FIELD IDENTIFICATION FIXED POINTS IN $N=2$ COSET THEORIES

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ABSTRACT

We discuss $N=2$ coset models with fixed points in their field identification. We determine the corrections to the modular transformations that are needed to resolve the fixed points, and show how to obtain the correct fusion rules. Furthermore we obtain the complete set of chiral Ramond ground states for the Grassmannian coset models, and compute the resulting string spectra (number of generations N_g and anti-generations N_a) for diagonal invariants. A useful tool in these computations is the Poincaré polynomial of the chiral ring, extended with an extra variable to characterize the intersection of the spinor current orbit and the chiral ring. For the non-Grassmannian coset theories it is straightforward to compute the Poincaré polynomial, but there is an additional technical complication in obtaining the extended polynomial. We show how to circumvent this problem and obtain N_g and N_a also for tensor products of these cosets.

1. Introduction

More than two years ago a large class of new $N = 2$ superconformal field theories was proposed by Kazama and Suzuki [1], using the coset construction [2]. Although the main interest of these theories is in constructing new superstring theories in four (or other) dimensions, very little work on this application has been presented so far. The reason for this (apart from a general sense of desperation in view of the enormous number of already known solutions) is that even computations of the spectrum are quite complicated in these models. Partly, these complications are just technical. In particular there is no efficient and practical way to handle the null-state conditions on the states of these theories in order to compute the branching functions. In principle it is completely straightforward to compute them to any level, but in practice the computations are extremely laborious, and even getting the integer part of the conformal weight of the ground state is hard. If one is only interested in the number of (anti)-generations of a given compactification it suffices to compute the chiral Ramond characters, for which an efficient method does exist [3].

The other major problem is more fundamental. Even if one knows how to compute the branching functions, one still does not know how to compute the characters. The branching functions are in one-to-one correspondence with pairs of representations (Λ, λ) of the Kac-Moody algebras G and H that form the coset G/H , but they are not in one-to-one correspondence with the characters of the (putative) conformal field theory belonging to that coset. The branching functions are defined as the trace of $\exp i\pi\tau(L_0 - \frac{c}{24})$ over the Hilbert space of states obtained by regarding the states in the G -representation Λ modulo the action of the subgroup H . These Hilbert spaces are certainly well-defined, but are three problems in associating them with modules of a conformal field theory. In order of increasing severeness, they are the following:

- (i). In general, some pairs of G and H representations do not have any states associated with them, so that the corresponding characters vanish.
- (ii). In general, some pairs of G and H representations are related to the same sets of states. In particular there may be more than one pair (Λ, λ) corresponding to the identity.
- (iii). In general, a pair (Λ, λ) may be associated to a reducible representation of the conformal field theory, so that the corresponding branching function is the sum of several characters of the conformal field theory.

It is usually (and quite reasonably) assumed that in unitary coset theories with $c > 0$, problem (i) occurs if and only if the occurrence of the representation λ in the representation Λ and its excitations is forbidden by a selection rule on the conjugacy classes of G and H that is specific to the embedding. Under this assumption one can easily show that problem (ii) is in fact a consequence of problem (i). These two problems have been addressed in [4], [5] and [3], and the solution is known as “field identification”: branching functions corresponding to the same conformal field theory representation are identified with each other.

Problem (iii) occurs when the order of the field identification (*i.e.* the number of mutually identified fields) is not the same for all fields. In particular it may happen that some fields are identified only with themselves. This has serious consequences, as was pointed out in [3]: it appears that the presence of such states makes it impossible to normalize the canonical modular invariant partition function. In other words, one cannot obtain in a straightforward way a partition function in which the identity appears only once, and all other fields an *integer* number of times. A solution to this problem was proposed in [6], and consists of two ingredients.

The first one is a logical and inevitable consequence of dealing properly with problem (i). The naive coset modular transformation matrix $S_G S_H^*$ is clearly unacceptable, since it relates vanishing and non-vanishing branching functions. To project out the former, one should consider instead of the diagonal invariant a new “D-type” invariant of the form^{*}

$$\sum_a \left| \sum_{\alpha=1}^p \mathcal{X}_{J^\alpha a} \right|^2 + p \sum_f |\mathcal{X}_f|^2 . \quad (1.1)$$

Here J^α are the “identification currents”, which are zero spin fields corresponding to the branching functions that are identified with the identity. These fields are simple currents [7] with the property that all vanishing branching functions correspond (formally) to fields that are non-local with respect to them, and are therefore projected out. Conversely, one can show that any modular invariant partition function in which the vanishing branching functions do not appear must have the identification currents in its chiral algebra. If one does not extend the algebra any further than necessary one arrives inevitably at (1.1)

* For simplicity we consider here a single identification orbit of order p , where p is prime.

(apart from possible fusion rule automorphisms). The fact that the algebra is extended by simple currents fixes the form of the modular invariant partition function completely: all identification orbits a of fields local w.r.t. J must appear, and all fixed-point fields f must appear with an extra factor p .

Although the physical interpretation of this extension of the chiral algebra is quite different from that of an extension by positive integer spin currents, the mathematics is the same. In particular, the fixed-point fields have to be split into p fields if one wishes to obtain a sensible fusion algebra, exactly as in the well-known case of the D -invariants of $SU(2)$ at level $4m$ [8-10]. We call this “resolving the fixed points”. The fact that this phenomenon may occur in coset theories has also been pointed out in [4].

In the absence of fixed points (1.1) can be used to define a new modular transformation matrix S , obtained by projecting $S_G S_H^*$ on the non-vanishing identification orbits. This procedure cannot be the correct one if there are fixed points, since the columns and rows corresponding to the fixed-point fields have to be split into p components each. Unitarity of the resulting matrix $\tilde{S}$ necessitates corrections for the entries among fixed points. These corrections are parametrized by a matrix $\hat{S}$, that appears in the following way if p is prime (for more details see [6] and section 2)

$$\tilde{S}_{f_i, g_j} = \frac{1}{p} S_{f_g} E_{ij} + (\hat{S})_{fg} (\delta_{ij} - \frac{1}{p} E_{ij}) ,$$

where $E_{ij} = 1$, $i, j = 1, \dots, p-1$, and S denotes the matrix $S_G S_H^*$ with the aforementioned projections. Together with the matrix $\hat{T}$ (defined as T restricted to the fixed points), $\hat{S}$ forms an N_f dimensional representation of the modular group, where N_f is the number of fixed points. In addition $\hat{S}$ is constrained by the fact that S must yield positive integer fusion coefficients when substituted into Verlinde’s formula [11].

Interestingly, the matrix $\hat{S}$ itself usually yields positive integer fusion coefficients, although nothing seems to require that. In many cases $\hat{T}$ and $\hat{S}$ can be recognized as modular transformation matrix of some unitary or non-unitary conformal field theory. Therefore we will often refer to this set of matrices as the modular transformation matrices of a “fixed-point conformal field theory”. Whether such a theory actually exists is not relevant for our purposes.

