ERRATUM

Due to a transmission error, the lines corresponding to Wick contractions were omitted from the following equations

SUPERMATRIX MODELS

$$(-)^o \underbrace{\Phi \Phi \dots \Phi}_{\text{}} (-)^o \underbrace{\Phi \Phi \dots \Phi}_{\text{}} (-)^o \underbrace{\Phi \Phi \dots \Phi}_{\text{}} \dots \quad (25)$$

$$\underbrace{\Phi \Phi \Phi \Phi \Phi \dots}_{\text{}} \quad (26)$$

$$\underbrace{\Phi \Phi}_{\text{}} \equiv \langle \Phi_{\mu\nu} \Phi_{e\lambda} \rangle = \frac{1}{2\beta} (-)^o \delta_{\lambda\mu} \delta_{\nu e} \quad (27)$$

$$\underbrace{\Phi_{\mu\nu} \dots \Phi_{e\lambda} \dots}_{\text{}} \rightarrow \dots \alpha_{\mu} \bar{\alpha}_{\nu} \dots \alpha_{\bar{e}} \bar{\alpha}_{\lambda} \dots \quad (31)$$

$$\underbrace{(\alpha\alpha)(\alpha\alpha) \dots (\alpha\alpha)}_{\text{}} (\bar{\alpha}\alpha)(\bar{\alpha}\alpha) \dots (\bar{\alpha}\alpha) \dots \quad (32)$$

$$(-)^{\# \text{fermionic loop indices}} \alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha \dots \quad (33)$$

ABSTRACT

We consider a very general class of supermatrix models. The result of our analysis is rather surprising but disappointing: all these models are strictly equivalent to theories based on ordinary matrices. This suggests that using supermatrices is not the right way to find a discretization of 2-D supergravity.

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SUPERMATRIX MODELS

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ABSTRACT

We consider a very general class of supermatrix models. The result of our analysis is rather surprising but disappointing: all these models are strictly equivalent to theories based on ordinary matrices. This suggests that using supermatrices is not the right way to find a discretization of 2-D supergravity.

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The double scaling limit [1],[2],[3] provides an efficient method to construct explicitly the sum over surfaces of arbitrary genus necessary in the calculation of string perturbation theory for non-critical strings in dimensions $d \leq 1$. The sum over triangulations performs very efficiently the integration over moduli space necessary to compute string amplitudes in the continuum limit for a given order of string perturbation theory. This is reasonable because the sum over triangulations in terms of large N matrix models gives an explicit procedure to sum over metrics modulo diffeomorphisms on surfaces of arbitrary topology (for details and references on the sum over triangulations and large N -limits see [4]). One might expect that similar arguments for supermoduli spaces should be possible. So far there is no construction of the lattice analogue of a superconformal model coupled to $2d$ -supergravity. In the continuum limit the two-dimensional supergeometry requires, together with the gravitational field, a gravitino field and an auxiliary fields [5]. In the same way that ordinary matrices provide triangulations of two-dimensional geometries, one might imagine that the sum over supermatrices should generate a discretization of supergeometries. In a $(N|1)$ -supermatrix the standard geometry would be represented by the $N \times N$ bosonic submatrix, and the gravitino and auxiliary fields could be represented by the other entries of the supermatrix. One can instead consider $(N|M)$ -supermatrices to simulate different fermionic geometries. In particular taking $N = M$ we would naively have the same number of degrees of freedom for the bosonic and fermionic coordinates of the supergeometry. This can be made manifest by carrying out the explicit integrations over supermatrices in components (see below). What is surprising is that the models based on supermatrices are completely equivalent to ordinary matrix models for $(N - M|0)$ -matrices. If we choose $N = M$ the theory is completely trivial and reduces for any N to a constant. One might speculate that in this case one is seeing effectively the cancellation between bosons and fermions of their contribution to the space-time cosmological constant. To make sure that this is the correct interpretation would require computing correlation functions in supermatrix theories for bosonic and fermionic operators. Within the formalism we develop below it is not yet clear how to identify space-time fermionic operators in terms of supermatrices. Further work is required to elucidate this question. The standard operators $\text{tr} M^n$ appearing in matrix models are here replaced by graded traces $\text{Str} M^n$, and their correlators are the same as those of the $(N - M|0)$ pure matrix theory. These operators do not exhaust the possible operators one could write down in supermatrix theories, but it is nevertheless disappointing to find such a drastic trivialization of the theory.

