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INSTANTON VACUUM IN THERMAL QCD

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Abstract

The effects of instanton-anti-instanton interactions in high-temperature QCD with colour SU(2) are studied with and without quarks. The instanton interaction potential proves to have no repulsive core, thus creating troubles with the definition of non-perturbative contributions to physical quantities. Also the chiral condensate is calculated, and it is shown that if the number of quark flavours $N \geq 2$ there is no chiral-symmetry breaking for sufficiently high temperature. For $N_f = 1$, however, chiral symmetry remains broken.

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1 INTRODUCTION

Considerable progress has been made over the last few years in the study of non-zero-temperature QCD (for a recent review, see Refs. [1], [2]). From lattice Monte Carlo simulations we learn that pure gauge theory exhibits a phase transition from the confinement phase to the plasma one with increasing temperature. When quarks are introduced in the theory the critical temperature T_c for the deconfinement phase transition actually coincides with the chiral-symmetry restoration one [3].

However, despite the progress of lattice calculations in thermal QCD we still need an analytical ‘explanation’ of this critical behaviour. In particular, we still have no adequate description of the dynamics in the confinement phase of the theory.

It is common belief now that to understand the long-range behaviour of zero-temperature QCD one has to go beyond the perturbative loop expansion. However, non-perturbative effects in QCD-like theories are hard to study in an analytical way since semiclassical methods do not work in strong-coupling theories. Therefore, it is an attractive possibility to have a certain parameter in the theory that enables one to pass from the weak-coupling limit, where the theory is easy to study semiclassically, to the strong-coupling limit. Temperature seems to be a natural parameter of this kind.

Indeed, temperature provides the scale for gauge theory. In particular, at the high temperature $T \gg \Lambda_{\text{QCD}}$ the effective coupling constant is small, $g^2(T) \ll 1$, and non-perturbative effects can be studied in the semiclassical way. Of course, semiclassical methods become invalid when one approaches T_c . Nevertheless, one may hope that if certain non-perturbative effects are essentially relevant to the dynamics in the low-temperature phase their contribution to physical quantities has to grow as one approaches T_c .

In this paper, we wish to examine whether this is true for instantons. Instanton effects in high-temperature QCD have been well studied in Ref. [2] in the dilute instanton gas approximation. Here we would like to take into account instanton–antiinstanton ($\bar{\Pi}\bar{\Pi}$) interactions.

We calculate $\bar{\Pi}\bar{\Pi}$ interactions using the general streamline (or valley) approach suggested in Ref. [4] for the quantum mechanical problem and in Ref. [5] where it is generalized to gauge theory. It enables one to give a rigorous and ansatz-independent notion of the $\bar{\Pi}\bar{\Pi}$ configuration and its interacting potential (we briefly recall this idea in Section 2). The $\bar{\Pi}\bar{\Pi}$ interaction potential turns out to be of the dipole–dipole type in three-dimensional space, since at high values of T the width of the temperature strip $\beta = T^{-1} \ll \Lambda$, and the theory becomes essentially three-dimensional. This potential proves to have no repulsive core.

In Section 3 we calculate the $\bar{\Pi}\bar{\Pi}$ contribution to the free energy and immediately face the following problem. Owing to the absence of the repulsive core, $\bar{\Pi}$ and $\bar{\Pi}$ attract each other to short distances and then collapse into the perturbative vacuum. This problem is not new: it was actually recognized in simple quantum mechanical theories with instantons [6], [4]. It means the mixture of the $\bar{\Pi}\bar{\Pi}$ contribution to the free energy with high orders of its loop expansion (see Section 3). In particular, the $\bar{\Pi}\bar{\Pi}$ free energy is defined by means of analytic continuation. It turns out to have an imaginary part whose presence reflects the badly divergent (non-Borel summable) behaviour of perturbative series.

In Section 4 we introduce quarks and calculate the quark-induced part of the $\bar{\Pi}\bar{\Pi}$ interacting potential. The presence of quarks drastically changes the qualitative picture of the instanton vacuum. Without quarks at high temperature this vacuum looks like a dilute instanton gas with $\bar{\Pi}\bar{\Pi}$, $\bar{\Pi}\bar{\Pi}\bar{\Pi}$, ... $\bar{\Pi}^n\bar{\Pi}^m$... interaction effects taken into account as a

virial expansion in the instanton density $n \sim \exp[-8\pi^2/g^2(T)] \ll 1$. In QCD with N_f flavours of massless quarks, instantons form a $\bar{\text{I}}\bar{\text{I}}$ molecular phase, since the pure dilute gas contribution ('atomic' contribution) is suppressed owing to the presence of fermionic zero modes.

In Section 5 we rewrite the instanton-induced $\bar{\text{I}}\bar{\text{I}}$ interacting potential in terms of the effective field theory using the Polyakov method [7]. Doing this we arrive at the 't Hooft effective $2N_f$ -fermion vertex [8], [9]. Note that the 't Hooft vertex has been derived at the one-instanton level and has then been extrapolated to the processes involving many I's and $\bar{\text{I}}$'s. We now check if this extrapolation is correct.

In Section 6 we calculate the chiral condensate $\langle \bar{\psi}\psi \rangle$ in the instanton vacuum and show that for the $N_f = 1$ case it remains non-zero for arbitrarily high temperature. On the contrary, if $N_f > 1$ it turns out to be zero in the molecular phase. Thus one concludes that chiral-symmetry breaking may take place only after the phase transition from the molecular to the liquid phase of the instanton vacuum at low temperature. However, our methods become invalid in this strong-coupling phase¹⁾. Our results are in qualitative agreement with the numerical simulations of Ref. [11].

2 $\bar{\text{I}}\bar{\text{I}}$ INTERACTING POTENTIAL

In this section we shall consider pure SU(2) gauge theory. At finite temperature, the theory is defined in its Euclidean version on a strip in the 'time' direction of width $\beta \equiv T^{-1}$. Bosonic fields have to satisfy periodic boundary conditions $A_\mu(0, \vec{x}) = A_\mu(\beta, \vec{x})$ for a detailed introduction to thermal QCD, see Refs. [1], [2]. The I's ($\bar{\text{I}}$'s) are defined as self-dual (anti-self-dual) periodic solutions of equations of motion with a topological charge on the temperature strip

$$Q = g^2/16\pi^2 \int_0^\beta dt \int d^3\vec{x} \text{Tr} (G_{\mu\nu} G_{\mu\nu}^*), \quad (2.1)$$

which is equal to 1 (−1) [2]. They can be constructed using the 't Hooft solution for the multi-instanton [12] in the $\mathbb{R}^4$ space with all instantons having identical orientations and sizes ρ , and being situated at $x_n = (t_c + n\beta, \vec{x}_c)$. Namely [13],

$$A_\mu^{\bar{\text{I}}}(x) = -\frac{1}{g} \bar{\eta}_{\mu\nu}^a \frac{\tau^a}{2} \ln \Pi(x), \quad (2.2)$$

where

$$\Pi(t, \vec{x}) = 1 + \sum_{n=-\infty}^{\infty} \frac{\rho^2}{(t - t_c + n\beta)^2 + (\vec{x} - \vec{x}_c)^2} = 1 + \frac{\pi\rho^2}{\beta r} \frac{\sinh 2\pi r/\beta}{\cosh 2\pi r/\beta - \cos 2\pi\tau/\beta}. \quad (2.3)$$

Here $(t_c, \vec{x}_c)$ refers to the centre of the instanton, and $\tau = t - t_c$, $r = |\vec{x} - \vec{x}_c|$. It is clear that solution (2.2) together with (2.3) is periodic and has unit topological charge on the temperature strip.