Although this enables us in principle to obtain the correct fusion rules of the coset theory, there are still problems with the characters. By extending the algebras with integer spin currents we have only made the normalization problem worse, since the identity now appears with multiplicity p^2 . Dividing by p^2 , and splitting the fixed-point characters into p distinct ones, we obtain instead of (1.1)

$$\sum_a \left| \frac{1}{p} \sum_{\alpha=1}^p \chi_{J^{\alpha_a}} \right|^2 + \sum_i \sum_f \left| \frac{1}{p} \chi_f \right|^2. \quad (1.2)$$

Unfortunately, the “naive” fixed-point characters $\frac{1}{p} \chi_f$ have in general a q -expansion with fractional coefficients. However, we are not forced to divide χ_f equally over its p components. Modular invariance allows a more general solution. The G and H modular transformations only give us information about χ_f , which is the sum of p characters. In principle, each of these p characters may be different, as long as the sum, χ_f , remains the same. The corrections to the “naive” characters $\frac{1}{p} \chi_f$ are tightly constrained: they must themselves be proportional to characters $\hat{\chi}_f$ of the matrices $\hat{S}$ and $\hat{T}$.

Instead of (1.2) we then finally get

$$\sum_a \left| \frac{1}{p} \sum_{\alpha=1}^p \chi_{J^{\alpha_a}} \right|^2 + \sum_i \sum_f \left| \frac{1}{p} (\chi_f + v_i \hat{\chi}_f) \right|^2, \quad (1.3)$$

where $\sum_i v_i = 0$ (as the notation already suggests, v_i depends on i , but must be independent of f to preserve modular invariance). This gives us a set of extra parameters v_i that may be used to try and obtain new fixed-point characters with integer coefficients. Note that the diagonal partition function is finally changed by more than just a normalization factor: there is an additional term that is proportional to the diagonal invariant of the fixed-point CFT.

The present status of the fixed-point resolution problem in coset theories can be summarized as follows. The three main steps in any solution to the problem are:

1. Find the matrix $\hat{S}$ that modifies the “naive” modular transformation matrix $S_G S_H^*$, replacing it by a new matrix $\tilde{S}$. The latter must satisfy $(\tilde{S}\tilde{T})^3 = \tilde{S}^2 = \tilde{C}$ and must be unitary.

2. Verify that $\tilde{S}$ yields positive integer fusion coefficients. If there is more than one solution to point 1, this can be used to reduce the amount of ambiguity.
3. Verify that the characters of $\hat{S}$ can be used as corrections to the branching functions, so that the new characters have positive integer coefficients. Should there be any remaining ambiguity in the solutions to point 1. and 2, this condition may remove it.

In [12], solutions to point 1. were found for all Kac-Moody algebras except B_n and C_{2n} at even level. It turns out that the matrices $\hat{S}$ and $\hat{T}$ are themselves modular transformation matrices of another Kac-Moody algebra, associated with the fixed points.*

At present, we can only verify point 2. by explicit computation. It would be very useful to have a general proof that the matrix $\hat{S}$ mentioned above always yields positive integer fusion coefficients for $\tilde{S}$. In any case, explicit verification is only needed for Kac-Moody algebras: in the next section we will show that any solution to points 1. and 2. in Kac-Moody algebras implies a solution to the analogous problem in tensor products and coset theories involving those Kac-Moody algebras. Note that it has not been proved that there cannot be other solutions.

Point 3. is very hard to verify explicitly and in general, since as a first step this would require a full computation of the branching functions. Up to now it has been verified only in a few examples. However, we would like to emphasize that the character modification described above is unavoidable if one accepts the existence of a conformal field theory associated to every pair of (non-conformal) embeddings $H \subset G$. The problem is that the modular transformation properties of G and H characters and the G/H branching functions provide insufficient information about the coset theory. In [6] it was shown how the missing information (i.e. the matrices $\hat{S}$ and $\hat{T}$) can be supplied by requiring proper modular transformations and fusion rules for the characters and fields of the coset theory. The most general way of supplying this information is precisely the set of solutions to points 1. and 2. If none of these solutions allows acceptable character corrections, the corresponding coset model is probably incurably ill.

Superconformal field theories offer us an opportunity of a partial verification of character modifications, since at least the chiral Ramond characters are easily computable. Of course

* The problem with B_n and C_{2n} at even level is that the matrix $\hat{T}$ is not that of a Kac-Moody algebra. So far, we have only been able to find the corresponding matrix $\hat{S}$ in some simple cases.

these characters are constant, so we will learn nothing about character modifications at higher excitation levels, but on the other hand they are the characters that are of most interest. Having solved this problem, we can for the first time correctly compute quantities of importance in spectrum calculations, such as the Poincaré polynomial of the chiral ring [3] of these theories, and the fusion rules of the Ramond ground states.

An issue of particular importance is that of the fusion rules between the Ramond ground states and a special Ramond ground state that is related to the identity by spectral flow. The latter state is a simple current of the theory, which we will call the “spinor current”. This current, the Ramond ground state with highest (or lowest) $N = 2$ charge, is used in building the space-time supersymmetry generator in string theories. In order to compute the number of generations and anti-generations we need to know the intersections of its orbit with the chiral ring. This information can be represented by extending the definition of the Poincaré polynomial by means of an extra variable (In the following we will call this the “extended Poincaré polynomial”). This is discussed in section 3. The material in section 3 has in principle nothing to do with field identification fixed points, except that in theories with such fixed points one has to resolve them before one can compute the extended Poincaré polynomial.

The ordinary Poincaré polynomial is much easier to obtain. Since the sum of the p fixed-point characters must be equal to the branching function $\mathcal{X}_f$, and since the multiplicity of the ground state of a chiral Ramond branching function is either zero or one [3], it is obvious that precisely one of the p fixed-point characters must be a Ramond ground state. Hence there is either zero or one Ramond ground state per field identification orbit, independent of the length of that orbit, and one can use the formula of [3] to determine whether or not there is a Ramond ground state in a given orbit.

This rather simple rule unfortunately does not tell us *which* of the p fields into which a fixed point splits becomes the Ramond ground state. This question (that depends partly, but only partly, on a choice of basis), is crucial in determining the fusion rules of the fixed-point fields. It turns out that to answer it we need to know the Ramond ground states not only of the original coset theory, but also of the theory associated with the fixed points. In some cases the latter (or at least its chiral Ramond sector) is itself a superconformal $N = 2$ coset theory. Of course we can then once again use the results of [3] to solve the problem. In the remaining cases the matrix $\hat{S}$, although usually known explicitly, is not of a familiar

type. Obtaining the Ramond ground states for these matrices is more tedious, but if the matrix $\hat{S}$ is explicitly known one can always get them by solving numerically the eigenvalue problem in the chiral Ramond sector. Fortunately it turns out that there is another way to deduce the correct answer.

The field identifications of all the $N = 2$ coset theories found in [1] are determined in section 4. Fixed-point problems exist in most of them. There turns out to be just one class of cosets where the fixed-point CFT is itself an $N = 2$ coset, namely the Grassmannian models, $SU(n + m)_k / SU(n)_{k+m} \times SU(m)_{k+n} \times U(1)$. The fixed-point CFT of these cosets is itself another Grassmannian coset model, with n , m and k divided by a common factor p . In section 5.1 we discuss how to compute, in principle, the extended Poincaré polynomials for all $N = 2$ coset models with fixed points. In practice, for the reasons explained above, we have only performed the calculation in this way for Grassmannian cosets. This is discussed in detail in section 5.2, and a few examples of extended Poincaré polynomials are given in section 5.3.