Throughout this letter we shall restrict our attention to Hermitian supermatrix models, our analysis being easily generalized to other cases. Our $(N|M)$ supermatrices will have

the following block decomposition [6]

$$\Phi = \begin{pmatrix} A & \Psi \\ \bar{\Psi} & \Omega \end{pmatrix} \quad (1)$$

where the indices in $\Phi_{\mu\nu}$ run from 1 to $N + M$, and

$$\begin{aligned} \Phi_{ij} &= A_{ij} & i, j &= 1, \dots, N \\ \Phi_{\alpha\beta} &= \Omega_{\alpha\beta} & \alpha, \beta &= N + 1, \dots, N + M \\ \Phi_{i\alpha} &= \Psi_i^\alpha & \Phi_{\alpha i} &= \bar{\Psi}_i^\alpha \end{aligned} \quad (2)$$

A and Ω are Hermitian bosonic matrices, whereas Ψ is fermionic. Although our results will be valid for general multi-supermatrix models, in order to simplify the notation we consider first the following single supermatrix model

$$Z(N|M) \equiv \int D\Phi \exp[-\beta V(\Phi)] \quad (3)$$

where $V(\Phi)$ is a general $SU(N|M)$ invariant potential

$$V(\Phi) = \text{Str} \Phi^2 + \sum_{k \geq 2} \frac{1}{k} g_k \text{Str} \Phi^k \quad (4)$$

Str is the supertrace, defined by

$$\text{Str} \Phi = \sum_{i=1}^N \Phi_{ii} - \sum_{\alpha=N+1}^{N+M} \Phi_{\alpha\alpha} = \sum_{\mu} (-)^{\sigma_{\mu}} \Phi_{\mu\mu} \quad (5)$$

where $\sigma_{\mu} = 0$ (1) for $\mu = 1, \dots, N$ ($\mu = N + 1, \dots, N + M$). The measure in (3) can be decomposed into a product $D\Phi = DA D\Omega D\bar{\Psi} D\Psi$, where

$$\begin{aligned} DA &= \prod_i dA_{ii} \prod_{i < j} 2d\text{Re} A_{ij} \prod_{i < j} d\text{Im} A_{ij} \\ D\Psi &= \prod_{i,\alpha} d\Psi_i^\alpha \end{aligned} \quad (6)$$

with similar expressions for $D\Omega$ and $D\bar{\Psi}$.

For $M = 0$, Φ is an ordinary Hermitian matrix and $Z(N|0)$ can be computed by first diagonalizing M and then using the method of orthogonal polynomials [7], [8].

There is no known generalization of this method for arbitrary $(M|N)$. However, there is one case where $Z(N|M)$ can be computed directly, namely when $N = M = 1$ and the potential V is cubic. In this case we have

$$Z_3(1|1) = \int d\lambda d\omega [2 - 3g(\lambda + \omega)] \exp\{-\lambda^2 + \omega^2 + g(\lambda^3 - \omega^3)\} \quad (7)$$

where we have already performed the integral over the Grassmann variables $\bar{\psi}$, ψ , and have used

$$\begin{aligned} Str \Phi^2 &= \lambda^2 - \omega^2 - 2\bar{\psi}\psi \\ Str \Phi^3 &= \lambda^3 - \omega^3 - 3\bar{\psi}\psi(\lambda + \omega) \end{aligned} \quad (8)$$

with $g \equiv -\frac{1}{3}g_3$, and

$$\Phi = \begin{pmatrix} \lambda & \psi \\ \bar{\psi} & \omega \end{pmatrix} \quad (9)$$

