, -\frac{k_y}{Xk_i})\end{aligned}\tag{4.1}$$

They are thus normal to the photon momentum, with components (k_x, k_y, Xk_i) .

The current is calculated in the low disruption approximation. We take a linear approximation for the bending to write the current at z (and $x_{0\perp}$). Since the current has both transverse and longitudinal components, it is now clearer not to use scaled variables, unless explicitly written. One has:

$$j(x) = ((k_{fx} + k'_x)(1 + \tilde{z}) + k_x \tilde{z}, (k_{fy} + k'_y)(1 + \tilde{z}) + k_y \tilde{z}, (2 - X)k_i)\tag{4.2}$$

Similarly, we calculate the current at point $z'(x_{0\perp})$

$$\begin{aligned}j(x') &= \\ (k_{fx}(1 + \tilde{z}') - k_x(1 - \tilde{z}') + k'_x \tilde{z}', & \quad k_{fy}(1 + \tilde{z}') - k_y(1 - \tilde{z}') + k'_y \tilde{z}', \quad (1 - X)(2 - X'')k_i)\end{aligned}\tag{4.3}$$

The radiation matrix elements associated with the first emission are:

$$\begin{aligned}\mathcal{M}_x^{(1)} &= -|\Delta|(\tilde{z} - \tilde{Z}) \\ \mathcal{M}_y^{(1)} &= -2\frac{k_y}{X}\end{aligned}\tag{4.4}$$

We have chosen the bunch recoil momentum Δ in the x plane ($\Delta_y = 0$). This is the

classical bending plane and we can accordingly neglect k_y in front of k_x since their typical ratio is of order $\frac{\ell_c}{L_b}$ [1]. This has been done in (4.4).

The radiation matrix elements associated with the second emission are now calculated with polarization vectors

$$\begin{aligned}\epsilon'_x &= (1, 0, -\frac{k'_x}{X'k_i}) \\ \epsilon'_y &= (0, 1, -\frac{k'_y}{X'k_i})\end{aligned}\tag{4.5}$$

The first matrix element now reads:

$$\mathcal{M}_x^{(2)} = -\frac{1}{X}(\Delta_x(1 + \tilde{z}')X'' + 2k_xX'' + 2k'_x)$$

We use (3.20) and the value of $\tilde{Z}'$ (3.22)

$$\tilde{Z}' = -1 - \frac{2q_\perp \cdot \Delta}{X''\Delta^2}\tag{4.6}$$

and, neglecting again components orthogonal to the bending plane in front of components in the bending plane, we find:

$$\mathcal{M}_x^{(2)} = -|\Delta|(\tilde{z}' - \tilde{Z}')\tag{4.7}$$

The second matrix element is using (3.20):

$$\mathcal{M}_y^{(2)} = -2\frac{q_y}{X''}\tag{4.8}$$

The different matrix elements have, as expected, the same form, when expressed in corresponding variables. The two radiation processes occur independently of each other. The first one reflects on the second one only through a change in the electron direction and energy.

The radiation amplitudes are obtained as integrals over the two sets of variables $x_\perp, z$ and $x'_\perp, z'$. They read:

$$\mathcal{M}_{ij} = e^2 \int \frac{1}{4\pi|\vec{x}'' - \vec{x}|} e^{i\phi} \mathcal{M}_i^{(1)} \mathcal{M}_j^{(2)} d^3\vec{x} d^3\vec{x}'\tag{4.9}$$

The transverse integrals can be done using the stationary phase approximation. The stationary points are selected according to the final momenta ((3.12),(3.15)). Their separation is of order ϵR . To leading order in ϵ one can separately integrate over $d^2x_\perp$ and over

$\epsilon^2 d^2(\delta'_\perp - \delta_\perp)$ with a determinant unity. We have to collect the leading quadratic terms in $x_\perp^2$ in the phase and also those of order $\epsilon(\delta'_\perp - \delta_\perp)^2$. We go back to (3.7) and find

$$-\frac{N\alpha}{2}(1+\tilde{z}) - \frac{N\alpha}{2}(1-\tilde{z}') - \frac{N\alpha}{2}(\tilde{z}' - \tilde{z}) = N\alpha,$$

for the coefficient of $x_\perp^2$, and $(1-X)\frac{N\alpha\epsilon}{\tilde{z}'-\tilde{z}}$ for the coefficient of $(\delta'_\perp - \delta_\perp)^2$. To leading order in ϵ , we can also take $|\vec{x} - \vec{x}'| = (\tilde{z}' - \tilde{z})L_b/2$.