Although we have not attempted to obtain the correct ground states of the fixed-point CFT for non-Grassmannian cosets, we have succeeded in obtaining the extended Poincaré polynomials of these cosets in a different way, by solving consistency conditions they have to satisfy. These consistency conditions follow by comparing two ways of computing the “Euler number” χ of a $c = 3n + 6$ tensor product: one using simple currents, whose action is encoded in the extended Poincaré polynomial, and the other using modular invariance of the chiral partition function [13], from which one can derive a simple formula involving the *unextended* Poincaré polynomial [14]. These consistency conditions turn out to be very powerful, especially since they should be satisfied for $c = 3n + 6$, with any integer value of n . This is explained in section 5.4.

These computations enable us to obtain the number of (anti)-generations for any tensor product of Kazama-Suzuki coset models, as well as all possible simple current twists of these theories. The results for minimally twisted (“diagonal”) theories are listed in table 2.

2. Simple currents and fixed points

Field identification in a coset model G/H is a consequence of selection rules on the branching rules for the embedding of H representations in G representations. These selection rules take the form of restrictions on the elements of the center of H that can appear in combination with elements of the center of G . It was shown in [6] that for every such selection rule there is a set of simple currents “dual” to it, in the sense that the allowed pairs of G and H representations are completely characterized by vanishing monodromy-charges with respect to this set of currents. These spin-0 currents are called the identification currents of the coset theory. It is easy to show that the orbits of these currents consists of pairs of G and H representations whose branching functions are identical, and that must be identified.

From the point of view of modular transformations and fusion rules, coset theories G/H may be thought of as tensor products $G \times H^c$, where H^c is the complement of H (a CFT whose matrices S and T are the complex conjugate of those of H). Field identification can thus be thought of as an extension of the algebra of this tensor product.

In this section we will show how to relate the resolution of fixed points in tensor products to the analogous problem in the factors of the tensor product. Furthermore, in subsections 2.2 and 2.3 we will discuss the simple currents of the coset theory after fixed-point resolution, and their action on the other fields in the theory.

2.1 Fixed-point resolution for tensor products and cosets

A field f is called a fixed point of a simple current J if $Jf = f$. In this paper we will only deal with the simplest possible fixed points, occurring for simple currents J of prime order p .[★]

We would like to prove that a solution to the fixed-point resolution problem for two conformal field theories with currents J_1 and J_2 of order p implies a solution to the corresponding problem for the tensor product of those theories, when the algebra is extended by

★ If the order is not prime, one may have a hierarchy of fixed points, each of which defines a new fixed-point CFT and is itself a fixed point of the previous fixed-point CFT. Presumably these fixed points can always be resolved by extending the algebra in steps, each time enlarging it by a prime factor. Among the $N = 2$ cosets only the Grassmannians can have a non-trivial hierarchy of fixed points, namely if the parameters n, m and k have more than one prime factor in common. Fortunately any coset with this property has $c > 9$, so that there is not much motivation to study this problem in detail.

the current (J_1, J_2) .[†] Here we denote fields in the tensor product by (a_1, a_2) , where a_1 and a_2 are fields of the first and second theory respectively.

A field (a_1, a_2) is a fixed point of (J_1, J_2) if and only if a_1 is a fixed point of J_1 and a_2 of J_2 . Fixed points can exist only if J_1 and J_2 have integer or half-integer spin.

In theories with extensions of the algebra by integer spin currents J the fields of the original theory that are not fixed points, and that are local w.r.t J are organized into sets $(a, Ja, J^2a, \dots, J^{p-1}a)$. To each such set belongs one primary field of the new theory (*i.e.* the theory obtained by extending the chiral algebra with the current J and its powers), which we will denote by a label a . Fixed-point fields f behave in exactly the opposite way: they are not grouped with other fields of the original theory, but on the other hand each fixed-point field yields p different fields in the new theory. Those fields will be labelled $f_i, i = 1, \dots, p-1$.

One can define a set of modular transformation matrices $\tilde{S}$ and $\tilde{T}$ for the fields of the new theory. To some extent, these matrices can be derived from the matrices S and T of the original theory. For $\tilde{T}$ the relation is simply

$$\tilde{T}_{a_i, b_j} = T_{ab} \delta_{ij} ,$$

where $i = 1, \dots, N_0/N_a$, and N_a is the length of the a -orbit (*i.e.* $N_a = 1$ for fixed points and $N_a = N_0 = p$ for other fields). For S one obtains [6]

$$\tilde{S}_{a_i, b_j} = \frac{N_a N_b}{N_0} S_{ab} E_{ij} + \Gamma_{ij}^{ab} ,$$

where $E_{ij} = 1$, independent of i and j . The matrix Γ is not directly determined by S , but there are many constraints on it. First of all it must be symmetric and satisfy

$$\sum_i \Gamma_{ij} = \sum_j \Gamma_{ij} = 0 . \tag{2.1}$$

Therefore the matrix Γ appears only if both a and b are fixed points. Furthermore the

[†] Note that this is not the same as extending the algebra by the currents $(J_1, 1)$ and $(1, J_2)$. In this case the fixed-point resolution problem can be solved in each theory *before* tensoring.

relations $(\tilde{S}\tilde{T})^3 = (\tilde{S})^2 = \tilde{C}$ and $\tilde{S}\tilde{S}^\dagger = \mathbf{1}$ impose restrictions on it:

$$\begin{aligned}\Gamma_{ij}^{ab}T_{bb}\Gamma_{jk}^{bc} &= T_{aa}^{-1}\Gamma_{ik}^{ac}T_{cc}^{-1} \\ \Gamma_{ij}^{ab}\Gamma_{jk}^{bc} &= C_{ac}C_{il}^aP_{lk} \\ \Gamma_{ij}^{ab}\Gamma_{jk}^{bc*} &= \delta_{ac}P_{ik} .\end{aligned}\tag{2.2}$$

Here C_{il}^a is an *a priori* unknown contribution to the new charge conjugation matrix, and P is the projection operator

$$P_{ij} = \delta_{ij} - \frac{1}{p}E_{ij} .$$

Finally, Γ is restricted by the requirement that $\tilde{S}$ yield correct fusion coefficients. The latter condition leads to rather complicated constraints on Γ , to be listed below.

If all of these requirements are met for some set of matrices Γ_{ij}^{ab} , we call this a solution to the fixed-point resolution problem. There is no general proof that there is always a unique solution to the conditions formulated so far. Therefore we cannot be certain that the canonical solution for the tensor product discussed below is the only one, even if the solutions in the two factors of the tensor product are unique.

A plausible ansatz for solutions is

$$\Gamma_{ij}^{ab} = \hat{S}_{ab}P_{ij} ,\tag{2.3}$$

which can be derived from (2.1) if $p = 2$. Furthermore one gets this form automatically if one assumes that the dependence on i and j is the same for any set of fixed points a and b , *i.e.* that Γ factorizes. All solutions for Kac-Moody algebras presented in [12] are obtained for the ansatz (2.3). Universality of the i, j -dependence of the matrix Γ suggests the following choice for the matrix Γ of the tensor product. Suppose (f_1, f_2) and (g_1, g_2) are fixed points of (J_1, J_2) . A natural choice for the fixed-point resolution matrix is

$$\Gamma_{i,j}^{(f_1,g_1),(f_2,g_2)} = \hat{S}_{f_1,g_1}^1 \hat{S}_{f_2,g_2}^2 P_{ij} .$$

Note that Γ cannot simply be the tensor product of the corresponding matrices Γ^1 and Γ^2 of the two factors (it has the wrong number of i and j indices), so that the ansatz (2.3) is

essential to make this work. The matrix T and the rest of the matrix S are computed in the usual way for tensor products.