The integral (8) can be easily computed by first doing a Wick rotation $\omega \rightarrow i\omega$, and then going to polar coordinates $re^{i\theta} = \lambda + i\omega$. After expanding the exponent in powers of g and doing the θ integral we are left with

$$Z(1|1) = 2\pi i \left[1 + 2 \sum_{p=1}^{\infty} g^{4p} A_p \int_0^{\infty} r dr e^{-r^2} (r^{12p} - 6pr^{12p-2}) \right] \quad (10)$$

where

$$A_p = \frac{3^{3p}}{4^{4p}} \frac{1}{p!(3p)!} \quad (11)$$

The r integrals are in fact Γ functions (change variable $u = r^2$), and the result is

$$Z_3(1|1) = 2\pi i \left[1 + \sum_{p=1}^{\infty} g^{4p} A_p (\Gamma(6p+1) - 6p\Gamma(6p)) \right] \quad (12)$$

which, after using the identity $\Gamma(z+1) = z\Gamma(z)$, reduces to

$$Z_3(1|1) = 2\pi i \sim Z_3(0|0) \quad (13)$$

This is a very simple result which suggests an exact cancellation between bosonic and fermionic degrees of freedom. In fact, by using a representation in terms of moments, it is

possible to use this result (eq. (13)) to compute $Z_3(N|1)$. This is done in the Appendix, where we obtain

$$Z_3(N|1) \sim Z_3(N-1|0) \quad (14)$$

supporting the idea of a cancellation between fermionic and bosonic degrees of freedom.

The methods used in the Appendix rely crucially on the purely cubic character of the potential and the fact that there is only one pair $(\bar{\Psi}, \Psi)$ of Grassmann vectors. In order to evaluate $Z(N|M)$ in general, we have to use some indirect methods. The first one is based on the application of the Virasoro constraints to the partition function (3) , and sheds some light on the nature of the mechanism involved. The second method considers the structure of the Feynman diagrams in detail, and is applicable to the more general case of multi-supermatrix models.

In order to derive the form of the Virasoro constraints on $Z(N|M)$, we consider the change of variables

$$\Phi \rightarrow \Phi' + \epsilon \Phi'^{n+1} \quad , \quad n = -1, 0, 1, 2, \dots \quad (15)$$

The Jacobian to first order in ϵ is easily computed

$$D\Phi = D\Phi' \left[1 - \epsilon \sum_{k=0}^n Str \Phi'^k Str \Phi'^{n-k} \right] \quad (16)$$

where $Str \Phi^0 = Str 1 = N - M$, and for $n = -1$ the sum vanishes. This implies (dropping the prime in Φ')

$$\sum_{k=0}^n \langle Str \Phi^k Str \Phi^{n-k} \rangle = \beta \sum_{j=1}^{\infty} g_j \langle Str \Phi^{j+n} \rangle \quad (17)$$

In order to write (16) we have introduced the couplings g_1 and g_2 , so that $V(\Phi)$ now reads

$$V(\Phi) = g_1 Str \Phi + \frac{1}{2} g_2 Str \Phi^2 + \sum_{j \geq 3} \frac{1}{j} Str \Phi^j \quad (18)$$

By using

$$\langle Str \Phi^k Str \Phi^{n-k} \rangle = \frac{k(n-k)}{\beta^2} \frac{\partial^2 Z}{\partial d_k \partial g_{n-k}} \quad (19)$$

and

$$\langle Str \Phi^k \rangle = \frac{k}{\beta} \frac{\partial Z}{\partial g_k} \quad (20)$$

we convert (16) into the set of conditions

$$L_n Z(N|M) = 0 \quad , \quad n \geq -1 \quad (21)$$

where

$$\begin{aligned} L_n = & \sum_{k=1}^n \frac{k(n-k)}{\beta^2} \frac{\partial^2}{\partial g_k \partial g_{n-k}} - \frac{2n(N-M)}{\beta} \frac{\partial}{\partial g_n} \\ & + \sum_{k=1}^{\infty} (k+n) g_k \frac{\partial}{\partial g_{n+k}} + (N-M)^2 \delta_{0,n} - g_1 \beta (N-M) \delta_{-1,n} \end{aligned} \quad (22)$$