Let us now consider the $\bar{\text{I}}\bar{\text{I}}$ interaction and briefly recall the valley approach of Refs. [4] and [5] to this problem. The $\bar{\text{I}}\bar{\text{I}}$ configuration is not an exact solution of the equation of motion; nevertheless at large separations R_μ between I and $\bar{\text{I}}$ it seems to be relevant in the physical vacuum. It can be considered as a valley with a small gradient and steep

1) In ref. [10], $\langle \bar{\psi}\psi \rangle$ was estimated to be non-zero in the liquid phase of a certain ansatz-dependent model of the instanton vacuum

walls in the functional space. Indeed, this valley is parametrized by collective coordinates τ_i of I and $\bar{I}$ (τ_i being positions, sizes, orientations), and the action varies slowly with the variation of τ_i , whereas it changes rapidly with variations of the profile function of each instanton (it corresponds to variations along non-zero modes). The $\bar{I}\bar{I}$ configuration is the bottom line of this valley (streamline). It satisfies the equation for the gradient line of steepest descent in the functional space [5]:

$$\frac{\delta S}{\delta A_\nu^a(x, \tau_i)} \equiv D_\mu G_{\mu\nu}^a = \sum_i \varepsilon_i^2(\tau) P_{\nu\alpha}^\perp \frac{\partial A_\alpha(x, \tau_i)}{\partial \tau_i}, \quad (2.4)$$

where $P_{\mu\nu}^\perp = \delta_{\mu\nu} - D_\mu D^{-2} D_\nu$ is the transverse projector, and $\varepsilon_i^2(\tau)$ are small parameters which measure the slope of the valley $\varepsilon_i^2(R \rightarrow \infty) \rightarrow 0$. This equation has to be solved with the boundary condition

$$A_\mu(x, \tau_i)|_{R \rightarrow \infty} = A_\mu^I(x) + A_\mu^{\bar{I}}(x) \quad (2.5)$$

to select the $\bar{I}\bar{I}$ gradient line. The $\bar{I}\bar{I}$ interaction potential is defined as

$$U_{\text{int}}^{\bar{I}\bar{I}}(\tau_i) = S[A_\mu(\tau_i)] - 16\pi^2/g^2 + O[\varepsilon^2(\tau_i)]. \quad (2.6)$$

Solving Eqs. (2.4) and (2.5) perturbatively in powers of $\varepsilon^2(\tau)$, one can calculate $U_{\text{int}}^{\bar{I}\bar{I}}$ as an expansion of powers of $1/|R|$.

This program has been performed for zero-temperature gauge theory in Ref. [5]. Two leading terms in $U_{\text{int}}^{\bar{I}\bar{I}}(R)$ have been calculated: i) the $O(1/R_\mu^4)$ term, which coincides with the Callan–Dashen–Gross dipole–dipole potential [14] and ii) the $O(1/R_\mu^6)$ correction. The expansion in powers of $\varepsilon^2(\tau)$ for Eq. (2.4) turns out to be similar to the conventional perturbation theory for the Schrödinger equation. The leading (zero-order) term for $A_\mu(x, \tau)$, which is nothing but a boundary condition (2.5) in the singular gauge, determines the $O(\varepsilon^2)$ term in potential $U_{\text{int}}^{\bar{I}\bar{I}}$; the $O(\varepsilon^2)$ correction for $A_\mu(x, \tau)$ determines the $O(\varepsilon^4)$ correction for $U_{\text{int}}^{\bar{I}\bar{I}}$ and so on [5], [15].

Here we shall calculate the leading term in $U_{\text{int}}^{\bar{I}\bar{I}}$ for high-temperature QCD, keeping in mind that the semiclassical approximation is justified at high T provided $g^2(T) \ll 1$. For the actual computation of the leading term in $U_{\text{int}}^{\bar{I}\bar{I}}$ we need not solve the streamline equation (2.4). We can simply use the sum of I and $\bar{I}$ fields (2.5) as a streamline of the zero-order approximation. Nevertheless, we shall use the conceptual framework of the valley approach to study the qualitative behaviour of $U_{\text{int}}^{\bar{I}\bar{I}}(R)$.

Equation (2.5) for periodic I and $\bar{I}$ with arbitrary relative orientation yields

$$A_\mu^k = \frac{i}{2g} \left\{ [u_1 \bar{u}_R (\sigma_\mu \bar{\sigma}_\nu - \delta_{\mu\nu}) u_R \bar{u}_1]_\ell^k \partial_\nu \ln \Pi_I + [u_2 (\bar{\sigma}_\mu \sigma_\nu - \delta_{\mu\nu}) \bar{u}_2]_\ell^k \partial_\nu \ln \Pi_{\bar{I}} \right\}. \quad (2.7)$$

Here we use matrix notations $\sigma_\mu u^{kk} = (1, -i\tau^a)$, $\bar{\sigma}_\mu k^k = (1, i\tau^a)$, while the orientation matrices for I and $\bar{I}$ are $u_2^{kk} = u_{2\mu} \sigma_\mu^{kk}$, $u_1^{kk} = u_1 \sigma_\mu^{kk}$ with unit four-vectors $u_{1\mu}, u_{2\mu}$ ($u_{1\mu}^2 = u_{2\mu}^2 = 1$). The matrix $u_R = \sigma_\mu u_{R\mu}$, $u_{R\mu}^2 = 1$ is a fixed unitary matrix. Actually the product $(u_1 u_R)_\ell^k$ plays the role of the orientation matrix of I . The expressions $\Pi_I(x, x_I)$ and $\Pi_{\bar{I}}(x, x_{\bar{I}})$ are determined by Eq. (2.3), where $\rho_I, x_{I\mu} = (t_I, \vec{x}_I)$ and $\rho_{\bar{I}}, x_{\bar{I}\mu} = (t_{\bar{I}}, \vec{x}_{\bar{I}})$ have to be substituted for the sizes and positions of I and $\bar{I}$ respectively.

Calculating the action on the trajectory (2.7) and taking into account Eq. (2.6) one obtains in the limit $|\vec{R}| = |\vec{x}_I - \vec{x}_{\bar{I}}| \gg |t_I - t_{\bar{I}}| \sim \beta; \rho_{\bar{I}} \sim \rho_I \sim \beta$:

$$U_{\text{int}}^{\bar{I}\bar{I}} = \frac{16\pi^3}{g^2} \frac{\rho_I^2 \rho_{\bar{I}}^2}{\beta |\vec{R}|^3} \left[|\vec{v}|^2 - 3 \frac{(\vec{v} \vec{R})^2}{|\vec{R}|^2} \right] + O\left(\frac{1}{g^2 |\vec{R}|^4}\right), \quad (2.8)$$

where $v_\mu = (v_0, \vec{v})$, and

$$v_\mu = \frac{1}{2} \text{Tr}(u_R \bar{u}_1 u_2 \bar{\sigma}_\mu), \quad v_\mu^2 = 1 \quad (2.9)$$

defines the relative orientation of I and $\bar{I}$.

The potential (2.8) is nothing but a dipole–dipole interaction in three-dimensional space. It has $1/|\vec{R}|^3$ behaviour and becomes zero after averaging over relative $\bar{I}\bar{I}$ orientation. It is instructive to compare result (2.8) with the zero-temperature $\bar{I}\bar{I}$ interaction potential

$$(U_{\text{int}}^{\bar{I}\bar{I}})_{T=0} = \frac{32\pi^2}{g^2} \frac{\rho_1^2 \rho_{\bar{1}}^2}{R^4} \left[1 - 4 \frac{(uR)^2}{R^2} \right] + O\left(\frac{1}{g^2 R^6}\right), \quad (2.10)$$

where the relative $\bar{I}\bar{I}$ orientation matrix is defined using the unit four-vector $u_\mu : u^{kk} = \sigma_\mu^{kk} u_\mu$. Equation (2.10) represents a dipole–dipole potential in the four-dimensional space. Comparing Eqs. (2.8) and (2.10) one finds that our result (2.8) has a natural origin: at high temperature the width β of the temperature strip goes to zero and the theory becomes essentially three-dimensional.