It is easy to show that (2.2) is satisfied by Γ if it is satisfied by Γ^1 and Γ^2 . In the remainder of this section we focus therefore on the fusion rules.

Four kinds of fusion rules have to be distinguished, depending on whether the fields involved are fixed points (F) or not fixed points (N). The formulas for the fusion coefficients are in those four cases (with a, b, c denoting normal fields, and f, g, h denoting fixed-point fields)

(NNN)-fusion rules:

$$\tilde{N}_{a,b,c} = \sum_{\alpha=0}^{p-1} N_{a,b,J^\alpha c} . \quad (2.4)$$

Here $\tilde{N}$ denotes a fusion coefficient of the new theory, and N a fusion coefficient of the original theory. The sum is over the complete orbit of the current acting on the field c .

(NNF)-fusion rules

$$\tilde{N}_{a,b,f_i} = N_{a,b,f} E_i , \quad (2.5)$$

where $E_i = 1$.

(NFF)-fusion rules

$$\tilde{N}_{a,f_i,g_j} = \frac{1}{p} N_{a,f,g} E_{ij} + \sum_h \frac{S_{ah}}{S_{0h}} \hat{S}_{fh} \hat{S}_{gh} P_{ij} . \quad (2.6)$$

(FFF)-fusion rules

$$\begin{aligned} \tilde{N}_{f_i,g_j,h_k} &= \frac{1}{p^2} E_{ijk} [N_{fgh} - M_f - M_g - M_h - P_{fgh}] \\ &\quad + \frac{1}{p} [\delta_{kj} E_{ij} M_f + \delta_{ik} E_{jk} M_g + \delta_{ji} E_{ki} M_h] \\ &\quad + \delta_{ijk} P_{fgh} , \end{aligned} \quad (2.7)$$

where $E_{ijk} = 1, \delta_{ijk} = 1$ if $i = j = k$ and zero otherwise, and no sum on i, j or k is implied

in the second line. Furthermore

$$M_f \equiv \sum_r \left(\frac{S_{fr}}{S_{0r}} - \frac{\hat{S}_{fr}}{S_{0r}} \right) \hat{S}_{gr} \hat{S}_{hr} , \quad (2.8)$$

and similarly for M_g and M_h , and

$$P_{fgh} = \sum_r \frac{\hat{S}_{fr} \hat{S}_{gr} \hat{S}_{hr}}{S_{0r}} . \quad (2.9)$$

In (2.8) and (2.9) the sum is over all fixed points r . Note that the denominators in both equations are S_{0r} and not $\hat{S}_{0r}$.

The (NNN) and (NNF) fusion coefficients are manifestly positive integers, but for the other ones this is not manifest at all. In particular it does not follow from (2.2) or from the fusion coefficients derived from the matrix $\hat{S}$ (which, remarkably, are often positive integers). Henceforth we will assume that all coefficients *are* positive integers, and show that then the same is true for any tensor product.

The (NNN) and (NNF) fusion coefficients in tensor products are related to those of the factors in a straightforward way:

$$\begin{aligned} \tilde{N}_{(a_1, a_2), (b_1, b_2), (c_1, c_2)} &= \sum_{\alpha=0}^{p-1} N_{a_1, b_1, J_1^\alpha c_1} N_{a_2, b_2, J_2^\alpha c_2} \\ \tilde{N}_{(a_1, a_2), (b_1, b_2), (f_1, f_2)_i} &= N_{a_1, b_1, c_1} N_{a_2, b_2, c_2} E_i . \end{aligned}$$

The (NFF) fusion coefficients are completely described by just two different numbers for given a, f and g , namely the diagonal and the off-diagonal coefficient. The former is $\tilde{N}_{a_q, f_{qi}, g_{qi}} = D_q$, where $q = 1, 2$ labels the factors in the tensor product, and i the components into which the fixed-point field decomposes. The off-diagonal fusion coefficient is $\tilde{N}_{a_q, f_{qi}, g_{qj}} = O_q$, where $i \neq j$. Using the factorization of Γ and the relation

$$N_{(a_1, a_2), (f_1, f_2), (g_1, g_2)} = N_{a_1, f_1, g_1} N_{a_2, f_2, g_2}$$

for the fusion coefficients *before* extension of the algebra, we derive then the following

expressions for the corresponding quantities D and O of the tensor product

$$\begin{aligned} D &= (p-1)O_1O_2 + D_1D_2 \\ O &= (p-2)O_1O_2 + O_1D_2 + D_1O_2 . \end{aligned} \tag{2.10}$$

These coefficients are manifestly positive integers.

The discussion of the (FFF) fusion coefficients is different for $p = 2$ and $p \geq 3$. Consider first the odd primes. There are five different types of terms in (2.7):

$$\begin{aligned} D &= N_{f_i, g_i, h_i} \\ F &= N_{f_i, g_j, h_j} \ (i \neq j) \\ G &= N_{f_j, g_i, h_j} \ (i \neq j) \\ H &= N_{f_j, g_j, h_i} \ (i \neq j) \\ O &= N_{f_i, g_j, h_k} \ (i \neq j, \ i \neq k, \ j \neq k) . \end{aligned}$$

We can express the corresponding coefficients for the tensor product in terms of those of the factors, labelled D_1, D_2 , etc. The result is

$$D = (p-1)(p-2)O_1O_2 + (p-1)(F_1F_2 + G_1G_2 + H_1H_2) + D_1D_2 ,$$

$$\begin{aligned} F &= (p-3)(p-2)O_1O_2 + (p-2)[(G_1 + H_1)O_2 + (G_2 + H_2)O_1 + F_1F_2] \\ &\quad + F_1D_2 + F_2D_1 + G_1H_2 + H_1G_2 , \end{aligned}$$

and the analogous expressions obtained by permuting F, G and H , and finally

$$\begin{aligned} O &= (p-3)^2O_1O_1 + (p-3)[(F_1 + G_1 + H_1)O_2 + (F_2 + G_2 + H_2)O_1] \\ &\quad + F_1G_2 + F_1H_2 + G_1F_2 + G_1H_2 + H_1F_2 + H_1G_2 + O_1E_2 + E_1O_2 + O_1O_2 . \end{aligned}$$

These expressions are manifestly positive integers if $p > 2$, assuming that $D_i, F_i \dots$ are positive integers.