Equations (21) and (22) were obtained by Matsuo [9] for ordinary matrix models ($M = 0$). They can be solved iteratively, yielding the usual perturbation expansion in $g_3, g_4, \dots$ unambiguously. Alternatively, we can first set $\beta = N \rightarrow \infty$ and solve the theory in the planar limit, obtaining the different types of critical behaviour. Since the values of $(M|N)$ enter (22) only through the difference $(M - N)$, we conclude that

$$Z(N|M) \sim Z(N - M|0) \quad (23)$$

The planar and double scaling limits of the supermatrix model would be obtained by setting $\beta \sim N - M \rightarrow \infty$ in the appropriate way.

We now turn to an argument which can be trivially generalized to models of several supermatrices. For the sake of simplicity we first give it for the single supermatrix model, indicating at the end the very few changes necessary to account for the general case.

The perturbative expansion of the matrix partition function can be represented as a set of double-lined Feynman diagrams (see for example ref. [7]). For Hermitian models the lines are oriented, but this will not play a role in our discussion below. Each closed line in the graphs represents an *index loop*. Since each index can take on N different values, a diagram is proportional to N raised to the number of index loops it contains.

For $Z(N|M)$ the set of Feynman diagrams involved is obviously identical, the only difference being the type of indices that can run on each index loop. In what follows we shall show that each fermionic value of an index running on a loop contributes a minus sign to the diagram, or in other words that, for fixed values of the indices, each diagram has an overall sign equal to $(-1)^{\# \text{fermionic loop indices}}$. As a consequence, each supermatrix Feynman diagram will be proportional to $(N - M)$ raised to the number of index loops, and we will have shown that $Z(N|M) \sim Z(N - M|0)$.

As we know from ordinary Quantum Field Theory, each fermionic loop carries with it a minus sign. In our case this is somewhat harder to prove due to the fact that the field variables are the two-indexed objects $\Phi_{\mu\nu}$, whereas we are interested in one-index loops. We thus have to analyze carefully the origin of minus signs in the Feynman diagram. This will correspond to a set of Wick contractions carried on a generic term in the expansion of the exponential in (3)

$$(Str\Phi^{n_1})(Str\Phi^{n_2})\dots \quad (24)$$

where some exponents n_i may be repeated. A particular Feynman diagram may be schematically represented as

$$(-)^\sigma \Phi\Phi \dots \Phi (-)^\sigma \Phi\Phi \dots \Phi (-)^\sigma \Phi\Phi \dots \quad (25)$$

where the $(-)^\sigma$ are the signs from the supertraces (see (5)), and some Wick contractions have been indicated. In order to evaluate (25) we first commute the Φ s into the standard form

$$\Phi\Phi\Phi\Phi\Phi\Phi \dots \quad (26)$$

and then compute the Wick contractions according to

$$\Phi\Phi \equiv \langle \Phi_{\mu\nu} \Phi_{\rho\lambda} \rangle = \frac{1}{2\beta} (-)^\sigma \delta_{\lambda\mu} \delta_{\nu\rho} \quad (27)$$

(This can be easily obtained from (5)). Therefore, we have to take into account three different sources of minus signs due to the presence of fermionic variables:

- (i) The $(-)^\sigma$ in the supertraces (equation (5)).
- (ii) The $(-)^\sigma$ in the propagator (27) .
- (iii) The minus signs from the permutation of fermionic matrix elements.

The last source of minus signs requires a more careful analysis. This is done by means of the following trick. To each supermatrix element $\Phi_{\mu\nu}$ we assign a product of numbers $\alpha_\mu \bar{\alpha}_\nu$ with the same Grassmann characters as the corresponding indices. To a product like (24) containing a total of p Φ s we assign a string of $2p$ α s, *all of them different*. For

instance, to

$$Str\Phi^3 Str\Phi^4 = (-)^{\sigma_\mu} \Phi_{\mu\nu} \Phi_{\nu\rho} \Phi_{\rho\mu} (-)^{\sigma_\lambda} \Phi_{\lambda\delta} \Phi_{\delta\tau} \Phi_{\tau\gamma} \Phi_{\gamma\lambda} \quad (28)$$

the following product is assigned

$$(-1)^{\sigma_1} \alpha_1 \bar{\alpha}_2 \alpha_3 \bar{\alpha}_4 \alpha_5 \bar{\alpha}_6 (-)^{\sigma_7} \alpha_7 \bar{\alpha}_8 \alpha_9 \bar{\alpha}_{10} \alpha_{11} \bar{\alpha}_{12} \alpha_{13} \bar{\alpha}_{14} \quad (29)$$