To conclude this section let us discuss the general properties of $U_{\text{int}}^{\bar{I}\bar{I}}(R)$. Although we have actually calculated only the leading term of the interaction potential at large R we can argue, on general grounds, that it has no repulsive core. Let us choose, for any separation $\vec{R}$, a particular relative orientation $\vec{v}_0$ [the single one for all $\vec{R}$ or at least the continuous function $\vec{v}_0(\vec{R})$] which minimizes the value of $U_{\text{int}}^{\bar{I}\bar{I}}(R, v)$. Then the above statement means that $U_{\text{int}}^{\bar{I}\bar{I}}(R, v_0)$ is an attractive potential. Using Eqs. (2.6) and (2.4) one finds

$$\frac{\partial U_{\text{int}}^{\bar{I}\bar{I}}(|\vec{R}|)}{\partial |\vec{R}|} = \left(\frac{\delta S}{\delta A_\mu} \times \frac{\partial A_\mu}{\partial |\vec{R}|} \right) = \varepsilon^2(R) \left(P_{\mu\alpha}^\perp \frac{\partial A_\alpha}{\partial |\vec{R}|} \times P_{\mu\beta}^\perp \frac{\partial A_\beta}{\partial |\vec{R}|} \right). \quad (2.11)$$

From Eq. (2.8) we learn that, for large $|\vec{R}|$, $U_{\text{int}}^{\bar{I}\bar{I}}$ is an attractive potential at least for $\vec{v} = \vec{R}/|\vec{R}|$. Then Eq. (2.11) shows that the only way to get a repulsive potential at some intermediate R is to have $\varepsilon^2(R_0) = 0$ for some particular separation R_0 . This means that the $\bar{I}\bar{I}$ configuration $A_\mu(x, R_0)$ is an exact solution of the equations of motion [see Eq. (2.4)]. However, the common belief is that there are no non-trivial, stable, exact solutions of Yang–Mills equations with finite action and zero topological charge. This means that the streamline $A_\mu(x, R, v_0)$ connects infinitely separated I and $\bar{I}$ at large R with the trivial vacuum $A_\mu = 0$ (up to gauge transformation) at $R = 0$ and $U_{\text{int}}^{\bar{I}\bar{I}}(R, v_0)$ and falls with a decrease of $|\vec{R}|$ from zero value at $|\vec{R}| \rightarrow \infty$ to $-16\pi^2/g^2$ at $R = 0$. Note that for the $T = 0$ case a similar statement, namely the absence of the repulsive core for the streamline configuration, has been proved rigorously using the conformal invariance of the theory at the classical level [5].

The above arguments concern only the classical part of the $\bar{I}\bar{I}$ interaction potential given by Eq. (2.6). Quantum corrections of Eq. (2.6) come from $1/|\vec{R}|$ corrections to determinants in the external field (2.7) and from loop corrections as well. These could, in principle, produce repulsion. However, they are suppressed by the extra power of g^2 , and thus could stabilize the system only at small distances $|\vec{R}| \sim \beta$, producing a contribution to the effective action $S \sim 1 \ll 16\pi^2/g^2$. Therefore, this configuration is a perturbative fluctuation above the trivial vacuum $A_\mu = 0$ rather than the $\bar{I}\bar{I}$ field. It is clear, on general grounds, that the possible effect of such a ‘flucton’ (the term was first used by E.V. Shuryak) cannot be separated from the ordinary loop expansion.

3 $\bar{\text{II}}$ FREE ENERGY

In this section we shall calculate the $\bar{\text{II}}$ contribution to the free energy. In the semiclassical approximation the instanton contribution to the partition function is given by

$$Z = \sum_{N,M} \frac{1}{N!M!} \int \prod_{n=1}^N d\mu_n \prod_{m=1}^M d\mu_m \exp \left(- \sum_{n,m} U_{nm} \right). \quad (3.1)$$

Here U_{nm} is the interacting potential of the n^{th} I and the m^{th} I. The instanton measure can be factorized into the product of a single instanton one provided weak coupling $g^2(T) \ll 1$ and large distance $|\vec{R}_{nm}| \gg \beta$ limits are assumed. For example (cf. refs. [15] and [5]),

$$d\mu_{nm} = d\mu_n d\mu_m \left(1 + O(1/|\vec{R}_{nm}|^3) \right). \quad (3.2)$$

Note that the leading term in U_{nm}^{int} given by Eq. (2.8) is proportional to $(1/g^2)(1/|\vec{R}|^3)$: thus it is much larger than the quantum corrections we have neglected in Eq. (3.2). The single-instanton measure has been calculated in Refs. [2] and [9]. The one-loop result for the SU(2) case reads

$$\begin{aligned} d\mu_n &\equiv d^4x_n \frac{d\rho_n}{\rho_n^5} \frac{d^3u_n}{2\pi^2} c_1 e^{-2c_2} \left[\frac{8\pi^2}{g^2} \right]^4 \exp \left\{ -\frac{4}{3} \left[\pi^2 \rho^2 T^2 - \ln \left(1 + \frac{\pi^2}{3} \rho^2 T^2 \right) \right] \right\} \\ &\times \exp[-8\pi^2/g^2(\rho_0)] = d^4x_n d\rho_n \frac{d^3v_n}{2\pi^2} n(\rho_n). \end{aligned} \quad (3.3)$$

Here $c_1^{\overline{\text{MS}}} \equiv 0.466$, $c_2^{\overline{\text{MS}}} \simeq 1.54$. The temperature-dependent factor $\sim \exp(-4/3\pi^2\rho^2T^2)$ makes the ρ -integral convergent and dominated by $\rho \sim \beta$. This ensures the weak coupling. The large-distance limit also has to be justified in order for Eqs. (2.8) and (3.1) to be valid. We shall discuss this below.

Performing the Mayer virial expansion in Eq. (3.1), one can calculate the free energy of the theory

$$F = - \frac{1}{\beta V^{(3)}} \ln Z \quad (3.4)$$

as a cluster expansion with respect to instanton density

$$F = F^{\text{PT}} + F^{\text{I}} + F^{\bar{\text{II}}} + F^{\text{III}} + \dots$$

Here F^{PT} is the perturbative contribution, F^{I} is the dilute gas one $F^{\text{I}} = O(e^{-8\pi^2/g^2(T)})$, and so on. The dilute gas contribution has been calculated in Ref. [2]:

$$\begin{aligned} F^{\text{I}} &= - \frac{2}{\beta V^{(3)}} \int d\mu \simeq -2c_1 e^{-2c_2} \left[\frac{\Lambda}{T} \right]^{22/3} \frac{T^4}{\pi^{10/3}} \left[\frac{8\pi^2}{g^2} \right]^4 \int_0^\infty d\lambda \lambda^{7/3} \\ &\times \left(1 + \frac{\lambda^2}{3} \right)^{4/3} e^{-4/3\lambda^2}, \end{aligned} \quad (3.5)$$

where $\lambda = \pi\rho T$, $\Lambda = \Lambda_{\overline{\text{MS}}}$.