For $p = 2$ one cannot have three different labels, so that O does not exist. It is instructive to write the other four kinds of fusion coefficients in terms of matrix elements of S and $\hat{S}$. From (2.7) we get

$$D = \frac{1}{4} \left[N_{fgh} + \sum_r \left(\left(\frac{S_{fr}}{S_{0r}} \right) \hat{S}_{gr} \hat{S}_{hr} + \left(\frac{S_{gr}}{S_{0r}} \right) \hat{S}_{hr} \hat{S}_{fr} + \left(\frac{S_{hr}}{S_{0r}} \right) \hat{S}_{fr} \hat{S}_{hr} \right) \right] . \quad (2.11)$$

$$F = \frac{1}{4} \left[N_{fgh} + \sum_r \left(\left(\frac{S_{fr}}{S_{0r}} \right) \hat{S}_{gr} \hat{S}_{hr} - \left(\frac{S_{gr}}{S_{0r}} \right) \hat{S}_{hr} \hat{S}_{fr} - \left(\frac{S_{hr}}{S_{0r}} \right) \hat{S}_{fr} \hat{S}_{hr} \right) \right] , \quad (2.12)$$

and analogously for G and H . Note that the trilinear terms in $\hat{S}$ cancel.

In tensor products the four coefficients are related to those of the two factors in the following way.

$$D = D_1 D_2 + F_1 F_2 + G_1 G_2 + H_1 H_2 , \quad (2.13)$$

and

$$F = F_1 D_2 + F_2 D_1 + G_1 H_2 + G_2 H_1 , \quad (2.14)$$

plus permutations of F , G and H . Clearly they are all positive integers if the coefficients of the two factors in the tensor product are. These results agree with the ones for $p > 2$ if one sets $p = 2$ and ignores the relation for O .

This completes the proof of integrality and positivity of the fusion coefficients of a tensor product, except that we have only considered fields (a_1, a_2) where a_1 and a_2 are local with respect to J_1 and J_2 . There are, however, also fields in the theory with both a_1 and a_2 non-local with respect to J_1 and J_2 , but (a_1, a_2) local with respect to (J_1, J_2) . It is easy to see that this does not affect the (NNN) and (NNF) fusion rules. In the discussion of (NFF) fixed points, if a_1 and a_2 have non-zero charge with respect to J_1 and J_2 , then $N_{a_1, f_1, g_1} = N_{a_2, f_2, g_2} = 0$ because of charge conservation ($Q_J(f) + Q_J(g) = 0 \bmod 1$ for any two fixed points of the same current J), and $S_{a, f} = 0$ if f is a fixed point of J and a has non-zero charge with respect to J . Hence the (NFF) coefficients are zero in this case.

The (FFF) fusion coefficients can involve non-integer charge fields only if the original theories have fixed points with non-integer charges. This is possible only if J_1 and J_2 have half-integer spin. Indeed, in that case fixed-point fields f_i *must* have charge $\frac{1}{2}$. For the reasons mentioned above N_{fgh} and S_{fg} are then equal to zero, so that the fusion coefficients F, G, H and D vanish for the tensor product as well as for its two factors.

2.2 The center of a coset theory

The center of a conformal field theory is defined as the discrete group generated by the simple currents of the theory under fusion. The center of a coset theory G/H *after* field identification and fixed-point resolution is determined by the following contributing factors:

- (i) The centers of G and H .
- (ii) The simple currents that are projected out by field identification.
- (iii) Identification of some of the remaining currents.
- (iv) Simple currents originating from resolved fixed points.

If the order of the center of G and H is $C(G)$ and $C(H)$, and if there are N fields identified with the identity, then the center is in most cases a discrete group of order $C(H)C(G)/N^2$. One factor N is simply due to field identification itself, and is always present. The second factor represents the removal of simple currents that do not satisfy the G/H selection rules. This factor is only present if for every identification current there is another current that has non-zero charge with respect to it. In that case one can find a closed set of N independent simple currents whose charges form precisely the complete set of allowed monodromies with respect to the identification currents. Any other current can be multiplied with currents in this set in such a way that the product is local w.r.t. the identification currents. This reduces the center by N .

If on the other hand the set of identification currents contains currents that are local with respect to all other simple currents, then there is no corresponding reduction of the number of simple currents of the coset, and the center will be bigger than $C(G)C(H)/N^2$. This phenomenon occurs always if a coset theory has field identification fixed points, since only currents that are local w.r.t. any other simple currents can have fixed points (see [15]).

[Remarkably, for Kac-Moody algebras (and their tensor products) the converse is also true: every current that is local with respect to all other simple currents has at least one

fixed point (note that such a current must have integer or half-integer spin to be local with respect to itself). This can be shown by inspection, using the results presented in [12].^{*} It is tempting to speculate that this is true for all conformal field theories, but it seems equally difficult to find a proof as it is to find counter examples.]

The second reason why the center could be larger than the naive order is the possibility that a fixed point f is resolved into simple currents. The simplest way to verify this is to compute the quantum dimension $q_f = S_{0f}/S_{00}$ of the fixed-point field f in the original theory. This quantity is equal to 1 for a simple current, and larger than 1 otherwise. After extension of the algebra with an orbit of simple currents of total order N , S_{00} is replaced by $\tilde{S}_{00} = NS_{00}$, but $\tilde{S}_{0f} = S_{0f}$, so that $\tilde{q}_f = \frac{1}{N}q_f$.

If f is a fixed point of an integer spin current of order N , q_f must be larger than or equal to N , so that $\tilde{q}_f \geq 1$. If f is a fixed point of a half-integer spin current (that necessarily has even order $N = 2M$), we can conclude that $q_f \geq \sqrt{2}M$. If this were not true one could tensor the theory with the Ising model and extend the algebra by the product of the two half-integer spin currents. The product $f' = f\sigma$, where σ is the Ising spin field, would then violate the inequality $q_{f'} \geq 1$.

By inspection of low values for the rank and the level, the only fixed points of simple non-exceptional KM-algebras for which these bounds are saturated occur for $SO(n)$ level 1 for odd $n \geq 3$ (for the spin- $\frac{1}{2}$ current J_v), $SO(n)$ level 2 for all $n \geq 3$ (for the spin-1 current J_v), A_3 level 3 and C_4 level 1 (in both cases the current has spin 1). Obvious isomorphism between Lie-algebras ($A_3 \sim D_3$, $C_2 \sim B_2$ and A_1 level $2k \sim B_1$ level k) and triality in D_4 provide the remaining solutions. The quantum dimensions of the fixed points turn out to grow rapidly with the level, so that this list is probably complete.

This entire list can be understood in terms of the conformal embeddings $SO(m)_1 \times SO(n)_1 \subset SO(m+n)$, $SO(n)_2 \subset SU(n)$, $SU(3)_3 \subset SO(8)$ and $Sp(8) \subset E_6$. By extending the algebra by a spin-1 current (formed out of two spin- $\frac{1}{2}$ currents in the first case) one inevitably obtains a larger KM-algebra with the same central charge, in which the algebras on the left-hand side are conformally embedded. Therefore the algebras on the right-hand side must have level 1. If furthermore they are simply laced, all their primary fields are

^{*} This explains precisely the two cases found in [12] where a (half)-integer spin current has no fixed points. This occurs for the current J_v of D_n at odd levels, and J_s of D_{2n} at odd levels. These currents are non-local with respect to J_s and J_v respectively.

simple currents, and hence all fixed points must necessarily be resolved into simple currents. It is not *a priori* clear why this should give all solutions, but apparently it does.

