CERN - TH - 6101/91

A DERIVATION OF THE NAMBU-GOTO ACTION FROM INVARIANCE PRINCIPLES

D.R. Grigore*

Theory Division, CERN - Geneva

ABSTRACT

We show that the Nambu-Goto action of string theory can be obtained from the invariance with respect to the reparametrization of the world sheet and the invariance with respect to the Poincaré transformations. The proof is based on the geometrical Lagrangian approach of Souriau. In a three-dimensional Minkowski space we obtain a new Lagrangian of a Chern-Simons nature.

CERN - TH - 6101/91

May 1991

* Permanent address: Department of Theoretical Physics, Institute of Atomic Physics, Bucharest-Magurele, Romania

1. Introduction

The Nambu-Goto action for the relativistic string is usually postulated in analogy to the relativistic action for a pointlike particle (see for instance [1]). On the other hand, it is known that the relativistic action for the pointlike particle can be derived rigorously using only the Poincaré invariance [2]-[3]. So, it is natural to expect that a similar statement is valid for the Nambu-Goto action as well. The purpose of this paper is to prove this point.

In Section 2 we will present the general formalism. This formalism uses only the Lagrangian density and is based on geometrical ideas originating in the work of Lagrange, used systematically in particle mechanics by Souriau [4] and generalized to dynamical systems with infinite number of degrees of freedom in [5]-[7]. Recently this formalism was used in the analysis of gauge invariance [8]. Here we use the same techniques to analyse the general case of a p -brane postulating the invariance with respect to reparametrizations of the world sheet and the usual Poincaré invariance.

In Section 3 we analyse the general conditions obtained in Section 2 in the simplest case $p = 1$, i.e. the relativistic particle. We derive the usual expression for the Lagrangian (of the homogeneous formulation). We have included the analysis of this case for two reasons. First, the proof is different from the proofs in [2]-[3] and does not use cohomology arguments like in [2]. Secondly, it is rather simple and it illustrates the general ideas, so it is a good guide for the more complicated case $p > 1$. Let us mention that an analysis using similar ideas was done in [9].

In Section 4 we analyse the case $p = 2$, i.e. the relativistic string. We are able to prove that the Poincaré invariance and the reparametrization invariance fix up to a multiplicative constant, the Lagrangian density in Minkowski spaces of dimension different from 3; namely, we obtain the Nambu-Goto Lagrangian. In a three-dimensional Minkowski space the Lagrangian is in general a sum of the Nambu-Goto term and a new term of a Chern-Simons nature.

2. The consequences of the reparametrization and of the Poincaré invariance for a p -brane

2.1 We present here the general formalism. Let S be a differentiable manifold of dimension $p + D$. To build a Lagrangian formalism we need an auxiliary object, namely the bundle of 1-jets of p -dimensional submanifolds of S , denoted $J_p^1(S)$. This manifold is

by definition:

$$J_p^1(S) \equiv \cup_{s \in S} J_p^1(S)_s$$

where $J_p^1(S)_s$ is the manifold of p -dimensional linear subspaces of the tangent space $T_s(S)$ at S in the point $s \in S$. This manifold is naturally fibered over S and we denote by π the canonical projection.

Let us choose a coordinate system (x^μ, ψ^A) on the open set $U \subseteq S$; here $\mu = 1, \dots, p$ and $A = 1, \dots, D$. Then, on the open set $V \subseteq \pi^{-1}(U)$ we shall choose the coordinate system $(x^\mu, \psi^A, \chi_\mu^A)$ where by definition, the p -plane in $T_{s_0}(S)$ corresponding to the set of numbers $(x_0^\mu, \psi_0^A, (\chi_\mu^A)_0)$ (here (x_0^μ, ψ_0^A) are the coordinates of $s_0 \in U$) is spanned by the tangent vectors:

$$\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + (\chi_\mu^A)_0 \frac{\partial}{\partial \psi^A}. \quad (2.1)$$

We will use systematically the convention of summation over the dummy indices.

The Lagrangian formalism describes p -dimensional immersed submanifolds usually given in a parametrized form $\Psi : P \rightarrow S$ (P is a p -dimensional manifold) We denote by $\dot{\Psi} : P \rightarrow J_p^1(S)$ the natural lift of Ψ . In the local coordinates above, if Ψ is given by $x^\mu \mapsto (x^\mu, \psi^A(x))$, then $\dot{\Psi}$ is given by $x^\mu \mapsto (x^\mu, \psi^A(x), \frac{\partial \psi^A}{\partial x^\mu}(x))$. A *Lagrangian function* (or *density*) is a smooth real function L defined on $J_p^1(S)$ or, more generally, on a subset of this manifold. Then the *Euler-Lagrange equations* for Ψ can be written in the local coordinates above as follows:

$$E_A(\Psi)(x) \equiv \frac{\partial L}{\partial \psi^A} \circ \dot{\Psi} - \frac{\partial}{\partial x^\mu} \left[\frac{\partial L}{\partial \chi_\mu^A} \circ \dot{\Psi} \right] = 0. \quad (2.2)$$

We define now the Poincaré-Cartan form associated to L to be the p -form θ_L given in the local coordinates above by:

$$\theta_L \equiv \varepsilon_{\mu_1, \dots, \mu_p} \sum_{k=1}^p \frac{1}{k!} \binom{p}{k} L_{A_1, \dots, A_k}^{\mu_1, \dots, \mu_k} \delta \psi^{A_1} \wedge \dots \wedge \delta \psi^{A_k} \wedge dx^{\mu_{k+1}} \wedge \dots \wedge dx^{\mu_p}. \quad (2.3)$$