Integrating over $d^2x_\perp$ and $\epsilon^2 d^2(\delta'_\perp - \delta_\perp)$, we fix the value of $x_\perp$ and $x'_\perp - x_\perp$ at the values used in the calculation of the matrix elements, which used the stationary conditions ((3.8),(3.10)), and simply collect extra factors associated with the coefficients of the quadratic terms. We can then write (4.9) as:

$$\begin{aligned} \mathcal{M}_{ij} = e^2 \int d\tilde{z} d\tilde{z}' \left(\frac{\pi R^2}{N\alpha} \right)^2 \frac{\epsilon}{1-X} \frac{1}{4\pi} \mathcal{M}_i^{(1)}(\tilde{z}) \mathcal{M}_i^{(2)}(\tilde{z}') \frac{L_b}{2} \\ \exp i \left(c_3(\tilde{z} - \tilde{Z})^3 + c_1(\tilde{z} - \tilde{Z}) + c'_3(\tilde{z}' - \tilde{Z}')^3 + c'_1(\tilde{z}' - \tilde{Z}') \right) \end{aligned} \quad (4.10)$$

where c_3, c_1, c'_3 and c'_1 are defined in (3.24), using the $X \rightarrow X''$ $\epsilon \rightarrow \frac{\epsilon}{1-X}$, $k_\perp \rightarrow q_\perp$ substitution in the latter case. The calculation can now proceed as in Ref. [1]. The z and z' integrals will give a product of an Airy function ($\mathcal{M}_y$) or of its derivative ($\mathcal{M}_x$). They provide a strong damping at large $k_y(\mathcal{M}^{(1)})$ and large $q_y(\mathcal{M}^{(2)})$ and practically limit the z and z' integrals to length $\ell_c(\mathcal{M}^{(1)})$ and $\ell_c(1-X)^{1/3}(\mathcal{M}^{(2)})$. The two integrals can be performed separately, if we limit ourselves to terms of order $(\alpha \frac{L_b}{\ell_c})^2$. They can be both extended to infinity, but we have to keep in mind the assumed ordering between z' and $z(z' > z)$. This eventually introduces a factor 1/2 in the rate, when integrating over phase space.

5. THE PRODUCTION RATE

From our previous discussion it follows that the modulus square of the amplitude, summed over polarizations, is simply the product of the ones obtained in the single photon case [1], times an extra factor

$$4 \left(\frac{\pi R^2}{N\alpha} \frac{\epsilon}{1-X} \frac{1}{4\pi} \right)^2$$

which we simply read from (4.10) remembering that, in the one-photon case, we only collect a factor $\frac{\pi R^2}{N\alpha}$ from the integral over the transverse co-ordinates. The extra factor 4 corresponds to the separate integrals over z and z' when we get twice the even part. The factor $\frac{1}{z'-z}$ in the propagator has been cancelled by a factor $z' - z$ appearing from the integral over $\delta'_\perp - \delta_\perp$ (3.15). This corresponds physically to the conservation of the electron flux as the electron radiates.

Since, in this leading term of order $(\alpha \frac{L_b}{\ell_c})^2$ the two photons are emitted a longitudinal distance of at least order ℓ_c away of each other, the corresponding bending of the electron and the practically tangential emission eliminate interference between the photons.