The permutation necessary to bring (24) into the standard form (26) will be decomposed into a set of operations on the corresponding string of α s.

First, we move each α at the beginning of a supertrace to the right of the last $\bar{\alpha}$ in the same supertrace. For instance, (29) becomes

$$(\bar{\alpha}_2 \alpha_3)(\bar{\alpha}_4 \alpha_5)(\bar{\alpha}_6 \alpha_1)(\bar{\alpha}_8 \alpha_9)(\bar{\alpha}_{10} \alpha_{11})(\bar{\alpha}_{12} \alpha_{13})(\bar{\alpha}_{14} \alpha_7) \quad (30)$$

This operation cancels all the minus signs from the supertraces. Note also that the two α s in a bracket $()$ have the same Grassmann character.

Next, to each Wick contraction between Φ s we associate a pair of lines between the corresponding α s according to

$$\dots \Phi_{\mu\nu} \dots \Phi_{\rho\lambda} \dots \rightarrow \dots \alpha_\mu \bar{\alpha}_\nu \dots \alpha_\rho \bar{\alpha}_\lambda \dots \quad (31)$$

and by moving around *only pairs* of α s in brackets we bring the string into the following form

$$(\bar{\alpha}\alpha)(\bar{\alpha}\alpha)\dots(\bar{\alpha}\alpha) (\bar{\alpha}\alpha)(\bar{\alpha}\alpha)\dots(\bar{\alpha}\alpha)\dots \quad (32)$$

This disentangles the different index loops. Since each product $(\bar{\alpha}\alpha)$ is necessarily bosonic, no new minus signs arise at this stage. Now, move the $\bar{\alpha}$ at the beginning of each index loop to the right of the last α in the same loop, and remove the brackets. After this operation (32) becomes

$$(-)^{\# \text{fermionic loop indices}} \alpha \bar{\alpha} \alpha \bar{\alpha} \alpha \bar{\alpha} \alpha \bar{\alpha} \dots \quad (33)$$

Next, move pairs $\alpha\bar{\alpha}$ so that the two pairs associated with each Wick contraction get together, i.e.,

$$\Phi_{\mu\nu} \Phi_{\rho\lambda} \rightarrow \alpha_\mu \bar{\alpha}_\lambda \alpha_\rho \bar{\alpha}_\nu \quad (34)$$

We do not get any additional minus sign from this. Finally, we permute (34) to bring the

α s in correspondence with the indices

$$\alpha_\mu \bar{\alpha}_l \alpha_\rho \bar{\alpha}_\nu = (-)^{\sigma_\nu} \alpha_\mu \bar{\alpha}_\nu \alpha_\rho \bar{\alpha}_\lambda \sim (-)^{\sigma_\nu} \Phi_{\mu\nu} \Phi_{\rho\lambda} \quad (35)$$

This last operation has brought the string of α s in complete correspondence with (26). Note that the sign in (35) exactly cancels the sign from the propagator (27). As a result, the overall sign is simply given by $(-)^{\# \text{ fermionic loop indices}}$ (from (33)), and the proof is complete.