Here $\varepsilon_{\mu_1, \dots, \mu_p}$ is the signature of the permutation $(1, \dots, p) \mapsto (\mu_1, \dots, \mu_p)$, $\delta \psi^A$ is by definition:

$$\delta \psi^A \equiv d\psi^A - \chi_\mu^A dx^\mu. \quad (2.4)$$

and $L_{A_1, \dots, A_k}^{\mu_1, \dots, \mu_k}$ is given by:

$$L_{A_1, \dots, A_k}^{\mu_1, \dots, \mu_k} \equiv \frac{1}{k!} \sum_{\sigma \in P_k} (-1)^{|\sigma|} \frac{\partial^k L}{\partial \chi_{\mu_{\sigma(1)}}^{A_1} \dots \partial \chi_{\mu_{\sigma(k)}}^{A_k}}. \quad (2.5)$$

(P_k is the group of permutation of the numbers $1, \dots, k$ and $|\sigma|$ is the signature of σ). Note that $L_{A_1, \dots, A_k}^{\mu_1, \dots, \mu_k}$ is completely antisymmetric in the upper indices and also in the lower indices.

The Lagrange-Souriau form σ_L is then, by definition:

$$\sigma_L \equiv d\theta_L. \quad (2.6)$$

This $(p+1)$ -form has some remarkable properties [5-7]:

(P1) σ_L given by (2.6) is globally defined;

(P2) The p -dimensional immersed submanifold $\Psi : P \rightarrow S$ satisfies the Euler-Lagrange equations (2.2) *iff*:

$$(\dot{\Psi})^*(i_Z \sigma_L) = 0$$

for any vector field Z on $J_n^1(S)$.

(P3) A Lagrangian L gives trivial Euler-Lagrange equations *iff*:

$$\sigma_L = 0$$

(P4) A transformation $\Phi \in \text{Diff}(J_p^1(S))$ is a *Noetherian symmetry* for L (i.e. leaves the action functional unchanged) *iff*:

$$\Phi^* \sigma_L = \sigma_L. \quad (2.7)$$

It is this last property which will be the key to our proof. Let us note also that usually one considers only *restricted Noetherian symmetries*, i.e., maps Φ which are of the form: $\Phi = \dot{\phi}$, where $\phi \in \text{Diff}(S)$ and $\dot{\phi} \in \text{Diff}(J_p^1(S))$ is the natural lift of ϕ to $J_p^1(S)$.

Finally, for practical computations, we need an explicit expression for σ_L . By differentiating (2.3) and rearranging the terms, one gets after some computations the following expression:

$$\begin{aligned} \sigma_L = \varepsilon_{\mu_1, \dots, \mu_p} \sum_{k=1}^p \frac{1}{k!} \binom{p}{k} [\sigma_{A_0, \dots, A_k}^{\mu_0, \dots, \mu_k} d\chi_{\mu_0}^{A_0} \wedge \delta\psi^{A_1} \dots \wedge \delta\psi^{A_k} \wedge dx^{\mu_{k+1}} \wedge \dots \wedge dx^{\mu_p} + \\ \rho_{A_0, \dots, A_k}^{\mu_1, \dots, \mu_k} \delta\psi^{A_0} \wedge \dots \wedge \delta\psi^{A_k} \wedge dx^{\mu_{k+1}} \wedge \dots \wedge dx^{\mu_p}]. \end{aligned} \quad (2.8)$$

where:

$$\sigma_{A_0, \dots, A_k}^{\mu_0, \dots, \mu_k} \equiv \frac{\partial L_{A_1, \dots, A_k}^{\mu_1, \dots, \mu_k}}{\partial \chi_{\mu_0}^{A_0}} - L_{A_0, \dots, A_k}^{\mu_0, \dots, \mu_k}. \quad (2.9)$$

and:

$$\rho_{A_0, \dots, A_k}^{\mu_1, \dots, \mu_k} \equiv \frac{1}{k+1} \left(\sum_{i=0}^k (-1)^i \frac{\partial L_{A_0, \dots, \hat{A}_i, \dots, A_k}^{\mu_1, \dots, \mu_k}}{\partial \psi_{A_i}} - \frac{\delta L_{A_0, \dots, A_k}^{\mu_0, \dots, \mu_k}}{\delta x^{\mu_0}} \right). \quad (2.10)$$

Here a hat over an index means, as usual, omission.

2.2 We particularize the general framework above for the case of a p -brane. We take in the general scheme from Section 2.1, $S = P \times M$. Here P is the manifold of the parameters characterizing our extended object, with coordinates denoted usually by τ^a ; $a = 1, \dots, p$. For instance, for a pointlike particle $p = 1$ and τ parametrize the world line. For a string, $p = 2$ and the parameters (τ^1, τ^2) describe the world sheet. If the string is open, then we can take $P = IR^2$, and if the string is closed, we can take $P = IR \times S^1$. In general P can be an arbitrary Riemann manifold. Up to a point, the analysis is rather insensitive to the specific choice of P . We have denoted by M the D -dimensional Minkowski space, identified as usual with IR^D and with coordinates denoted by X^μ ; $\mu = 1, \dots, D$. The coordinates on $J_p^1(S)$ are (τ^a, X^μ, U^μ_a) . We particularize now (2.1)-(2.10) above. We have:

$$\frac{\delta}{\delta \tau^a} \equiv \frac{\partial}{\partial \tau^a} + U^\mu_a \frac{\partial}{\partial X^\mu}. \quad (2.11)$$

$$\delta X^\mu \equiv dX^\mu - U^\mu_a d\tau^a. \quad (2.12)$$

$$\begin{aligned} \sigma_L = \varepsilon_{\mu_1, \dots, \mu_p} \sum_{k=1}^p \frac{1}{k!} \binom{p}{k} [\sigma_{\mu_0, \dots, \mu_k}^{a_0, \dots, a_k} dU^{\mu_0}_{a_0} \wedge \delta X^{\mu_1} \wedge \dots \wedge \delta X^{\mu_k} \wedge d\tau^{a_{k+1}} \wedge \dots \wedge d\tau^{a_p} + \\ \rho_{\mu_0, \dots, \mu_k}^{a_1, \dots, a_k} \delta X^{\mu_0} \wedge \dots \wedge \delta X^{\mu_k} \wedge d\tau^{a_{k+1}} \wedge \dots \wedge d\tau^{a_p}. \end{aligned} \quad (2.13)$$

where:

$$\sigma_{\mu_0, \dots, \mu_k}^{a_0, \dots, a_k} \equiv \frac{\partial L_{\mu_1, \dots, \mu_k}^{a_1, \dots, a_k}}{\partial U^{\mu_0}_{a_0}} - L_{\mu_0, \dots, \mu_k}^{a_0, \dots, a_k}. \quad (2.14)$$

$$\rho_{\mu_0, \dots, \mu_k}^{a_1, \dots, a_k} \equiv \frac{1}{k+1} \left(\sum_{i=0}^k (-1)^i \frac{\partial L_{\mu_0, \dots, \hat{\mu}_i, \dots, \mu_k}^{a_1, \dots, a_k}}{\partial X^{\mu_i}} - \frac{\delta L_{\mu_0, \dots, \mu_k}^{a_0, \dots, a_k}}{\delta \tau^{a_0}} \right). \quad (2.15)$$

and:

$$L_{\mu_1, \dots, \mu_k}^{a_1, \dots, a_k} \equiv \frac{1}{k!} \sum_{\sigma \in P_k} (-1)^{|\sigma|} \frac{\partial^k L}{\partial U^{\nu_1}_{\mu_{\sigma(1)}} \dots \partial U^{\nu_k}_{\mu_{\sigma(k)}}}. \quad (2.16)$$

2.3 Next, we analyse the reparametrization invariance. Let $\xi \in Diff(P)$. Then, the action of a reparametrization transformation on the manifold S is simply:

$$\phi_\xi(\tau, X) = (\xi(\tau), X). \quad (2.17)$$

One computes without difficulty the lift of this map to $J_p^1(S)$ and gets:

$$\dot{\phi}_\xi(\tau, X, U^\mu_a) = (\xi(\tau), X, A_a{}^b(\tau)U^\mu_b). \quad (2.18)$$

Here $A_a{}^b(\tau)$ is the inverse of the $p \times p$ matrix $\frac{\partial \xi^a}{\partial \tau^b}$, i.e.

$$\frac{\partial \xi^b}{\partial \tau^a}(\tau) A_b{}^c(\tau) = \delta_a^c. \quad (2.19)$$

Then it is clear that the theory is reparametrization invariant iff $\dot{\phi}_\xi$ is a Noetherian symmetry, i.e.:

$$(\dot{\phi}_\xi)^* \sigma_L = \sigma_L. \quad (2.20)$$

for $\forall \xi \in Diff(P)$.

It is not hard to prove this condition is equivalent to:

$$A_{a_0}{}^{b_0} \dots A_{a_k}{}^{b_k} \sigma_{\mu_0, \dots, \mu_k}^{a_0, \dots, a_k} \circ \dot{\phi}_\xi = (\det(A)) \sigma_{\mu_0, \dots, \mu_k}^{b_0, \dots, b_k}. \quad (2.21)$$

and

$$A_{a_1}{}^{b_1} \dots A_{a_k}{}^{b_k} \left(\rho_{\mu_0, \dots, \mu_k}^{a_1, \dots, a_k} \circ \dot{\phi}_\xi - \frac{1}{k+1} \frac{\partial^2 A_b{}^d}{\partial \tau^c} U^\nu{}_d \sigma_{\nu \mu_0, \dots, \mu_k}^{b c a_1, \dots, a_k} \circ \dot{\phi}_\xi \right) = (\det(A)) \rho_{\mu_0, \dots, \mu_k}^{b_1, \dots, b_k}. \quad (2.22)$$

In the following we will consider in (2.20) only the diffeomorphisms ξ belonging to the connected component of the identity in $Diff(P)$, i.e. $\xi \in (Diff(P))_0$. If $\xi \in (Diff(P))_0$ then without losing any information, we can consider in (2.21) and (2.22) that ξ is an infinitesimal transformation, i.e.:

$$\xi^a(\tau) = \tau^a + \theta^a(\tau). \quad (2.23)$$

with θ infinitesimally small. We insert (2.23) into (2.21) and (2.22) and we keep only the first order terms in θ . Taking into account that θ is an arbitrary function, it is straightforward to establish that $\sigma_{\dots}$ and $\rho_{\dots}$ do not depend on τ and also that they obey the following relations:

$$U^\nu{}_b \frac{\partial \sigma_{\mu_0, \dots, \mu_k}^{a_0, \dots, a_k}}{\partial U^\nu{}_c} = \delta_b^c \sigma_{\mu_0, \dots, \mu_k}^{a_0, \dots, a_k} - \sum_{i=0}^k \delta_b^{a_i} \sigma_{\mu_0, \dots, \mu_k}^{a_0, \dots, a_{i-1}, c, a_{i+1}, \dots, a_k}. \quad (2.24)$$

$$U^\nu{}_d \sigma_{\nu\mu_0,\dots,\mu_k}^{bca_1,\dots,a_k} + (b \leftrightarrow c) = 0. \quad (2.25)$$

$$U^\nu{}_b \frac{\partial \rho_{\mu_0,\dots,\mu_k}^{a_1,\dots,a_k}}{\partial U^\nu{}_c} = \delta_b^c \rho_{\mu_0,\dots,\mu_k}^{a_1,\dots,a_k} - \sum_{i=1}^k \rho_{\mu_0,\dots,\mu_k}^{a_1,\dots,a_{i-1},c,a_{i+1},\dots,a_k}. \quad (2.26)$$

2.4 The proper orthochronous Poincaré transformations act on S as follows:

$$\phi_{L,a}(\tau, X) = (\tau, LX + a). \quad (2.27)$$

Here $a \in M \cong IR^D$ and $L \in End(M)$ is a proper orthochronous Lorentz transformation. An elementary computation yields:

$$\dot{\phi}_{L,a}(\tau, X, U) = (\tau, LX + a, LU). \quad (2.28)$$

where:

$$(LU)^\mu{}_a \equiv L^\mu{}_\nu U^\nu{}_a. \quad (2.29)$$

Then the condition of Poincaré invariance is simply:

$$(\dot{\phi}_{L,a})^* \sigma_L = \sigma_L. \quad (2.30)$$

It is elementary to prove that (2.30) is equivalent to the fact that the functions $\sigma_{\dots}$ and $\rho_{\dots}$ do not depend on X and also they obey the following relations:

$$L^{\mu_0}{}_{\nu_0} \dots L^{\mu_k}{}_{\nu_k} \sigma_{\mu_0,\dots,\mu_k}^{a_0,\dots,a_k} \circ \dot{\phi}_\xi = \sigma_{\nu_0,\dots,\nu_k}^{a_0,\dots,a_k}. \quad (2.31)$$

$$L^{\mu_0}{}_{\nu_0} \dots L^{\mu_k}{}_{\nu_k} \rho_{\mu_0,\dots,\mu_k}^{a_1,\dots,a_k} \circ \dot{\phi}_\xi = \rho_{\nu_0,\dots,\nu_k}^{a_1,\dots,a_k}. \quad (2.32)$$

for any proper orthochronous Lorentz transformation L , i.e. they are Lorentz covariant tensor functions.

2.5 So, finally we get that the reparametrization and the Poincaré invariance are equivalent to the fact that the functions $\sigma_{\dots}$ and $\rho_{\dots}$ depend only on U and verify the relations (2.24)-(2.26) and (2.31)-(2.32). After one determines the most general form of $\sigma_{\dots}$ and $\rho_{\dots}$ verifying these equations, then one uses (2.14)-(2.16) to determine the most general form of L . This programme is feasible in principle for any p , but in practice we have succeeded in performing it, only for $p = 1$ and $p = 2$. These two cases will be treated in detail in the following two sections.

3. The relativistic pointlike particle

3.1 First, we particularize the formulae (2.14) and (2.15) to the case $p = 1$. From (2.13) it is clear that in this case we have only three functions to analyse: $\sigma_{\mu\nu}$, ρ_μ and $\rho_{\mu\nu}$. (Because the indices a can take only one value we have suppressed them.) From (2.14) we have for $k = 1$:

$$\sigma_{\mu\nu} = \frac{\partial^2 L}{\partial U^\mu \partial U^\nu}. \quad (3.1)$$

and from (2.15) for $k = 1$ and $k = 2$ respectively:

$$\rho_\mu = \frac{\partial L}{\partial X^\mu} - \frac{\delta L_\mu}{\delta \tau}. \quad (3.2)$$

$$\rho_{\mu\nu} = \frac{1}{2} \left(\frac{\partial^2 L}{\partial U^\mu \partial U^\nu} - (\mu \leftrightarrow \nu) \right). \quad (3.3)$$

3.2 We determine first $\sigma_{\mu\nu}$. From the Lorentz covariance (2.31) and the symmetry in μ and ν (see (3.1)) it follows that $\sigma_{\mu\nu}$ is of the following form:

$$\sigma_{\mu\nu}(U) = G_{\mu\nu} f(\|U\|^2) + U^\mu U^\nu g(\|U\|^2). \quad (3.4)$$

Here $G_{\mu\nu}$ is the Minkowski bilinear form on M , f and g are smooth functions of the variable:

$$\zeta \equiv \|U\|^2 \equiv G_{\mu\nu} U^\mu U^\nu. \quad (3.5)$$

and we have taken into account that, according to Section 2.5, $\sigma_{\mu\nu}$ is a function only of U .