We denote by $|M_2|^2$ the matrix element thus obtained, when squared and summed over all polarizations. The radiation rate associated with the graph of Fig. 1a is then, to order $(\alpha \frac{L_b}{\ell_c})^2$:

$$\mathcal{J}_2 = \frac{1}{16k_i} \int |M_2|^2 (2\pi) \delta(k_i - k_f - k - k') \frac{d^3 \vec{k}_f d^3 \vec{k} d^3 \vec{k}'}{(2\pi)^9 k_f k k'} \quad (5.1)$$

We use the δ -function to integrate over the longitudinal momentum of the electron:

$$\mathcal{J}_2 = \frac{1}{16k_i^2} \frac{1}{(2\pi)^8} \int |M_2|^2 d^2 k_{f\perp} d^2 k_\perp d^2 q_\perp \frac{dX dX'}{XX'(1-X-X')} \quad (5.2)$$

This two-photon production rate has the dimension of an inverse momentum squared, as read from (4.7), (4.10) and (5.2). This is fine since it should be proportional to an area. If we normalize it to the bunch area πR^2 , the differential rate (probability of radiation) is a dimensionless quantity

$$I_2(X, X') = \frac{1}{\pi R^2} \frac{d\mathcal{J}_2}{dX dX'} = \frac{1}{\pi R^2} (1-X) \frac{d\mathcal{J}_2}{dX dX'} \quad (5.3)$$

It is natural to relate it to the product of two one-photon differential radiation rates [1]

$$I_1(X) = \frac{1}{\pi R^2} \frac{1}{8k_i^2} \frac{1}{(2\pi)^5} \int |M_1|^2 d^2 k_{f\perp} d^2 k_\perp \frac{1}{X(1-X)} \quad (5.4)$$

However, if we wish to express that in term of probabilities, we should not refer to the full

probabilities of emission, but rather to those pertaining to the same impact parameter.

The kinematic indeed fixes $\tilde{x}_\perp = \tilde{\Delta}$ (3.12) and the two successive emissions occur at the same value of $\tilde{x}_\perp$, up to a difference of order ϵ . One should then calculate

$$\begin{aligned}\mathcal{I}_2(X, X'', \tilde{r}_\perp) &= \frac{1}{2\tilde{\Delta}} \frac{dI_2(X, X'')}{d\tilde{\Delta}} \\ \mathcal{I}_1(X, \tilde{r}_\perp) &= \frac{1}{2\tilde{\Delta}} \frac{dI_1(X)}{d\tilde{\Delta}}\end{aligned}\tag{5.5}$$

thus obtaining, in both cases, the differential rate with respect to $\tilde{\Delta}$, and normalizing it to the relevant annular impact area $2\pi R^2 \tilde{\Delta} d\tilde{\Delta}$. We recall that $\tilde{\Delta}$ is equal to the scaled classical impact parameter $r/R = \tilde{r}$.

We also have to introduce specifically the one-photon emission rate when the electron has only energy $k_i(1 - X)$ instead of k_i . This is written as:

$$\mathcal{I}_1(X''; 1 - X, \tilde{r}_\perp) = \frac{1}{2\tilde{\Delta}} \frac{dI_1(X''; 1 - X)}{d\tilde{\Delta}}\tag{5.6}$$

In the Appendix, we shall obtain the simple, compact and intuitive result:

$$\mathcal{I}_2(X, X'', \tilde{r}_\perp) = \frac{1}{2} \mathcal{I}_1(X; 1, \tilde{r}_\perp) \mathcal{I}_1(X''; 1 - X, \tilde{r}_\perp)\tag{5.7}$$

The probability for two-photon emission is, up to a factor $1/2$, the product of the probabilities for two one photon processes, granting the fact that the electron loses energy in between and the emission probability changes accordingly. The extra factor $1/2$ comes from the integration over $k_\perp$ and $q_\perp$, taking into account the ordering $Z' > Z$. Since, in the dominant (order $(\alpha \frac{L_b}{\ell_c})^2$) contribution, the two photons are produced at a distance at least ℓ_c apart of each other, there is no interference.

Relation (5.7) is our main result and will be exploited in another paper [8]. It corresponds to the successive emission of two photons, and is readily generalizable to higher orders.

It should be emphasized that factorization holds only for each value of impact parameter.

The mean energy loss is given by

$$\delta_2 = \int I_2(X, X'') \left(X + \frac{X''}{1-X} \right) dX dX'' \quad (5.8)$$

with

$$I_2(X, X'') = \int_0^1 \mathcal{I}_2(X, X'', \tilde{r}_\perp) d\tilde{r}_\perp^2 \quad (5.9)$$

where X and X'' vary between 0 and 1.