Let us calculate $F^{\bar{\text{II}}}$ now. One has

$$F^{\bar{\text{II}}} = - \frac{1}{\beta V^{(3)}} \int d\mu_I d\mu_{\bar{\text{I}}} \left[\exp(-U_{\text{int}}^{\bar{\text{II}}}) - 1 \right].$$

To integrate over the relative orientation four-vector v_μ let us proceed to angular variables

$$\int \frac{d^3v}{2\pi^2} = \int \frac{v^2 dv \sin \theta d\theta d\varphi}{\pi^2 \sqrt{1-v^2}} = 2 \int_0^1 \frac{v^2 dv}{\pi^2 \sqrt{1-v^2}} \int_0^1 d\alpha \int_0^{2\pi} d\varphi, \quad (3.6)$$

where $v \equiv |v|$. Introducing a new variable

$$t = 3 \frac{16\pi^3}{\beta |\vec{R}|^3} \rho_1^2 \rho_1^2 v^2 \quad (3.7)$$

instead of $|\vec{R}|$, one gets

$$F^{\bar{\Pi}} = -48\pi^8 T^4 \left[c_1 e^{-2c_2} \int \frac{d\lambda}{\lambda^3} \left[\frac{8\pi^2}{g^2} \right]^4 \left(\frac{\lambda}{\pi} \right)^{22/3} \left(1 + \frac{\lambda^2}{3} \right)^{4/3} e^{-4/3\lambda^2} \right]^2 \times e^{-16\pi^2/g^2(T)} I(g^2), \quad (3.8)$$

where

$$I(g^2) = \int_0^\infty \frac{dt}{t^2} \int_0^1 d\alpha \left[e^{t/g^2(\alpha^2-1/3)} - 1 \right]. \quad (3.9)$$

Let us divide the contributions coming from repulsive and attractive relative orientations of an $\bar{\Pi}$ pair in Eq. (3.9):

$$I(g^2) = I^{\text{rep}}(g^2) + I^{\text{att}}(g^2), \quad (3.10)$$

where $I^{\text{rep}}(g^2)$ is the same integral as in Eq. (3.9) with the α -integration restricted to $0 \leq \alpha \leq 1/\sqrt{3}$, whereas $I^{\text{att}}(g^2)$ corresponds to $1/\sqrt{3} < \alpha < 1$. There is no problem to justify the large-distance approximation in the repulsive part of the free energy. Indeed, the integral over t is convergent at $t \rightarrow \infty$ (it corresponds to $|\vec{R}| \rightarrow 0$) and is dominated by $t \sim g^2$. Thus

$$|\vec{R}| \sim \left[\frac{1}{g^2} \frac{\rho_1^2 \rho_1^2}{\beta} \right]^{1/3} \sim \frac{\beta}{g^{2/3}} \gg \beta.$$

Let us now consider $I^{\text{att}}(g^2)$: the integral over t is divergent and dominated by $t \rightarrow \infty$. However, our method is invalid at small $\bar{\Pi}$ separations $|\vec{R}| \rightarrow 0$. Moreover, it is shown, in the previous section, that there is no repulsive core for the $\bar{\Pi}$ system. Instead, the $\bar{\Pi}$ configuration tends to collapse into the trivial vacuum $A_\mu = 0$ at $|\vec{R}| \rightarrow 0$. Therefore, in trying to carry out the integration over $\vec{R}$ [in Eq. (3.9)] we face a problem of the mixing of the non-perturbative $\bar{\Pi}$ contribution to the free energy, which arises from the region of large separations $|\vec{R}|$ in (3.9), with the perturbative one coming from $|\vec{R}| \sim \beta$. This mixing is due to the zero topological charge Q of the $\bar{\Pi}$ system. In general, $Q = 0$ fields give the following contributions to the free energy:

$$F = F^{\text{PT}} + F^{\bar{\Pi}} + F^{\bar{\Pi}\bar{\Pi}} + \dots = \sum_{n=0}^{\infty} C_n^{(0)} g^{2n}(T) + e^{-16\pi^2/g^2(T)} \sum_{n=0}^{\infty} C_n^{(1)} g^{2n}(T) + O(e^{-32\pi^2/g^2(T)}). \quad (3.11)$$

Equation (3.11) contains perturbative series in powers of g^2 as well as exponentially small corrections. Since the perturbative expansion is a badly divergent one (cf. Refs. [6]), one has to define rigorously what the first term in Eq. (3.11) means. We shall define $F^{\text{PT}}(g^2)$ as

a function which does not contain the essential singularity at $g^2 = 0$ with the expansion $F^{\text{PT}} = \Sigma C_n^{(0)} g^{2n}$. In a similar way, one can define the sum of g^{2n} corrections to other exponentially small terms in Eq. (3.11).

Now we can calculate the $\bar{\Pi}$ non-perturbative contribution to F , which we identify with the $O(e^{-16\pi^2/g^2})$ term in Eq. (3.11). To do this one has to perform the following program [6], [4]. Let us change the sign of g^2 : $-g^2 > 0$ in the integral $I^{\text{att}}(g^2)$. It becomes convergent at $t \rightarrow \infty$ (small $|\vec{R}|$) and gives the contribution $O(e^{16\pi^2/g^2})$. It is exactly the second term in Eq. (3.11) since the first one is negligible at $g^2 < 0$. Performing the analytic continuation from negative g^2 to positive ones, as a final answer one obtains $F^{\bar{\Pi}}$. This program has been carried out for quantum mechanical problems [4], [6]. $F^{\bar{\Pi}}$ turns out to have an imaginary part reflecting the non-Borel summable behaviour of F^{PT} at positive g^2 [6]:

$$F^{\text{PT}} \sim \sum_n n! g^{2n}. \quad (3.12)$$

We should next like to address the following question: does the same pathology (the imaginary part of $F^{\bar{\Pi}}$) arise in gauge theory?

The answer is ‘yes’, and we now demonstrate it. The only modification in high-temperature QCD as compared with quantum mechanics (see, for example, Ref. [4]) is the presence of the $\bar{\Pi}$ orientation. The point is that the sum of I^{rep} and I^{att} in Eq. (3.9) is convergent at $t \rightarrow 0$ (large R limit) owing to the dipole-dipole type of the orientation dependence of the potential (2.8). Namely,

$$\int_0^1 d\alpha (\alpha^2 - 1/3) = 0. \quad (3.13)$$

However, once we consider I^{rep} and I^{att} separately the logarithmic divergence of both pieces arises at $t \rightarrow 0$. Surely it will cancel in the final sum of I^{rep} and I^{att} . However, to do the actual computations one has to introduce a regularization. For example, let

$$I^{\text{rep}}(g^2) = \int_\epsilon^\infty dt/t^2 \int_0^{1/\sqrt{3}} d\alpha \left[e^{t/g^2(\alpha^2 - 1/3)} - 1 \right] \quad (3.14)$$

and

$$I^{\text{att}}(\bar{g}^2) = \int_\epsilon^\infty dt/t^2 \int_{1/\sqrt{3}}^1 d\alpha \left[e^{-t/\bar{g}^2(\alpha^2 - 1/3)} - 1 \right], \quad (3.15)$$

where $\bar{g}^2 \equiv -g^2$ is taken to be positive in Eq. (3.15). Performing the integrations and making the substitution $\bar{g}^2 = -g^2$ afterwards, one obtains the finite result for $I(g^2)$

$$\begin{aligned} I(g^2) &= \frac{1}{g^2} \left[\int_0^{1/\sqrt{3}} d\alpha \left(\frac{1}{3} - \alpha^2 \right) \ln \frac{1/3 - \alpha^2}{g^2} - \int_{1/\sqrt{3}}^1 d\alpha \left(\alpha^2 - \frac{1}{3} \right) \ln \frac{\alpha^2 - 1/3}{-g^2} \right] \\ &= -\frac{1}{g^2} \left[\frac{2}{9} - \frac{2}{9\sqrt{3}} \ln \frac{1 + \sqrt{3}}{1 - 1/\sqrt{3}} \pm \frac{i\pi 2}{9\sqrt{3}} \right]. \end{aligned} \quad (3.16)$$

Here the sign of the imaginary part depends on the way the analytic continuation goes from negative g^2 to positive ones.