Now we use the reparametrization invariance as expressed by the formulae (2.24) and (2.25). These two relations became in our case:

$$U^\omega \frac{\partial \sigma_{\mu\nu}}{\partial U^\omega} = -\sigma_{\mu\nu}. \quad (3.6)$$

$$U^\mu \sigma_{\mu\nu} = 0. \quad (3.7)$$

Inserting (3.4) into (3.7) we get easily that:

$$f(\zeta) = -\zeta g(\zeta)$$

i.e. instead of (3.4) we can take:

$$\sigma_{\mu\nu}(U) = g(\|U\|^2)(U_\mu U_\nu - G_{\mu\nu} \|U\|^2). \quad (3.8)$$

Inserting this relation into (3.6) we obtain a differential equation for the function g :

$$2\zeta g'(\zeta) + 3g(\zeta) = 0$$

which implies:

$$\zeta^3 [g(\zeta)]^2 = \text{const.} \quad (3.9)$$

We have three possibilities:

(a) $\text{const} > 0$. In this case L must be defined only on the domain:

$$D_+ \equiv \{(\tau, X, U) | \|U\|^2 > 0\}$$

Then (3.9) gives:

$$g(\zeta) = \frac{m}{\zeta^{3/2}}. \quad (3.10)$$

for some $m \in \mathbb{R}$.

(b) $\text{const} < 0$. In this case L must be defined on the domain:

$$D_- \equiv \{(\tau, X, U) | \|U\|^2 < 0\}$$

and (3.9) gives:

$$g(\zeta) = \frac{m}{(-\zeta)^{3/2}}. \quad (3.11)$$

for some $m \in \mathbb{R}$.

(c) $\text{const} = 0$. The Lagrangian can be defined everywhere and (3.9) gives:

$$g = 0. \quad (3.12)$$

3.3 We analyse separately these three cases.

(a) From (3.1) and (3.10) we have:

$$\frac{\partial^2 L}{\partial U^\mu \partial U^\nu} = \frac{m}{(\|U\|^2)^{3/2}} (U_\mu U_\nu - G_{\mu\nu})$$

This system can be integrated easily, and we obtain:

$$L(\tau, X, U) = -m(\|U\|^2)^{1/2} + U^\mu l_\mu(\tau, X) + l(\tau, X). \quad (3.13)$$

with l_μ and l arbitrary smooth functions of τ and X .

(b) In (3.13) one must change the sign of the expression under the square root.

(c) In (3.13) one must put $m = 0$.

3.4 We analyse now the functions ρ_μ and $\rho_{\mu\nu}$.

(a) First we compute these functions using (3.2), (3.3) and (3.13). One gets:

$$\rho_\mu = U^\nu \left(\frac{\partial l_\nu}{\partial X^\mu} - (\mu \leftrightarrow \nu) \right) + \frac{\partial l}{\partial X^\mu} - \frac{\partial l_\mu}{\partial \tau}. \quad (3.14)$$

$$\rho_{\mu\nu} = \frac{1}{2} \left(\frac{\partial l_\nu}{\partial X^\mu} - (\mu \leftrightarrow \nu) \right). \quad (3.15)$$

Because ρ_μ and $\rho_{\mu\nu}$ must depend only on U , $\rho_{\mu\nu}$ is in fact constant and ρ_μ is linear in U :

$$\rho_\mu = 2\rho_{\mu\nu}U^\nu + c_\mu. \quad (3.16)$$

We use now (3.26). This relation gives for $k = 0$:

$$U^\nu \frac{\partial \rho_\mu}{\partial U^\nu} = \rho_\mu. \quad (3.17)$$

and for $k = 1$:

$$U^\omega \frac{\partial \rho_{\mu\nu}}{\partial U^\omega} = 0. \quad (3.18)$$

The relation (3.18) is identically fulfilled and (3.17) yields:

$$c_\mu = 0. \quad (3.19)$$

Finally, the Lorentz covariance (2.32) imposes that $\rho_{\mu\nu}$ is a Lorentz invariant tensor. Because from (3.15) it follows that $\rho_{\mu\nu}$ must be antisymmetric also, we have two cases:

1) $D \neq 2$. Then $\rho_{\mu\nu} = 0$ so the last two terms in (3.13) give a null contribution to σ_L . According to the property (P3) these terms form a total divergence and so they can be discarded. It follows that L is equivalent to:

$$L(\tau, X, U) = -m(\|U\|^2)^{1/2}. \quad (3.20)$$

If the Minkowski bilinear form $G_{\mu\nu}$ is given by: $\text{diag}(G) = (1, -1, \dots, -1)$, then (3.20) is the usual Lagrangian for the relativistic particle in the homogeneous formulation.

2) $D = 2$. In this case we can have:

$$\rho_{\mu\nu} = \kappa \varepsilon_{\mu\nu}. \quad (3.21)$$

with $\varepsilon_{\mu\nu}$ the completely antisymmetric tensor in two dimensions. In this case we note that we can take:

$$l = 0; l_{\mu\nu} = -\kappa\varepsilon_{\mu\nu}X^\nu. \quad (3.22)$$

and L becomes:

$$L(\tau, X, U) = -m(\|U\|^2)^{1/2} + \kappa\varepsilon_{\mu\nu}X^\mu U^\nu. \quad (3.23)$$

Any other solution for l_μ and l satisfying (3.22) can only add a total divergence to the expression above.

(b) One must change the sign of the expression under the square root in (3.20) and (3.23). This case corresponds to tachionic particles.

(c) One can get a non-trivial Lagrangian only for $D = 2$, namely the last term in (3.23).

We have reobtained rather easily the results of [2]-[3].