If an integer spin current J of prime order $p > 2$ is non-trivial in several factors of a tensor product, it can only have a fixed point if each of its components in the individual factors has one. The total quantum dimension of this fixed point *before* extension of the algebra is the product of the quantum dimensions in each factor, each of which is at least p . After extension this quantity is divided by p , but the result is never equal to 1 if J is non-trivial in more than one factor. If $p = 2$ we have to include also the possibility that the components of J have half-integer spin. By an analogous argument one sees then that the fixed-point field can be resolved into simple currents only if J is non-trivial in at most *two* factors (indeed, this phenomenon occurs in the tensor product of two Ising models). Of course, the lower bound $\sqrt{2}$ on the quantum dimension of the fixed point must be saturated in both factors.

In Grassmannian $N = 2$ cosets $SU(m+n)/SU(m) \times SU(n) \times U(1)$ the only current that has fixed points is non-trivial in all three non-Abelian factors, so that none of the fixed points yields additional simple currents. In all other $N = 2$ cosets the lower bound on the quantum dimension is never saturated for the denominator of the coset.

Thus we may conclude that the order of the center of a Kazama-Suzuki coset CFT is larger than $C(G)C(H)/N^2$ if and only if there are field identification fixed points. If the order of the field identification is N , and a current of order p has fixed points, then the center has order $C(G)C(H)p/N^2$.

2.3 Simple currents acting on fixed points

The action of simple currents (other than the one producing the fixed point) on fixed-point fields is governed by the (NFF) fusion coefficients. For a simple current K the coefficient N_{Kf}^g is of course either zero or one. After resolving the fixed points one gets

$$\tilde{N}_{K,f_i}^{g_j} = E_{ij} \frac{1}{p} [N_{Kf}^g - \sum_h \frac{S_{Kh}}{S_{0h}} \hat{S}_{fh} \hat{S}_{hg}^*] + \delta_{ij} \sum_h \frac{S_{Kh}}{S_{0h}} \hat{S}_{fh} \hat{S}_{hg}^* .$$

Even if K is not a simple current, there is only one consistent answer if $N_{Kf}^g < p - 1$,

namely

$$\sum_h \frac{S_{Kh}}{S_{0h}} \mathcal{S}_{fh} \mathcal{S}_{hg}^* = N_{Kf}^g .$$

Only for this choice are all fusion coefficients positive integers. The fusion coefficients are then simply

$$\tilde{N}_{K,fi}^{g_j} = \delta_{ij} N_{Kf}^g . \quad (2.15)$$

If $N_{Kf}^g = p - 1$ there is a second solution,

$$\sum_h \frac{S_{Kh}}{S_{0h}} \mathcal{S}_{fh} \mathcal{S}_{hg}^* = -1 ,$$

with fusion coefficients

$$\tilde{N}_{K,fi}^{g_j} = E_{ij} - \delta_{ij} . \quad (2.16)$$

Note that in either case the total number of fields on the left-hand side of the fusion product is equal to N_{Kf}^g , but that it is distributed differently over the fields g_i .

Returning to simple currents K we conclude that for $p \geq 3$ they act diagonally on the fixed points (*i.e.* as in (2.15)), while for $p = 2$ they may act diagonally, or they may interchange the two distinct fields that belong to each fixed point.

Because of the relation

$$\frac{S_{Kh}}{S_{0h}} = e^{2\pi i Q_K(h)}$$

and the simple current fusion rule

$$N_{Kf}^g = \delta_{g,Kf} .$$

we see that for $p \geq 3$

$$\sum_h e^{2\pi i Q_K(h)} \mathcal{S}_{fh} \mathcal{S}_{hg}^* = \delta_{Kf,g} .$$

Because $\hat{S}$ is unitary this implies

$$e^{2\pi i Q_K(h)} \hat{S}_{fh} = \hat{S}_{Kf,h} . \quad (2.17)$$

This demonstrates that the matrix $\hat{S}$ inherits a simple current structure from the original theory. If there is a consistent way of assigning one of the fixed points f_0 to the identity element of some conformal field theory with modular transformations $\tau \rightarrow -\frac{1}{\tau}$ generated by $\hat{S}$, then Kf_0 is associated with a simple current of that conformal field theory.

If $p = 2$ (2.17) may hold with positive as well as negative coefficients, and although the simple current structure of $\hat{S}$ still exists, it may be more complicated. There is furthermore considerable freedom in choosing these signs, because if $p = 2$ there are many equivalent choices for $\hat{S}$. If one replaces $\hat{S}$ by $D\hat{S}D$, where D is a diagonal matrix with eigenvalues 1 or -1 , the modular group properties are preserved, and also the fusion coefficients remain positive integers. One may check, using (2.11) and (2.12), that the only effect of a $-$ sign in column and row f is to flip the two components of f in the fusion rules. It may sometimes be possible to choose these signs in such a way that all simple currents act diagonally, but in general this cannot be done.

In tensor products the current combination (K_1, K_2) , acting on a fixed point $(f_1, f_2)_i$ with $i = 1$ or 2 , may flip the value of i . Given a sign choice for $\hat{S}$ in each of the factors in the tensor product, the action of (K_1, K_2) is completely determined by that of its separate components. If K_1 and K_2 change the index i of $(f_1)_i$ and $(f_2)_i$ in the same way, (K_1, K_2) will not change i ; If K_1 and K_2 act differently, (K_1, K_2) will flip i . This follows directly from (2.10).

3. Simple currents and the chiral ring

The method of fixed-point resolution explained in [6] and in the previous section makes it possible to obtain the fusion rules, as well as the simple current orbits and charges for any coset theory, even for those with fixed points. These calculations can be carried out in a straightforward way on a computer. Computing the characters is far less straightforward, since one first has to determine the character representations of the fixed-point CFT. All this information would be needed if one wanted to compute the complete spectrum of some four-dimensional string theory.

Fortunately one is usually not interested in the *complete* spectrum. In a four-dimensional $(2,2)$ string theory there are only a few quantities that are really of interest, namely (roughly in order of importance and computational accessibility) the difference of the number of generations and anti-generations, the values of each of these number separately, the number of gauge bosons and the number of massless scalars. For the first two of these quantities we only need the chiral Ramond ground states. Furthermore we do not have to know the complete simple current orbits of these fields, but only the points where the orbit reaches another chiral Ramond ground state, or its world-sheet supersymmetric partner. For clarity we should mention that we work with primary fields with respect to integer spin algebras only, and that we never use the notion of super-primary fields. Thus we treat a field and the one related to it by world-sheet supersymmetry as two separate primary fields in the theory. In particular, the world-sheet supersymmetry generator is a primary field, albeit a special one, a simple current.

Another simple current of interest is the one corresponding to the Ramond chiral field that is connected to the identity by spectral flow, the spinor current. There are two such fields, and they are each other's conjugate. We will denote them as S and S^c . If one bosonizes the $U(1)$ generator of the $N=2$ algebra as $i\sqrt{c/3}\partial\phi$, these currents are

$$S = e^{(i/2)\sqrt{c/3}\phi} \quad \text{and} \quad S^c = e^{(-i/2)\sqrt{c/3}\phi} . \quad (3.1)$$

The charge of any field with respect to this simple current is equal to the fractional part of the $N=2$ charge (up to normalization; the $N=2$ charge is sometimes defined to be twice ours).