We now consider a model with R supermatrices of type $(M|N)$, allowing for *totally arbitrary* invariant couplings among themselves. Thus, the most general quadratic part of the potential will be given by

$$\sum_{p,q=1}^R g_{pq} \text{Str}(\Phi_p \Phi_q) \quad (36)$$

with the following propagator

$$\langle \Phi_{p,\mu\nu} \Phi_{q,\rho\lambda} \rangle = \frac{1}{2\beta} (g^{-1})_{pq} (-)^{\sigma_\nu} \delta_{\lambda\mu} \delta_{\nu\rho} \quad (37)$$

Note that (36) is more general than the *chain models* considered in [10],[11],[12]. Our only restriction on g is $\det g \neq 0$. Similarly, we allow for arbitrary interactions among the Φ_p s. For instance, we can have terms of the form

$$\text{Str}(\Phi_p^{n_p} \Phi_q^{n_q} \Phi_r^{n_r} \dots) \quad (38)$$

The corresponding theory of $(N|0)$ matrices has not been solved in this very general case. However, since the sources of minus signs are the same as before, our previous proof readily extends to this case, and we still have

$$Z(N|M) \sim Z(N-M|0). \quad (39)$$

Although we have been referring to $D=0$ supermatrix models, our main result (39) applies also to models in one dimension, since these can be considered as the $R \rightarrow \infty$ limit of the chain models.

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APPENDIX

Here we compute the partition function $Z_3(N|1)$ directly. This is given by (3) with $V(\Phi) = \text{Str}(\Phi^2 - g\Phi^3)$. After integration over the fermionic variables we are left with

$$Z_3(N|1) \sim \int d\lambda_1 \dots d\lambda_N d\omega \Delta^2(\lambda) \prod_{i=1}^N [2 - 3g(\lambda_i + \omega)] \exp\left\{-\sum_{i=1}^N (\lambda_i^2 - g\lambda_i^3) + \omega^2 - g\omega^3\right\} \quad (40)$$

where $(\lambda_1, \dots, \lambda_N)$ are the eigenvalues of the matrix A , and $\Delta(\lambda)$ is the Vandermonde determinant. The fact that we write $\sim$ instead of $=$ in (40) and other equations below means that we are neglecting factors arising from the integration over the unitary group [8], i.e.,

$$\int DA = \frac{(2\pi)^{N(N-1)/2}}{\prod_1^N p!} \int d\lambda_1 \dots d\lambda_N \Delta^2(\lambda) \quad (41)$$

In order to evaluate (40) we shall have to use some results from the theory of moments and orthogonal polynomials. A complete discussion can be found, for instance, in [13] and in [14]. Here we shall just give a summary of the facts needed for our computation.

The general bosonic partition function $Z(N|0)$ can be written in terms of the moments μ_k

$$\mu_k \equiv \int_{-\infty}^{\infty} d\lambda \lambda^k e^{-V(\lambda)} \quad (42)$$

as

$$Z(N|0) \sim \int d\lambda_1 \dots d\lambda_N \Delta^2(\lambda) \exp\left[-\sum_{i=1}^N V(\lambda_i)\right] = N! D_{N-1} \quad (43)$$

where D_{N-1} is the Hadamard determinant defined by

$$D_n = \det \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n} \end{pmatrix} \quad (44)$$

If we define the generating function

$$G(\lambda) = \sum_{k=0}^{\infty} \frac{\mu_k}{\lambda^{k+1}} \quad (45)$$

then the following identity is obtained [13],[14]

$$G'(\lambda) + V'(\lambda)G(\lambda) = \sum_p p g_p \lambda^{p-1} \sum_{k=0}^{p-2} \frac{\mu_k}{\lambda^{k+1}} \quad (46)$$

where $V(\lambda) = \sum_p g_p \lambda^p$. For the cubic potentials $V(\lambda) = \lambda^2 - g\lambda^3$, eq. (46) yields the following set of recursion relations when expanded in powers of λ

$$\begin{aligned} 2\mu_1 - 3g\mu_2 &= 0 \\ (k-1)\mu_{k-2} - 2\mu_k + 3g\mu_{k+1} &= 0 \quad k \geq 2 \end{aligned} \quad (47)$$