6. CONCLUSIONS AND DISCUSSION

Whilst the factorization property looks intuitive, its derivation is far from being so since it relies on a mixed X and z history of the successive radiations, each successive radiation probability per unit length depending on the energy kept by the electron.

Computing the full radiation yield to order $(\alpha \frac{L_b}{\ell_c})^2$ we have to consider also corrections to the single photon rate, where one has now to impose that no extra radiation has occurred before and after the emission taking place at a particular point (up to ℓ_c) along the bunch.

As a concrete example let us consider the “super” case [3] where $L_b/\ell_c = 60$. In this case the one-photon calculation gives [1]

$$\delta_1 = 0.78 \frac{\alpha L_b}{\pi \ell_c} = 10.9\% \quad (6.1)$$

Using the factorization property it is readily found that

$$\delta_2 = 2.81(\delta_1)^2 = 3.3\% \quad (6.2)$$

But the diagrams of Fig. 1b, when combined with the one-photon diagram, give a reduction of

$$\delta'_1 = -3.2(\delta_1)^2 = -3.8\% \quad (6.3)$$

Therefore δ_2 and δ'_1 almost compensate each other giving only a very slight reduction. In this case

$$\delta_1 + \delta_2 + \delta'_1 = 10.4\%. \quad (6.4)$$

As previously said, the realistic Dirac case introduces only some extra X and X'' dependence when one neglects helicity flip. The factorization property (5.7) then also applies.

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APPENDIX

In this appendix we give some details of the derivation of the factorization property. The factorization property has been used for rates summed over polarization but it applies to each separate polarization configuration. Indeed, the different polarization rates differ only by a numerical factor [1]. We can therefore sketch the derivation of (5.7) using one particular polarization, taken as the transverse (y) one.

The space integral of the amplitude in our former one-photon case $\mathcal{M}_y$ gives an Airy function and one finds:

$$\int_{-\infty}^{+\infty} |\mathcal{M}_y|^2 dk_y = e^2 \left(\frac{\pi R^2}{N\alpha} \right)^2 L_b^2 \frac{8(1-X)(3/2)\tilde{\Delta}B}{(N\alpha\epsilon)(\frac{N\alpha X}{12(1-X)}\tilde{\Delta}^2)^{2/3}} \left(\frac{2N\alpha}{R} \right)^3 \quad (\text{A1})$$

with

$$B = \int_0^{\infty} (A_i(u^2))^2 u^2 du \quad (\text{A2})$$

whilst the production rate is given by (5.4). One introduces (A1) and (5.4) and simplifies the calculation making use of two relations which follow from the definition of ϵ (1.2) and ℓ_c [1].

$$\frac{L_b}{\epsilon N\alpha k_i} = \left(\frac{R}{N\alpha} \right)^2 \quad (\text{A3})$$

$$\frac{1}{k_i(N\alpha\epsilon)^{2/3}} = \frac{1}{\ell_c} \left(\frac{R}{N\alpha} \right)^2 \quad (\text{A4})$$

The integration over k_x gives a factor $X\Delta$ and the integration over $k_{f\perp}$ can be transformed into an integration over Δ , which varies from 0 to $\frac{2N\alpha}{R}$. This readily gives:

$$I_{1y}(X) = \frac{\alpha}{\pi} \frac{9}{\pi} (12)^{2/3} B \frac{L_b}{\ell_c} \left(\frac{1-X}{X} \right)^{2/3} \quad (\text{A5})$$

We may, however, stop short of the integration over $\tilde{\Delta} = \frac{R}{2N\alpha}\Delta$ and normalize to an area

$2\pi R^2 \tilde{\Delta} d\tilde{\Delta}$ instead of πR^2 . This gives:

$$\mathcal{I}_{1y}(X, \tilde{r}_\perp) = \frac{\alpha}{\pi} \left(\frac{8}{\pi} \right) (12)^{2/3} (3/2) B \left(\frac{1-X}{X} \right)^{2/3} \left(\frac{L_b}{\ell_c} \right) \tilde{\Delta}^{2/3} \quad (\text{A6})$$