Four-dimensional type-II or heterotic strings are built out of $N = 2$ theories by first tensoring enough of them to get $c = 9$ and finally tensoring the result with a D_5 level 1 Kac-Moody algebra (we ignore an irrelevant E_8 factor here). This D_5 may either play the rôle of a gauge symmetry (in heterotic strings), or it represents the space-time sector plus ghosts of type-II or heterotic strings, as explained in [16] (this “bosonic string map” has been applied to tensor products of minimal $N = 2$ models in [17]). To preserve world-sheet supersymmetry in the tensor product one has to extend the algebra by all the bilinear combinations of the supercurrents of any two factors of the tensor products. In particular this includes the D_5 factor, where the supercurrent belongs to the conjugacy class (v) . This extension of the algebra is necessary for consistency, but only in the space-time sectors of heterotic and type-II theories (in the gauge sector one may choose entirely different extensions to obtain $(2,0)$ theories, as was discussed in [18].)

To get space-time supersymmetry one extends the algebra further by a simple current $S_{\text{tot}} = (s, S_1, \dots, S_k)$, where s is the D_5 spinor and S_i is the spinor current S in the i^{th} factor. If one makes the space-time and the world-sheet supersymmetry extensions in both the left- and right-moving sectors one obtains a so-called $(2, 2)$ theory.

To compute the number of generations or anti-generations of such a theory one proceeds as follows. Consider all fields of the form $(s, R_1, \dots, R_k)$ in the right-moving sector, where R_i is any Ramond ground state. Remove all fields whose charge with respect to the simple current S_{tot} is non-integral. Then consider the S_{tot} -orbit of the remaining fields. On the orbit one may encounter fields of the form $(s, (T_F)^{\epsilon_1} R'_1, \dots, (T_F)^{\epsilon_k} R'_k)$ or $(c, (T_F)^{\epsilon_1} R'_1, \dots, (T_F)^{\epsilon_k} R'_k)$, where T_F is the supercurrent in each factor, and ϵ_i is zero or one. These fields appear in the left-moving sector, paired with the right-moving field $(s, R_1, \dots, R_k)$. They are massive unless all ϵ_i vanish. However, if one takes into account the extension of the chiral algebra, they are excitations of a massless ground state: since all bilinears of the world-sheet supercurrent are in the chiral algebra, we can bring all these fields to the form $(s, R'_1, \dots, R'_k)$ or $(c, R'_1, \dots, R'_k)$, with $h - \frac{c}{24} = 0$. The total number of states of the first kind is the number of generations, and the total of the second kind the number of anti-generations.

To do the necessary bookkeeping it is useful to introduce a polynomial that keeps track of the intersections of the S -orbits and the chiral ring. Consider a Ramond chiral field R of charge Q , and its S -orbit, consisting of fields $S^m \phi$. Suppose the order of S is K , *i.e.*

$S^K = 1$, and there is no other integer between 1 and K with that property. The following features of Ramond ground states are relevant for our purpose. First of all, R may be a fixed point of some power of S , or it may be a fixed point up to world sheet supersymmetry. We denote the smallest positive power of S that takes R to either R or $T_F R$ as N_R . Some of the other fields $S^m R$ on the orbit may themselves be Ramond chiral fields. This defines a set of numbers $\mathcal{F}_+ = \{0, m_1, \dots, m_i\}$, which can be restricted to the range $0, \dots, N_R - 1$. Other fields on the orbit may be generated from a Ramond chiral field by the action of the supercurrent T_F . This defines a second set of numbers $\mathcal{F}_- = \{n_1, \dots, n_i\}$, restricted to the same range. Note that the numbers in $\mathcal{F}_+$ and $\mathcal{F}_-$ are even integers, since an odd number of actions of S changes Ramond to Neveu-Schwarz fields.

The extended Poincaré polynomial is now defined as follows

$$\mathcal{P}(t, x) = \sum_R \frac{t^{-2N_S Q(R)}}{1 - \epsilon(R)x^{N_R}} \left[\sum_{m \in \mathcal{F}_+} x^m - \sum_{m \in \mathcal{F}_-} x^m \right], \quad (3.2)$$

where the first sum is over all chiral Ramond fields, and $Q(R)$ is the $N=2$ charge including the integer part. The factor $2N_S$ multiplying the charge ensures that all powers of t are integer; the sign is just a convention. Note that the order K of S is either equal to N_S or to $2N_S$, and that the allowed charges with respect to a simple current of order K are always equal to $\frac{q}{K}, q \in \mathbf{Z}$. Hence if $K = 2N_S$ all powers of t are in fact even integers. If $K = 2N_S$ $S^{N_S} = T_F$, with respect to which all Ramond fields have half-integer charge. Hence in this case all powers of t are odd integers. In the denominator $\epsilon(R) = 1$ if $S^{N_R} R = R$ and $\epsilon(R) = -1$ if $S^{N_R} R = T_F R$. By expanding in x one obtains thus for all fields all positive powers m so that $S^m R = R$ or $S^m R = T_F R$, where the sign of the coefficient of x^m decides between these two possibilities.

Note that with this definition there is a difference between the formally identical polynomials $(1 + x^n)/(1 - x^{2n})$ and $1/(1 - x^n)$. In the former case S^n acting on the field of interest yields a *different* Ramond ground state whereas in the former case it yields the *same* Ramond ground state. However, spectrum calculations are not sensitive to this difference, since in any case one must always sum over the entire orbit of the current S , and one counts only the multiplicity of the fields one encounters in doing this. Hence one may treat the polynomial as a genuine function of x , at the expense of some extra information that is irrelevant in spectrum calculations.

The highest and lowest powers of t in the polynomial correspond to the fields S and S^c , that have charges $Q = -\frac{c}{12}$ and $Q = \frac{c}{12}$ respectively. These terms have multiplicity 1. The denominators of their x -polynomials are equal to $1 \pm x^{N_S}$. The numerator contains in any case the terms $(1 + x^2)$ for S^c (because $S^2 S^c = S$) and $(1 \mp x^{N_S-2})$ for S (with opposite relative sign in comparison to the denominator). There may be additional terms in the numerator, and we will discuss their implications in a moment.

If one tensors two $N=2$ theories with polynomials $\mathcal{P}_1(t, x)$ and $\mathcal{P}_2(t, x)$, the polynomial $\mathcal{P}$ of the tensor product is obtained by an operation denoted by a “*”,

$$\mathcal{P}(t, x) = \mathcal{P}_1(t, x) * \mathcal{P}_2(t, x) , \quad (3.3)$$

that is defined as follows. First compute the ordinary product $\mathcal{P}_1(t, y)\mathcal{P}_2(t, z)$. Then remove all terms in which the power of y and z differs. In the remaining terms, set $yz = x$. The result will have a certain periodicity in x , and this can be used to bring it again in the form (3.2). It is easy to see that the product polynomial obtained this way has the right properties. The $N = 2$ charges in both factors are added automatically. The current $S = (S_1, S_2)$ intersects the chiral ring only if S_1 and S_2 do so simultaneously (hence only terms with the same powers of z and y contribute). Furthermore, the sign and multiplicity of each term is correct.