We have only made the analytic continuation for the I^{att} part of $I(g^2)$. However, taken by itself, it contains a divergence at $t \rightarrow 0$. Therefore, one may be afraid that our answer (3.16) is an artifact of the regularization scheme we have chosen. To check this we also perform the calculation of Eq. (3.9) in the dimensional regularization ($d^3R \rightarrow$

Figure 1: The free energy dependence on temperature. The solid line corresponds to the result of lattice Monte Carlo simulations for F/F_{SB} from Ref. [17]. The dilute gas contribution $F^{\text{I}}/F_{\text{SB}}$ is given by the dashed line. The dotted line corresponds to $F^{\text{II}}/F_{\text{SB}}$.

$\alpha^{3(1-\varepsilon)}R$, thus $dt \rightarrow dt t^\varepsilon$, $\varepsilon > 0$). The answer is the same. We conclude that in high-temperature QCD the II non-perturbative part of the free energy actually does contain the imaginary part given by Eqs. (3.8) and (3.16), similarly to the case of quantum mechanical problems with instantons.

Since the total free energy in a stable state has to be real $\text{Im } F^{\text{II}} \neq 0$ ensures the existence of the imaginary part of F^{PT} equal in the value and opposite in sign. It gives the large n behaviour of perturbative series:

$$F^{\text{PT}} \sim \sum_n (n+8)! \left(\frac{g^2(T)}{16\pi^2} \right)^n. \quad (3.17)$$

Thus the only sum of F^{PT} and F^{II} is real and therefore has a physical sense. However, the typical accuracy of perturbative calculations in field theories is that of one or two loops, whereas the imaginary part comes from high loops. This creates the problem of the unambiguous definition of non-perturbative effects for physical quantities. It is clear that the same problem arises for other quantities whenever different contributions with the same topological charge are taken into account.

As for the real part of F^{II} , it turns out to be very small for reasonable values of T . In Fig. 1 we have plotted the result of lattice Monte Carlo simulations for F/F_{SB} [17] ($F_{\text{SB}} = -\pi^2/15 T^4$). The dilute gas contribution in Eq. (3.4) is given by the dashed line. The dotted line corresponds to $F^{\text{II}}/F_{\text{SB}}$. It has the ‘right’ sign, but becomes valuable only at small T below T_c . This means that instantons have hardly anything to do with the nature of the phase transition. Note that perturbative corrections to F_{SB} are instead too large and ill-defined (owing to infrared divergences [18], [2]) and also cannot explain the

lattice curve above T_c [17].

4 THE THEORY WITH QUARKS

In this section we calculate the $\bar{\Pi}$ interacting potential induced by the presence of N_f sorts of quarks. We use the following two-component notations for quark fields:

$$\psi_D = \begin{pmatrix} \psi_L^{k\alpha} \\ -i\psi_{R\dot{\alpha}}^k \end{pmatrix}, \quad \bar{\psi}_D = (i\bar{\psi}_{Rk\alpha} - \bar{\psi}_L^{\dot{\alpha}k}), \quad (4.1)$$

where ψ_D means Dirac spinor, $\dot{\alpha}, \alpha = 1, 2$ are spin indices and $k = 1, 2$ are colour ones. The Euclidean fermion action reads

$$S = \sum_{i=1}^{N_f} \int_0^\beta dt \int d^3x \left[\bar{\psi}_L^{\dot{\alpha}(i)} i \bar{\nabla}_{\dot{\alpha}\alpha} \psi_L^{\alpha(i)} + \psi_R^{\dot{\alpha}(i)} i \bar{\nabla}_{\dot{\alpha}\alpha} \bar{\psi}_R^{\alpha(i)} \right], \quad (4.2)$$

where $\bar{\nabla}_{\dot{\alpha}\alpha} = \bar{\sigma}_{\mu\dot{\alpha}\alpha} \nabla_\mu$ and $\nabla_\mu = \partial_\mu - igA_\mu$. We restrict ourselves to the chiral limit of massless quarks.

The fermion operator $i\nabla$ has $2N_f$ zero modes in the background of the I field. Namely [2]

$$\begin{aligned} \psi_{R\dot{\alpha}k}^{(i)} &= \theta_R^i \left\{ 1/\sqrt{2\pi T} \right\} \left\{ \rho_I^{-1} \Pi_I^{1/2} \right\} \partial_\mu (\Pi_I^{-1} f_I) [\bar{\sigma}_\mu u_R \bar{u}_1]_{\dot{\alpha}k} \\ \bar{\psi}_{L\dot{\alpha}k}^{(i)} &= \bar{\theta}_L^i \left\{ 1/\sqrt{2\pi T} \right\} \left\{ \rho_I^{-1} \Pi_I^{1/2} \right\} \partial_\mu (\Pi_I^{-1} f_I) [\bar{\sigma}_\mu u_R \bar{u}_1]_{\dot{\alpha}k}, \end{aligned} \quad (4.3)$$

where $\theta_R^i, \bar{\theta}_L^i$ are Grassmann variables, while

$$f_I = \sum_{n=-\infty}^{\infty} (-1)^n \frac{\rho_I^2}{(t - t_I - n\beta)^2 + (\vec{x} - \vec{x}_I)^2} = (\Pi_I - 1) \frac{\cos \pi(t - t_I)/\beta}{\cosh(\pi|\vec{x} - \vec{x}_I|/\beta)}. \quad (4.4)$$

We see that the fermion zero mode had $e^{-\pi|\vec{x}|/\beta}$ fall-off at large $|\vec{x}|$. It is due to the antiperiodic boundary condition for the fermion fields at the temperature strip $\psi(\beta, \vec{x}) = -\psi(0, \vec{x})$ [20]. In the $\bar{I}$ background another pair of fermion fields is non-zero. Namely,

$$\begin{aligned} \psi_L^{(i)k}{}_\alpha &= \theta_L^i \left(1/\sqrt{2\pi T} \right) \left\{ \rho_{\bar{I}}^{-1} \Pi_{\bar{I}}^{1/2} \right\} \partial_\mu (\Pi_{\bar{I}}^{-1} f_{\bar{I}}) [u_2 \bar{\sigma}_\mu]^k{}_\alpha \\ \bar{\psi}_R^{(i)k}{}_\alpha &= \bar{\theta}_R^i \left(1/\sqrt{2\pi T} \right) \left\{ \rho_{\bar{I}}^{-1} \Pi_{\bar{I}}^{1/2} \right\} \partial_\mu (\Pi_{\bar{I}}^{-1} f_{\bar{I}}) [u_2 \bar{\sigma}_\mu]^k{}_\alpha, \end{aligned} \quad (4.5)$$

where $f_{\bar{I}}$ is given by Eq. (4.4) $\rho_I, t_I, \vec{x}_I$ is replaced by $\rho_{\bar{I}}, t_{\bar{I}}, \vec{x}_{\bar{I}}$, and θ_L^i, θ_R^i are a new set of Grassmann variables.