$\tilde{r}_\perp$ is the classical impact parameter associated with the recoil momentum $\tilde{\Delta}$. The two-photon rate, with both polarization transverse, or along y can be written as (4.10)

$$\left| \mathcal{M}_{yy} \right|^2 = 4 \left| \mathcal{M}_y(1) \right|^2 \left| \mathcal{M}_y(2) \right|^2 \left(\frac{\epsilon}{1-X} \right)^2 \left(\frac{1}{4\pi} \right)^2 \quad (\text{A7})$$

Here the first term $\mathcal{M}(1)$ corresponds to an incident momentum k_i and a photon fractional momentum X , whilst the second one, $\mathcal{M}(2)$, corresponds to an incident momentum $k_i(1-X)$ and a fractional momentum X'' (3.19).

The rate is now given by (5.2). The integrals over k_y and q_y are performed according to (A1). We are left with integrals over k_x and q_x which give $\frac{1}{2} X X'' \Delta^2$. We combine the conditions (3.22) and (4.6) with the condition $Z' > Z$. The integral over $k_{f\perp}$ is replaced by an integral over Δ varying from 0 to $\frac{2N\alpha}{R}$.

We remark that the factor $(\frac{\epsilon}{1-X})^2$ readily introduces a term proportional to $(k_i^2(1-X)^2)^{-1}$ which, together with the terms in $\mathcal{M}(2)$, will lump all the corresponding k_i dependence into a term including $(\ell_c(1-X)^{1/3})^{-1}$. At the same time the factor k_i^{-2} appearing in front of the expression for the rate (5.2) will combine with terms involving k_i in $\mathcal{M}(1)$, lumping the full corresponding k_i dependence into a term $(\ell_c)^{-1}$.

The production rate with the ordering $Z' > Z$ reads:

$$I_{2yy}(X, X'') = \frac{1}{2} \frac{\alpha^2}{\pi^2} \left(\frac{9}{\pi} \right) \frac{16}{15} (12)^{4/3} B^2 \frac{L_b^2}{\ell_c^2 (1-X)^{1/3}} \left(\frac{1-X}{X} \right)^{2/3} \left(\frac{1-X''}{X''} \right)^{2/3} \quad (\text{A8})$$

There is no factorization but it appears when one writes instead:

$$\begin{aligned} \mathcal{I}_{2yy}(X, X'', \tilde{r}_\perp) &= \frac{1}{2} \left(\frac{\alpha}{\pi} \right)^2 \left(\frac{8}{\pi} \right)^2 \left(\frac{3}{2} \right) B^2 (12)^{4/3} \\ &\quad \left(\frac{1-X}{X} \right)^{2/3} \left(\frac{1-X''}{X''} \right)^{2/3} \frac{L_b}{\ell_c} \frac{L_b}{\ell_c (1-X)^{1/3}} \tilde{\Delta}^{4/3} \end{aligned} \quad (\text{A9})$$

Combining (A6) and (A9), one finds (5.7) since the same factorization property applies to all polarizations and to the rate summed over polarizations.

Whilst the derivation is rather involved, the final result is particularly simple and provide a direct probabilistic interpretation of successive radiations at fixed impact parameter.

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FIGURE CAPTIONS

1. a) Radiation of two photons by an incident electron of momentum k_i leaving the bunch with momentum k_f . The crosses stand for the interaction with the bunch field. The energies of the photons are denoted by $k = Xk_i$ and $k' = X'k_i$, with $k_f = (1 - X - X')k_i$. The bunch transfers momentum but no energy.
- b) A contribution to the radiative correction to order $(\alpha \frac{L_b}{\ell_c})^2$ to single photon emission, through its interference with the lowest order term. The imaginary part of that amplitude can be related to the probability for the electron not to have radiated up to a certain point on the trajectory, with a location defined up to ℓ_c .
- c) Vertex correction. It contributes to the rate only to order $\alpha^2 \frac{L_b}{\ell_c}$ through its interference with the lowest order diagram.

a)

b)

c)