These can be solved in terms of μ_0 and μ_1 . If we denote by $\{P_n(\lambda)\}$ the set of monic orthogonal polynomials with respect to the measure $e^{-V(\lambda)}$,

$$\begin{aligned} \int d\lambda P_n(\lambda) P_m(\lambda) e^{-V(\lambda)} &= h_n \delta_{n,m} \\ P_n(\lambda) &= \lambda^n + \dots \end{aligned} \quad (48)$$

then we have the following identity

$$\begin{aligned} \langle \prod_{i=1}^N (\zeta - \lambda_i) \rangle_{\lambda} &\equiv \int d\lambda_1 \dots d\lambda_N \Delta^2(\lambda) \prod_{i=1}^N (\zeta - \lambda_i) e^{-\sum_{j=1}^N V(\lambda_j)} \\ &= N! \det \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_N \\ \mu_1 & \mu_2 & \dots & \mu_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{N-1} & \mu_N & \dots & \mu_{2N-1} \\ 1 & \zeta & \dots & \zeta^N \end{pmatrix} \sim Z(N|0) P_N(\lambda) \end{aligned} \quad (49)$$

Equations (43) , (47) and (49) are all we need in order to evaluate $Z_3(N|1)$. After

Wick rotating $\omega \rightarrow i\omega$ and defining

$$\zeta = \frac{2}{3g} - i\omega \quad (50)$$

we have

$$Z_3(N|1) \sim i(3g)^N \int d\omega \langle \prod_{i=1}^N (\zeta - \lambda_i) \rangle_\lambda \exp(-\omega^2 + i\omega^3) \quad (51)$$

If we define a second set of moments $\hat{\mu}_k$

$$\hat{\mu}_k \equiv \int d\omega \left(\frac{2}{3g} - i\omega \right)^k \exp(-\omega^2 + i\omega^3) \quad (52)$$

and use (49) , then (51) is given by the following determinant

$$Z(N|1)_3 \sim i(3g)^N N! \det \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_N \\ \mu_1 & \mu_2 & \dots & \mu_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{N-1} & \mu_N & \dots & \mu_{2N-1} \\ \hat{\mu}_0 & \hat{\mu}_1 & \dots & \hat{\mu}_N \end{pmatrix} \quad (53)$$

It is easy to see that formally the new moments $\hat{\mu}_k$ satisfy the same recursion relations (47) . These relations can be used to rearrange the determinant in (53) into the following form

$$(3g)^{-N} \det \begin{pmatrix} \mu_0 & \mu_1 & 0 & 0 & \dots & 0 \\ \mu_1 & \mu_2 & \mu_0 & \mu_1 & \dots & \mu_{N-2} \\ \mu_2 & \mu_3 & 2\mu_1 & 2\mu_2 & \dots & 2\mu_{N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{N-1} & \mu_N & (N-1)\mu_{N-2} & (N-1)\mu_{N-1} & \dots & (N-1)\mu_{2N-4} \\ \hat{\mu}_0 & \hat{\mu}_1 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (54)$$

and we obtain

$$Z_3(N|1) \sim 3ig N!(N-1)! D_{N-2} \det \begin{pmatrix} \mu_0 & \mu_1 \\ \hat{\mu}_0 & \hat{\mu}_1 \end{pmatrix} \quad (55)$$

This can be simplified further by using (43) and noting that, according to (53)

$$Z_3(1|1) = 3ig \det \begin{pmatrix} \mu_0 & \mu_1 \\ \hat{\mu}_0 & \hat{\mu}_1 \end{pmatrix} \quad (56)$$

and we finally get

$$Z_3(N|1) \sim Z_3(1|1)Z_3(N-1|0) \quad (57)$$

where we may substitute the value $Z_3(1|1) = 2\pi i$ obtained in the main text. Taking into account the normalization factors neglected so far, one can actually write

$$\frac{Z_3(N|1)_g}{Z_3(N|1)_{g=0}} \sim \frac{Z_3(N-1|0)_g}{Z_3(N-1|0)_{g=0}} \quad (58)$$

Note that equation (57) could be considered as non-perturbative in the sense that no expansion in powers of g or $1/N$ has been used in its derivation. However, for a cubic potential neither the integrals that give the partition function nor the moments are really well defined, and the above computation is formal.

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