The fermion-induced potential $U_{\bar{\Pi}}^{\text{ferm}}$ is calculated in a similar way to the boson-induced one. The zero modes (4.3) and (4.5) can be considered as $\bar{\Pi}$ streamline fermionic fields of zero-order approximation [5]. The Grassmann parameters $\theta_I, \bar{\theta}_I$ enter into the game on the same footing with the bosonic collective coordinates τ_i . Inserting Eqs. (4.3) and (4.5) in the fermion action (4.2), one gets after tedious calculation

$$\begin{aligned} U_{\bar{\Pi}}^{\text{ferm}} &= -16i\pi^3 \sum_{i=1}^{N_f} (\bar{\theta}_L^i \theta_L^i + \theta_R^i \bar{\theta}_R^i) \left(v_0 \sin \frac{\pi(t_I - t_{\bar{I}})}{\beta} + \right. \\ &\quad \left. \frac{(\vec{v}, \vec{R})}{|\vec{R}|} \cos \frac{\pi(t_I - t_{\bar{I}})}{\beta} \right) \frac{\rho_I \rho_{\bar{I}}}{\beta |\vec{R}|} e^{-\pi|\vec{R}|/\beta} (1 + O(1/|R|^2)). \end{aligned} \quad (4.6)$$

Figure 2: Instanton-anti-instanton molecular phase. Solid lines correspond to insertions of $U_{\text{II}}^{\text{ferm}}$, wavy lines stand for $\exp [-U_{\text{II}}^{\text{bos}}]$.

The coefficient in this equation depends on the scale of the parameters θ , which is fixed by our choice of the normalization of zero modes by $2\pi\beta$ in Eqs. (4.3) and (4.5).

We see that $U_{\text{I}}^{\text{ferm}}$ has exponential fall-off at large $|\vec{R}|$, reflecting the antiperiodic boundary conditions for fermion fields. This peculiar behaviour is the direct consequence of non-zero temperature.

The Mayer virial expansion (3.4) for the free energy contains now only contributions with zero topological charge, since only these configurations have no exact fermion modes. Thus F^{I} is absent and the expansion for the non-perturbative free energy starts from F^{II} . It yields

$$F^{\text{II}} \sim - \frac{1}{\beta V^{(3)}} \int d\mu_{\text{I}} d\theta_{\text{R}} d\bar{\theta}_{\text{L}} d\mu_{\text{I}} d\theta_{\text{L}} d\bar{\theta}_{\text{R}} \left[\exp - (U_{\text{II}}^{\text{bos}} + U_{\text{II}}^{\text{ferm}}) \right] , \quad (4.7)$$

where $U_{\text{II}}^{\text{bos}}$ and $U_{\text{II}}^{\text{ferm}}$ are given by Eqs. (2.8) and (4.6). Unfortunately, we cannot actually calculate F^{II} (4.7). The point is that, owing to the exponential fall-off of $U_{\text{II}}^{\text{ferm}}$, the $|\vec{R}|$ integration in (4.7) is dominated by $|\vec{R}| \sim \beta$ and unknown $1/|\vec{R}|$ corrections to Eqs. (2.8) and (4.6) become important²⁾.

To conclude this section let us make a qualitative comment on the structure of the instanton vacuum in this present case. Instantons form the II molecular phase shown in Fig. 2.

Vertices correspond to the $\text{I}(\bar{\text{I}})$ measure $d\mu_n^{\text{I}} d\theta_{\text{Rn}} d\bar{\theta}_{\text{Ln}} (d\mu_n^{\text{I}} d\theta_{\text{Ln}} d\bar{\theta}_{\text{Rn}})$. Solid lines correspond to insertions of $U_{\text{II}}^{\text{ferm}}$. Notice that each $\text{I}(\bar{\text{I}})$ emits exactly $2N_{\text{f}}$ solid lines. Wavy lines correspond to $\exp (-U_{\text{II}}^{\text{bos}})$. This II molecular gas is rather dilute at high

2) Remember that for the pure bosonic case (Section 3) we had $|\vec{R}| \sim \beta/g^{2/3}(T) \gg \beta$.

temperature, the distances between molecules being of the order of $\beta e^{16\pi^2/g^2(T)}$, whereas the distance between I and $\bar{I}$ within the single molecule is $|\vec{R}| \sim \beta$.

5 't HOOFT $2N_f$ EFFECTIVE VERTEX

In this section we study instanton-induced effects in the low-energy limit of the theory. Using the $\bar{I}\bar{I}$ interacting potential from Eq. (4.6), we can write down the grand partition function for the instanton vacuum at large distances

$$Z = \sum_{N,M} \frac{1}{N!M!} \int \prod_{n=1}^N d\tilde{\mu}_n \prod_{i=1}^{N_f} d\theta_{nR}^i d\bar{\theta}_{nL}^i \prod_{m=1}^M d\tilde{\mu}_m \prod_{i=1}^{N_f} d\theta_{mL}^i d\bar{\theta}_{mR}^i \exp - \left[\sum_{n,m} U_{nm}^{\text{ferm}} \right], \quad (5.1)$$

where U_{nm}^{ferm} is the potential (4.6) of the interaction of the n^{th} I with $\bar{I}$. The instanton measure modifies in the presence of quarks [3], [9]: ³⁾

$$d\tilde{\mu}_n = d\mu_n c_F^{-N_f} \left(\frac{\rho T}{2\pi} \right)^{N_f} \exp [N_f/3\pi^2 p^2 T^2 - N_f/6 \ln(1 + \pi^2 \rho^2 T^2/3)], \quad (5.2)$$

where $c_F^{\text{MS}} \equiv 1.166$. Note that we have neglected U_{nm}^{bos} in Eq. (5.1) [see Eq. (3.1)] in the large distance limit since it gives $O(\beta^3/g^2 R^3)$ corrections to Eq. (5.1) (we assume $|\vec{R}_{nm}|$ to be much larger than $\beta/g^{2/3}$ in this section).

We would now like to rewrite the ‘statistical mechanics of fermions’ (5.1) in terms of the equivalent field theory using the general Polyakov method [7]. This method can be applied to the statistical systems of particles interacting via the Coulomb potential (for a more detailed discussion of the application of this method to Coulomb systems of instantons, see Ref. [15]). The point is that, for example, the Coulomb potential in four-dimensional theory for bosonic particles has $1/x^2$ behaviour. It coincides exactly with the behaviour of the propagator for the bosonic field. This coincidence can be used to represent a Coulomb system in terms of effective field theory [7].

In the present case, our potential (4.6) also exactly resembles the propagator of the free fermion field. Indeed, in the low-energy limit this propagator has the form

$$\langle \psi(x)^\alpha, \bar{\psi}(0)^{\dot{\alpha}} \rangle = -\frac{iT^2}{2} \left\{ \sigma_0^{\alpha\dot{\alpha}} \sin \frac{\pi t}{\beta} + \sigma_i^{\alpha\dot{\alpha}} \frac{x_i}{|\vec{x}|} \cos \frac{t\pi}{\beta} \right\} \frac{e^{-\pi|\vec{x}|/\beta}}{|\vec{x}|}. \quad (5.3)$$

The similarity of Eqs. (4.6) and (5.3) enables us to consider Eq. (5.1) as a Coulomb statistical system of Fermi ‘particles’. We have

$$\begin{aligned} \exp \left(- \sum_{n,m} U_{nm}^{\text{ferm}} \right) &= c \int \prod_{i=1}^{N_f} d\psi_L^{(i)} d\bar{\psi}_R^{(i)} d\psi_R^{(i)} d\bar{\psi}_L^{(i)} \\ &\times \exp \left(- \int d^4x \left\{ \sum_{i=1}^{N_f} \left[\bar{\psi}_L^{(i)} i \not{\partial} \psi_L^{(i)} + \psi_R^{(i)} i \not{\partial} \bar{\psi}_R^{(i)} \right. \right. \right. \\ &- \sum_n 4\pi^{3/2} \sqrt{\beta} \rho_n \delta^{(4)}(x - x_n) [u_R \bar{u}_1]^k \alpha \left(\bar{\theta}_{Ln}^i \psi_{Lk}^{(i)\alpha} + \bar{\psi}_{Rk}^{(i)\alpha} \theta_{Rn}^i \right) \\ &\left. \left. \left. - \sum_m 4\pi^{3/2} \sqrt{\beta} \rho_m \delta^{(4)}(x - x_n) [\bar{u}_2]_{\dot{\alpha}k} \left(\bar{\theta}_{Rm}^i \psi_R^{(i)\dot{\alpha}k} + \bar{\psi}_L^{(i)\dot{\alpha}k} \theta_{Lm}^i \right) \right] \right\} \right) \end{aligned}$$

3) The Jacobian $(T/2\pi)^{N_f}$ is included in Eq. (5.2) due to our choice of the fermion zero mode normalization by $(2\pi/T)$.