4. The relativistic string

4.1 Like in section 3, we particularize the formulae (2.14) and (2.15) for the case $p = 2$. From (2.13) it follows that one must analyse the following functions: $\sigma_{\mu\nu}^{ab}$, $\sigma_{\mu\nu\omega}^{abc}$, ρ_μ , $\rho_{\mu\nu}^a$ and $\rho_{\mu\nu\omega}^{ab}$. We get from (2.14):

$$\sigma_{\mu\nu}^{ab} = \frac{1}{2} \left(\frac{\partial^2 L}{\partial U^\mu_a \partial U^\nu_b} + (\mu \leftrightarrow \nu) \right). \quad (4.1)$$

$$\sigma_{\mu\nu\omega}^{abc} = \frac{1}{2} \frac{\partial}{\partial U^\mu_a} \left(\frac{\partial^2 L}{\partial U^\nu_b \partial U^\omega_c} - (\nu \leftrightarrow \omega) \right). \quad (4.2)$$

and from (2.15):

$$\rho_\mu = \frac{\partial L}{\partial X^\mu} - \frac{\delta L_\mu^a}{\delta \tau^a}. \quad (4.3)$$

$$\rho_{\mu\nu}^a = \frac{1}{2} \left(\frac{\partial L_\nu^a}{\partial X^\mu} - \frac{\partial L_\mu^a}{\partial X^\nu} - \frac{\delta L_{\mu\nu}^{ba}}{\delta \tau^b} \right). \quad (4.4)$$

$$\rho_{\mu\nu}^a = \frac{1}{3} \left(\frac{\partial L_{\nu\omega}^{ab}}{\partial X^\mu} - \frac{\partial L_{\mu\omega}^{ab}}{\partial X^\nu} + \frac{\partial L_{\mu\nu}^{ab}}{\partial X^\omega} \right). \quad (4.5)$$

4.2 We determine first $\sigma_{\mu\nu}^{ab}$. From the Lorentz covariance property (2.31) and the symmetry in μ and ν (see (4.1)) it follows that $\sigma_{\mu\nu}^{ab}$ must be of the following form:

$$\sigma_{\mu\nu}^{ab} = U_\mu^1 U_\nu^1 f^{ab}(\|U_1\|^2, \|U_2\|^2, (U_1, U_2)) +$$

$$U_\mu^2 U_\nu^2 g^{ab}(\dots) + (U_\mu^1 U_\nu^2 + U_\nu^1 U_\mu^2) h^{ab}(\dots) + G_{\mu\nu} k^{ab}(\dots). \quad (4.6)$$

Here f , g , h and k are smooth functions of the Lorentz invariants:

$$\zeta_1 \equiv \|U_1\|^2, \zeta^2 \equiv \|U_2\|^2, \zeta_{12} \equiv (U_1, U_2) \equiv G_{\mu\nu} U^\mu{}_1 U^\nu{}_2. \quad (4.7)$$

Also the indices $a, b, \dots$ are raised or lowered with the metric δ_{ab} .

Like in Section 3 we use first the reparametrization invariance (2.24) and (2.25). In our case these relations are:

$$U^\omega{}_c \frac{\partial \sigma_{\mu\nu}^{ab}}{\partial U^\omega{}_d} = \delta_c^d \sigma_{\mu\nu}^{ab} - \delta_c^a \sigma_{\mu\nu}^{db} - \delta_c^b \sigma_{\mu\nu}^{ad}. \quad (4.8)$$

and:

$$U^\mu{}_c \sigma_{\mu\nu}^{ab} = 0. \quad (4.9)$$

If we insert (4.6) into (4.9) we can establish by a straightforward computation that, in fact $\sigma_{\mu\nu}^{ab}$, must be of the form:

$$\begin{aligned} \sigma_{\mu\nu}^{ab} = & [(U_\mu^1 U_\nu^2 + U_\nu^1 U_\mu^2)(U_1, U_2) - U_\mu^1 U_\nu^1 \|U_2\|^2 \\ & - U_\mu^2 U_\nu^2 \|U_1\|^2 - G_{\mu\nu}((U_1, U_2)^2 - \|U_1\|^2 \|U_2\|^2)] \sigma^{ab}(\|U_1\|^2, \|U_2\|^2, (U_1, U_2)). \end{aligned} \quad (4.10)$$

where σ^{ab} are smooth functions of the variables ζ_1 , ζ_2 and ζ_{12} . Inserting (4.10) into (4.8) we will obtain a system of partial differential equations for the functions σ^{ab} . It is more convenient to use instead of the variables ζ_1 , ζ_2 and ζ_{12} above, the variables ζ_1 , ζ_2 and:

$$\Delta \equiv (\zeta_{12})^2 - \zeta_1 \zeta_2. \quad (4.11)$$

(This change of variables is always possible in a convenient chart).

By some computations, one arrives at the following system:

$$2\zeta_1 \frac{\partial \sigma^{ab}}{\partial \zeta_1} + 2\Delta \frac{\partial \sigma^{ab}}{\partial \Delta} = -\sigma^{ab} - \delta_1^a \sigma^{1b} - \delta_1^b \sigma^{a1}. \quad (4.12)$$

$$2\zeta_{12} \frac{\partial \sigma^{ab}}{\partial \zeta_1} = -\delta_2^a \sigma^{1b} - \delta_2^b \sigma^{a1}. \quad (4.13)$$

$$2\zeta_{12} \frac{\partial \sigma^{ab}}{\partial \zeta_2} = -\delta_1^a \sigma^{2b} - \delta_1^b \sigma^{a2}. \quad (4.14)$$

$$2\zeta_2 \frac{\partial \sigma^{ab}}{\zeta_2} + 2\Delta \frac{\partial \sigma^{ab}}{\partial \Delta} = -\sigma^{ab} - \delta_2^a \sigma^{2b} - \delta_2^b \sigma^{a2}. \quad (4.15)$$

where, in (4.13) and (4.14) one supposes that ζ_{12} is expressed as a function of ζ_1 , ζ_2 and Δ (using (4.11)).