(5.4)

where C is the normalization constant. We have introduced here N_f sorts of Fermi colour and spinor doublet fields $\psi_L, \bar{\psi}_R, \bar{\psi}_L, \psi_R$. Now we carry out sums over N and M in eq. (5.1). We get

$$\begin{aligned}
Z = & C \int \prod_{i=1}^{N_f} d\psi_L^{(i)} d\bar{\psi}_R^{(i)} d\psi_R^{(i)} d\bar{\psi}_L^{(i)} \times \exp \left(- \int d^4x \left\{ \sum_{i=1}^{N_f} \left(\bar{\psi}_L^{(i)} i \not{\partial} \psi_L^{(i)} + \psi_R^{(i)} i \not{\partial} \bar{\psi}_R^{(i)} \right) \right. \right. \\
& - \int \tilde{n}(\rho) d\rho \frac{d^3v}{2\pi^2} \prod_{i=1}^{N_f} d\theta_R^i d\bar{\theta}_L^i \exp \left(4\pi^{3/2} \rho \sqrt{\beta} \left[\bar{\theta}_L^i (u_R \bar{u}_1 \psi_L^{(i)}) + (u_R \bar{u}_1 \bar{\psi}_R^{(i)}) \theta_R^i \right] \right) , \\
& \left. \left. - \int \tilde{n}(\rho) d\rho \frac{d^3v}{2\pi^2} \prod_{i=1}^{N_f} d\theta_L^i d\bar{\theta}_R^i \exp \left(4\pi^{3/2} \rho \sqrt{\beta} \left[\bar{\theta}_R^i (\bar{u}_2 \psi_R^{(i)}) + (\bar{u}_2 \bar{\psi}_L^{(i)}) \theta_L^i \right] \right) \right\} \right) . \quad (5.5)
\end{aligned}$$

Here brackets mean there is contraction of the colour and spin indices; for example $(\bar{u}_2 \psi_R) = \bar{u}_{2\dot{\alpha}k} \psi_R^{\dot{\alpha}k}$ and $\tilde{n}(\rho)$ is related to $n(\rho)$ [see Eq. (3.3)] via Eq. (5.2) (where the obvious substitution $d\mu \rightarrow n(\rho)$, $d\tilde{\mu} \rightarrow \tilde{n}$ has to be made. Equation (5.5) represents the statistical system (5.1) in terms of effective field theory. It is clear that fields ψ which enter into this field theory actually coincide with quark fields⁴⁾

Integration over Grassmann parameters θ can easily be done in Eq. (5.5) and we arrive at the following effective low-energy potential for quark fields:

$$\begin{aligned}
V^{\text{eff}} = & -4^4 \pi^6 \beta^{N_f} \int d^4x \int d\rho \rho^{2N_f} \tilde{n}(\rho) \frac{d^3v}{2\pi^2} \left[\prod_{i=1}^{N_f} (\bar{u}_2 \psi_L^{(i)}) (\bar{u}_2 \bar{\psi}_R^{(i)}) \right. \\
& \left. + \prod_{i=1}^{N_f} (u_R \bar{u}_1 \bar{\psi}_R^{(i)}) (u_R \bar{u}_1 \psi_R^{(i)}) \right] . \quad (5.6)
\end{aligned}$$

This is nothing other than the 't Hooft effective $2N_f$ vertex [9], which has been obtained from the one-instanton calculation. For example, for the case $N_f = 2$ it takes the more familiar form (after integration over orientations):

$$\begin{aligned}
V^{\text{eff}} = & \frac{-4^4 \pi^6 \beta^2}{24} \int d^4x \int d\rho \rho^4 \tilde{n}(\rho) \left\{ \left[\det_{ij} \left(\psi_L^{(i)} \bar{\psi}_R^{(j)} \right) - \left(\psi_L^{(1)} \psi_L^{(2)} \right) \left(\bar{\psi}_R^{(1)} \bar{\psi}_R^{(2)} \right) \right] \right. \\
& \left. + \left[\det_{ij} \left(\psi_R^{(i)} \bar{\psi}_L^{(j)} \right) - \left(\psi_R^{(1)} \psi_R^{(2)} \right) \left(\bar{\psi}_L^{(1)} \bar{\psi}_L^{(2)} \right) \right] \right\} . \quad (5.7)
\end{aligned}$$

This result is often quoted in the literature without the second terms in square brackets. Presumably the reason is that in the original 't Hooft paper [8] these terms have been omitted, since instanton-orientation effects have not been taken into account. However, in Ref. [9] the averaging over orientations has been correctly performed, and the result [Eqs. (5.6), (5.7)] has been obtained. We rederive it now, taking into account fermion-induced $\bar{\Pi}$ interactions. This proves rigorously that this vertex can be used in graphs involving many instanton vertices.

4) This can be shown rigorously. One has to calculate correlation functions of the quark field in the instanton vacuum and show that the coupling of the quark field to the field ψ in Eq. (5.5) is equal to unity (see Ref. [15]).

Note that result (5.7) obeys all physical symmetries⁵⁾ except the chiral one. The same conclusion can be drawn for the general result (5.6). Therefore, Eq. (5.6) shows the breakdown of the chiral group $[U_A(1)]^{N_f}$ to group Z_{2N_f} . Hence, instantons produce chiral-symmetry breaking. The question concerning the possible existence of binary-quark condensates is postponed to the next section.

If one is interested in the bosonization of the theory under consideration, it is necessary to rewrite the second terms in square brackets in Eq. (5.7) using Fierz identities,

$$\begin{aligned} (\psi_R^{(1)} \psi_R^{(2)}) (\bar{\psi}_L^{(1)} \bar{\psi}_L^{(2)}) &= -\frac{1}{2} \det (\psi_R^{(i)} \bar{\psi}_L^{(j)}) + \frac{1}{32} \{ (\bar{\psi}_L^{(1)} [\bar{\sigma}_\mu \sigma_\mu - \bar{\sigma}_\nu \sigma_\nu] \psi_R^{(1)}) \\ &\times [\bar{\psi}_L^{(2)} (\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu) \psi_R^{(2)}] - [\bar{\psi}_L^{(2)} (\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu) \psi_R^{(1)}] [\bar{\psi}_L^{(1)} (\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu) \psi_R^{(2)}] \}. \end{aligned} \quad (5.8)$$

Expression (5.8) contains four-fermion vertices $(\bar{\psi}\psi\bar{\psi}\psi)$. One can pass from these four-fermion vertices to Yukawa vertices $\bar{\psi}\psi\phi$ introducing Bose fields ϕ by means of a simple formula:

$$\exp \left[\int d^4x (1/m^2 \bar{\psi}\psi\bar{\psi}\psi) \right] \approx \int d\phi \exp \left[\int d^4x \left(\frac{m^2}{2} \phi^2 + \bar{\psi}\psi\phi \right) \right]. \quad (5.9)$$

The first term in Eq. (5.9) gives the Yukawa couplings with scalar Bose fields ϕ (without indices) which actually coincide with π and σ mesons (see for example Ref. [10]). The second term in Eq. (5.9) gives the Yukawa couplings with the vector fields $\phi_{\alpha\beta}$ ($\phi_{\alpha\beta} = -\phi_{\beta\alpha}$).

To conclude, we would like to stress the actual inconsistency of results (5.6), (5.7) at short distances. These 't Hooft potentials are derived by dropping bosonic $\bar{\Pi}\Pi$ interactions and are, hence, valid only in tree graphs in the low-temperature region. In loop corrections the region of small $R \sim \beta$ contributes to the integral and requires the bosonic $\bar{\Pi}\Pi$ interactions to be taken into account.

6 BINARY-QUARK CONDENSATE

In this section we would like to address the question of the existence of the binary-quark condensates for different N_f in the $\bar{\Pi}\Pi$ gas approximation for hot QCD. The binary-quark condensate is defined as

$$\begin{aligned} \langle \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \rangle &= \frac{1}{Z_0} \text{Tr} \left[e^{-\beta H} (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R) \right] = \sum_N \sum_M \frac{1}{N!} \frac{1}{M!} \prod_n^N (d\mu_n^I d\theta_{R_n} d\bar{\theta}_{L_n}) \\ &\times \prod_m^M (d\mu_m^I d\theta_{L_m} d\bar{\theta}_{R_m}) (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R) \exp \left[- \sum_n^N \sum_m^M (U_{nm}^{\text{bos int}} + U_{nm}^{\text{ferm int}}) \right]. \end{aligned} \quad (6.1)$$

To obtain a non-trivial binary condensate it is necessary to break down the chiral group $[U_A(1)]^{N_f}$ to Z_2 . We learn in Section 3 that I 's ($\bar{I}$'s) produce symmetry breaking $[U_A(1)]^{N_f} \rightarrow Z_{2N_f}$. For the $N_f = 1$ case $Z_{2N_f} = Z_2$, and the instanton-induced 't Hooft vertex (5.6) ensures the non-zero $\langle \bar{\psi}\psi \rangle$. Indeed, Eq. (5.6) for $N_f = 1$ gives

$$V^{\text{eff}} = -4^3 \pi^6 \beta \int d\rho \rho^2 \tilde{n}(\rho) \int d^4x [(\psi_L \bar{\psi}_R) + (\bar{\psi}_L \psi_R)]. \quad (6.2)$$

5) It is easy to check that flavour symmetry is respected by both the first and the second term in square brackets in Eq. (5.7).

Figure 3: Instanton (anti-instanton) induced contribution to the binary quark condensate for $N_f=1$.

Figure 4: Instanton-anti-instanton induced leading-order ($T/\Lambda \gg 1$) contribution to the binary quark condensate for the case $N_f = 2$.

Equation (6.2) is nothing other than a mass term for quark fields. Its presence signals the appearance of $\langle \bar{\psi}\psi \rangle$. Putting it in another way one can calculate $\langle \bar{\psi}\psi \rangle$ directly from graph in Fig. 3.

$$\langle \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \rangle_{N_f=1} \simeq 0.285 (\ln T/\Lambda_{\overline{\text{MS}}})^4 (\Lambda_{\overline{\text{MS}}}/T)^{20/3} T^3. \quad (6.3)$$

Here the numerical factor arises from the integration over the instanton size. Equation (6.3) represents the leading one-instanton contribution to the chiral condensate. One can calculate the correction coming from the $2\text{I} + \bar{\text{I}}$ system. It contains the additional small factor $0 < (\Lambda_{\overline{\text{MS}}}/T)^{40/3} \ll 1$. We would like to stress that Eq. (6.3) gives the non-zero value for $\langle \bar{\psi}\psi \rangle$ for arbitrarily high temperature.

To obtain a non-zero result for $\langle \bar{\psi}\psi \rangle$ at $N_f > 1$ it is necessary to break the group Z_{2N_f} down to Z_2 . The leading ($T/\Lambda \gg 1$) contribution to the binary-quark condensate, for the case $N_f = 2$, is shown in Fig. 4. It represents the infinite $\bar{\text{I}}\bar{\text{I}}$ chain. (It is impossible to construct any finite $\bar{\text{I}}\bar{\text{I}}$ configuration contributed to the binary-quark condensate). This contribution is proportional to

$$\int \dots \int d\rho_1 n(\rho_1) F(\rho_1, \rho_2) d\rho_2 n(\rho_2) F(\rho_2, \rho_3) \dots, \quad (6.4)$$

where

$$F(\rho_i, \rho_j) \equiv \int d^4(x_i - x_j) d^3 v_{ij} d^2 \theta (U_{ij}^{\text{ferm}})^2 e^{-U_{ij}^{\text{bos}}}, \quad (6.5)$$

and $n(\rho)$ is I ($\bar{\text{I}}$) density

$$n(\rho_i) \sim e^{-8\pi^2/g^2(\rho_i)} e^{-2(\pi\rho_i T)^2}. \quad (6.6)$$

As we have already mentioned in Section 4, we cannot actually compute integral (6.5) since the integration over $(x_i - x_j)$ is dominated by short distances $\sim \beta$, where our approximation is invalid. However, we need not know the exact value of expression (6.5). For our purpose the rough estimate

$$n(\rho_i) f(\rho_i, \rho_{i+1}) \sim e^{-8\pi^2/g^2(T)} e^{-2(\pi\rho_i T)^2} \quad (6.7)$$

is sufficient. It is valid at the high-temperature limit. Owing to Eq. (6.6), the main contribution to expression (6.4) comes from the region $\rho_1 \sim \rho_2 \sim \rho_3 \sim \dots \sim 1/T$ and we arrive at an exactly zero answer for the binary-quark condensate for $N_f = 2$.

It is possible to understand the phenomenon of high-temperature restoration of chiral invariance from the first principles. According to the index theorem [21], the binary-quark condensate can arise from the configuration with topological charge equal to $1/N_f$. For $N_f > 1$, $1/N_f$ is not an integer number. But configurations with fractional topological charge have an infinite Euclidean action and are suppressed in the thermodynamic limit. In the high-temperature phase, instanton contributions to physical quantities are given by the Mayer virial expansion in the instanton density. Each term represents the contribution of a cluster containing a finite number of instantons. These terms cannot contribute to $\langle \bar{\psi}\psi \rangle$. Therefore, we conclude that the chiral condensate is always zero in the high-temperature phase for $N_f \geq 2$. On the contrary, in the low-temperature phase, at $T < T_c$, instantons form a kind of liquid [10], [22]. It represents an infinite instanton cluster filling the whole volume with an action proportional to $\beta V^{(3)}$. Then the formation of the chiral condensate is not, in principle, ruled out by the index theorem. However, we cannot study the liquid instanton phase in this paper since semiclassical methods become invalid in the strong-coupling regime.

7 CONCLUSIONS

We have studied instanton effects in high-temperature QCD with $\bar{\text{I}}$ interactions taken into account. The $\bar{\text{I}}$ contribution to the free energy turns out to have an imaginary part. Its appearance is the consequence of a non-Borel summable behaviour of the perturbative loop expansion. Thus the problem arises of the mixing of non-perturbative contributions with high orders of perturbative series. To obtain real results for physical quantities one has somehow to add high orders of perturbative loop expansion to purely non-perturbative contributions. However, this is hardly possible to perform practically. As for the real part of $F^{\bar{\text{I}}}$, it turns out to be numerically negligible at $T > T_c$.

We are also studying the possibility of chiral-symmetry breaking due to instantons. For the $N_f = 1$ case $\langle \bar{\psi}\psi \rangle$ is non-zero at arbitrarily high temperature, whereas for $N_f \geq 2$ it is always zero in the high-temperature instanton medium.

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