Although the preceding system looks rather complicated, it is remarkable that it can be solved explicitly. The computations are tedious, but do not pose difficult problems so we give only the final results. As in Section 3, we have three possible choices for the domain of L .

(a) $D_+ \equiv \{(\tau, X, U) | \Delta > 0\}$ In this case we have:

$$\sigma^{11}(\zeta_1, \zeta_2, \Delta) = \frac{c\zeta_2}{\Delta^{3/2}}. \quad (4.16)$$

$$\sigma^{22}(\zeta_1, \zeta_2, \Delta) = \frac{c\zeta_1}{\Delta^{3/2}}. \quad (4.17)$$

$$\sigma^{12}(\zeta_1, \zeta_2, \Delta) = -\frac{c\zeta_{12}}{\Delta^{3/2}} + \frac{C}{\Delta}. \quad (4.18)$$

$$\sigma^{21}(\zeta_1, \zeta_2, \Delta) = -\frac{c\zeta_{12}}{\Delta^{3/2}} - \frac{C}{\Delta}. \quad (4.19)$$

Here c , C are real constants and as before, ζ_{12} is considered as a function of ζ_1 , ζ_2 and Δ . But from (4.1) it is clear that we must have $\sigma^{12} = \sigma^{21}$. This fixes $C = 0$ in the formulae above. If we revert to the old variables ζ_1 , ζ_2 and ζ_{12} it follows that we have:

$$\sigma^{11}(\zeta_1, \zeta_2, \zeta_{12}) = \frac{c\zeta_2}{(\zeta_{12}^2 - \zeta_1\zeta_2)^{3/2}}. \quad (4.20)$$

$$\sigma^{22}(\zeta_1, \zeta_2, \zeta_{12}) = \frac{c\zeta_1}{(\zeta_{12}^2 - \zeta_1\zeta_2)^{3/2}}. \quad (4.21)$$

$$\sigma^{12}(\zeta_1, \zeta_2, \zeta_{12}) = \sigma^{21}(\zeta_1, \zeta_2, \zeta_{12}) = -\frac{c\zeta_{12}}{(\zeta_{12}^2 - \zeta_1\zeta_2)^{3/2}}. \quad (4.22)$$

(b) $D_- \equiv \{(\tau, X, U) | \Delta < 0\}$ One must change the sign of the expression under the square root in (4.20)-(4.22).

(c) If L is defined everywhere, then necessarily:

$$\sigma^{ab} = 0. \quad (4.23)$$

4.3 Now, one considers (4.1) as a system for the function L . It is not very hard to integrate this system. One obtains the following results corresponding to the cases (a)-(c) above.

(a) In this case:

$$L(\tau, X, U) = c[(U_1, U_2)^2 - \|U_1\|^2 \|U_2\|^2]^{1/2} + \frac{1}{2} \varepsilon^{ab} c_{\mu\nu}(\tau, X) U^\mu_a U^\nu_b + d_\mu^a(\tau, X) U^\mu_a + l(\tau, X). \quad (4.24)$$

Here ε^{ab} is the completely antisymmetric symbol defined with the convention $\varepsilon^{12} = 1$, and $c_{\mu\nu}$, d_μ^a , l are smooth functions of τ and X .

(b) One changes the sign of the expression under the square root in (4.24).

(c) One takes $c = 0$ in (4.24).

4.4 Instead of analysing the functions $\sigma_{\mu\nu\omega}^{abc}$ we consider first ρ_μ and $\rho_{\mu\nu}$.

(a) From (4.3), (4.4) and (4.24) one derives by direct computation that:

$$\rho_\mu = \frac{1}{2} \varepsilon^{ab} c_{\mu\nu\omega} U^\nu_a U^\omega_b + d_{\mu\nu}^a U^\nu_a + c_\mu. \quad (4.25)$$

and

$$\rho_{\mu\nu}^a = \frac{1}{2} (\varepsilon^{ab} c_{\mu\nu\omega} U^\omega_b + d_{\mu\nu}^a). \quad (4.26)$$

where:

$$c_{\mu\nu\omega} \equiv \frac{\partial c_{\nu\omega}}{\partial X^\mu} - \frac{\partial c_{\mu\omega}}{\partial X^\nu} - \frac{\partial c_{\nu\mu}}{\partial X^\omega}. \quad (4.27)$$

$$d_{\mu\nu}^a \equiv \frac{\partial d_\nu^a}{\partial X^\mu} - \frac{\partial d_\mu^a}{\partial X^\nu} + \varepsilon^{ab} \frac{\partial c_{\mu\nu}}{\partial \tau^b}. \quad (4.28)$$

and

$$c_\mu \equiv \frac{\partial d_\mu^a}{\partial \tau^a} + \frac{\partial l}{\partial X^\mu}. \quad (4.29)$$

Because the functions $\rho_{\mu\nu}$ must depend only on U , it follows that $c_{\mu\nu\omega}$, $d_{\mu\nu}^a$, c_μ must in fact be constants.

We can analyse now the consequences of (2.26). For $k = 0$ and $k = 1$ we have respectively:

$$U^\nu_b \frac{\partial \rho_\mu}{\partial U^\nu_a} = \delta_b^a \rho_\mu. \quad (4.30)$$

and:

$$U^\omega_b \frac{\partial \rho_{\mu\nu}^c}{\partial U^\omega_a} = \delta_b^a \rho_{\mu\nu}^c - \delta_b^c \rho_{\mu\nu}^a. \quad (4.31)$$

Inserting (4.25) and (4.26) into (4.30) and (4.31) respectively we easily get that we must have:

$$d_{\mu\nu}^a = 0. \quad (4.32)$$

$$c_\mu = 0. \quad (4.33)$$

The coefficients $c_{\mu\nu\omega}$ remain quite arbitrary, the only constraint being the complete antisymmetry, as follows from (4.27).

Finally, (2.32) is equivalent to the Lorentz invariance of the tensor $c_{\mu\nu\omega}$. So we have two distinct cases.

1) $D \neq 3$ Then we must have:

$$c_{\mu\nu\omega} = 0. \quad (4.34)$$

The equations (4.27)-(4.29) must be considered now as a system for the functions $c_{\mu\nu}$, d_μ^a and l . It is not necessary to find the most general solution. Indeed, if these functions are such that (4.27)-(4.29) are identically satisfied, then it follows that the last three terms in (4.24) give a null contribution to σ_L , i.e. they form a total derivative. So, L is equivalent to the usual Nambu-Goto Lagrangian:

$$L(\tau, X, U) = c[(U_1, U_2)^2 - \|U_1\|^2 \|U_2\|^2]^{1/2}. \quad (4.35)$$

2) $D = 3$ In this case we can have:

$$c_{\mu\nu\omega} = \kappa \varepsilon_{\mu\nu\omega}. \quad (4.36)$$

for some real κ .

Again we must consider the system (4.27)-(4.29). As above, we need only a particular solution, e.g.:

$$c_{\mu\nu} = \frac{1}{3} \kappa \varepsilon_{\mu\nu\omega} X^\omega. \quad (4.37)$$

and d_μ^a and l can be taken to be zero. In this case (4.24) gives:

$$L(\tau, X, U) = c[(U_1, U_2)^2 - \|U_1\|^2 \|U_2\|^2]^{1/2} + \frac{1}{6} \kappa \varepsilon_{\mu\nu\omega} \varepsilon^{ab} U^\mu_a U^\nu_b X^\omega. \quad (4.38)$$

Any other solution of (4.27)-(4.29) only adds a total derivative to this expression.

One may ask if it is still necessary to analyse the conditions on the functions $\sigma_{\mu\nu\omega}^{abc}$ and $\rho_{\mu\nu\omega}^{ab}$. That it is not the case can be seen as follows. Both expressions (4.35) and (4.38) can be easily seen to verify the following relations:

$$L \circ \dot{\phi}_{\xi} = (\det(A))L. \quad (4.39)$$

for any $\xi \in (Diff(P))_0$, and

$$L \circ \dot{\phi}_{L,a} = L. \quad (4.40)$$

for any proper orthochronous Lorentz transformation (L, a) .

These two relations easily imply that (2.20) and (2.30) are identically satisfied, so indeed (4.35) and (4.38) are the most general solution of these invariance conditions, up to a total divergence.

(b) One changes the sign of the expression under the square root in (4.35) and (4.38).

(c) In this case we can have non-trivial Lagrangian defined everywhere *iff* $D = 3$, namely (4.38) with $c = 0$.

Remarks

1) The Lagrangian (4.35) appearing in case (a) corresponds to the usual relativistic string submitted to the condition that every point of it is moving with a subluminal velocity. The case (b) obviously corresponds to a "tachionic" string for which all points are travelling with supraluminal velocities.

2) It is interesting to note that the term of a Chern-Simons nature appearing in (4.38) verifies (4.39) for any $\xi \in Diff(P)$. This property is not shared by the usual Nambu-Goto Lagrangian (4.35).

5. Conclusions

We have succeeded in deriving the Nambu-Goto Lagrangian only from invariance considerations. A new expression of a Chern-Simons character appears in a three-dimensional Minkowski space. It is worth investigating the physical implications of this contribution.

Also it seems desirable to extend the analysis to the general case of a p -brane. Perhaps an ingenious combinatorial argument could solve this problem.

Finally it is clear that the formalism used above is powerful enough to analyse the case of two or more interacting objects, but the computations will be very complicated.

Nevertheless, it is easy to show that in this formalism one can have non-trivial interactions, so one is able to circumvent the well-known "no-interaction" theorems (see [9] and references cited there).

Acknowledgements: The author wishes to thank to Prof. G. Veneziano for stimulating discussions.

References

1. J.Scherk, Rev. Mod. Phys. 47(1975)123
2. J.M.Lévy-Leblond, Commun. Math. Phys. 12 (1969) 67
3. D.R.Grigore, "Symmetries in the Lagrange-Souriau Formalism", subm.for publication.
4. J.M.Souriau, "Structure des Systemes Dynamiques", Dunod, Paris,1970
5. D.Betounis, Phys. Rev. D29 (1984) 599
6. D.Betounis, J. Math. Phys. 28 (1987) 2347
7. H.Rund, Lect. Notes in Pure and Appl. Math. 100 (1985) 155
8. D.R.Grigore, "Gauge Theories in the Souriau-Lagrange Framework for Classical Field Theories", preprint CERN-TH-6039/91
9. H.P.Kuntzle, Symposia Math. 14 (1974) 53