If one has a theory with $c = 9$, it can be used as the internal sector of a heterotic or type-II theory. The charges of the D_5 -conjugacy classes (s) and (c) with respect to S_{tot} are $\frac{3}{4} \bmod 1$ and $\frac{1}{4} \bmod 1$ respectively. The highest and lowest $N=2$ charges in a $c = 9$ theory are $\pm\frac{3}{4}$. Since the total $N=2$ charge of any state must be integral, the internal states that can be combined with the D_5 conjugacy class (s) correspond to the $t^{-\frac{1}{2}N_S}$ and $t^{\frac{3}{2}N_S}$ terms in the polynomial. The states with D_5 conjugacy class (c) are simply the CPT-conjugates of the ones with (s).

There is only one term in the polynomial with maximal and minimal power of t , and in the four-dimensional string spectrum these terms contribute the space-time supersymmetry generator, the $E_6/SO(10) \times U(1)$ gauge bosons and their CPT-conjugates. If the x -polynomials of these terms are not of the standard form discussed above, (*i.e.* if the numerator has more than two terms) then this indicates the presence of additional gravitinos

in the space-time spectrum. Since the current S_{tot} extends both the left and the right algebra, any field appearing on its orbit will automatically extend the algebra as well. If this includes other Ramond ground states, they will lead to extra space-time supersymmetry generators. Then one must have either $N=2$ or $N=4$ supersymmetry.

If there is only $N=1$ supersymmetry, the chiral generations come from the terms in the polynomial proportional to $t^{-\frac{1}{2}N_s}$. To count them one expands the polynomial $\mathcal{P}(t, x)$ in x , keeping all terms proportional to $t^{-\frac{1}{2}N_s}$ and x^m , $0 \leq m < N$, where N is the order of S_{tot} . One is left with a finite polynomial, multiplying $t^{-\frac{1}{2}N_s}$

$$p(x) = \sum_{m=0}^{N-1} a_m x^m .$$

A term in the remaining polynomial contributes to the number of generations if $a_m > 0$ and $m = 0 \bmod 4$ or $a_m < 0$ and $m = 2 \bmod 4$, and otherwise to the number of anti-generations. (Note that if $m = 2 \bmod 4$ the action of S_{tot} changes the D_5 conjugacy class; if $a_m < 0$ one has to act with a combination of the space-time and internal world-sheet supersymmetry generator to obtain a massless state. This also changes the D_5 conjugacy class.) In other words,

$$\begin{aligned} N_g + N_a &= \sum_m |a_m| \\ N_g - N_a &= p(i) \quad (i^2 = -1), \end{aligned}$$

where N_g (N_a) is the number of (anti)-generations.

There is a rather different way to compute the difference between generations and anti-generations, which uses only the Poincaré polynomial. This polynomial is much more easily computable than its extended version, and furthermore the computation is based on far fewer assumptions about the details of fixed-point resolution. The only input in the computation is the fact that the sum of the distinct characters belonging to a certain fixed point is equal to the fixed-point branching function. This means that the number of Ramond ground states for a given $N=2$ charge is not altered by fixed-point resolution, and that to normalize properly one has to divide by the length of the *identification* orbit **for each orbit separately**, and not by the length of the *identity* orbit. This is perhaps counter-intuitive, since to normalize the diagonal partition function in a modular invariant way one has to

divide by the length of the identity orbit, but it is nevertheless in complete agreement with modular invariance when the possibility of character modifications is taken into account. Another way to state this result is that for branching functions with one Ramond ground state *there is precisely one Ramond ground state per identification orbit*, irrespective of the length of that orbit.

A formula for $|N_g - N_a|$ in terms of the Poincaré polynomial defined as[★]

$$P(t) = t^{cN_s/6} \mathcal{P}(t, 0) , \quad (3.4)$$

has been derived by Buturović [14]. Transformed to our conventions his formula is

$$|N_g - N_a| = \frac{1}{N_S} \sum_{r,s=0}^{\frac{1}{2}N_S-1} P(e^{2\pi i x/N_S}) , \quad (3.5)$$

where x is the greatest common divisor of r and s (defined to be r if $s = 0$ and vice-versa). This result gives a non-trivial check on the previous result, because it does not depend directly on the fusion rules needed to determine the S -orbit (indirectly it is related to the fusion rules via modular invariance, which is a crucial input in the derivation of (3.5)). Clearly the Poincaré polynomial $P(t)$ imposes thus some constraints on the x -dependence of its extended version $\mathcal{P}(t, x)$. It might be interesting to explore this further.

4. Field identifications and fixed points

The superconformal coset models of [1] have the general form $G \times SO(2d)/H$, where $d = \frac{1}{2}(\dim G - \dim H)$. The group H always contains a $U(1)$ factor, and to isolate the non-Abelian part we will sometimes write it as $H = \hat{H} \times U(1)$.

The determination of their identification currents is simply a matter of examining carefully the conjugacy class selection rules. We present the results below.[†] Simple non-Abelian

★ With this definition one obtains a polynomial in t^2 with first term equal to 1.

† The identification currents of $N = 2$ coset models have also been obtained recently by A. Sierra [19]. We agree with some of his results.

KM-algebras are denoted G_k , where G is the Lie-algebra and k the level. The basic currents of the non-Abelian factors are denoted as in [6]. For the $U(1)$ factor we choose the following notation. By U_m we mean a $c = 1$ torus conformal field theory with m primary fields. These fields can be labelled $q = 0, \dots, m-1$, and have conformal weight $h = q^2/2m$. For example, $SU(2)_1 \equiv U_2$.

There is some freedom in the normalization of $U(1)$ factor. Instead of a set of charges q , we can choose to work with a set of charges lq , $l \in \mathbf{Z}$. Note that to obtain the same conformal weight, we have to change m to ml^2 . At first sight this would seem to enhance the number of fields by a factor l , but this is not true: since all charges $q' \neq lq$ vanish, one should introduce an extra identification current J_m with charge $q' = m$, with respect to which all non-vanishing fields are local. This current identifies the non-vanishing fields in groups of l , so that one finally gets the correct answer. A natural choice for the $U(1)$ normalization in $N=2$ coset models turns out to be $U_m, m = C(G)C(\hat{H})(k+h)$, where $C(G)$, k and h are respectively the order of the center, the level and the dual Coxeter number of G and $C(\hat{H})$ is the order of the center of $\hat{H}$.

An identification current can have fixed points only if each of its components in the different factors of the coset model has one. For $N=2$ models this implies that it must have zero charge, and be a singlet with respect to the $SO(2d)$ factor. The currents that have fixed points are indicated below.

We also indicate below which modular transformation matrix $\hat{S}$ may be used as a correction matrix. The precise determination of $\hat{S}$ is done as follows. First we determine in G and each factor of $\hat{H}$ the fixed-point conformal field theory, using the table listed in [12]. For factors of H one should use the conjugate of the matrix $\hat{S}$. The correction matrix $\hat{S}$ at this stage is thus given by the tensor product

$$\hat{S} = \hat{S}(G) \otimes S(SO(2d)) \otimes \hat{S}^*(\hat{H}) \otimes S^*(U_n). \quad (4.1)$$

Note that the complete $SO(2d)$ and U_n matrices appear, since the fixed-point current is trivial in these factors.

The matrix $\hat{S}$ resolves the fixed points correctly if the chiral algebra is extended only with the fixed-point current. However, there are additional field identifications. If we restrict

ourselves to coset theories with only $\mathbf{Z}_p$ fixed points (where p is prime), the remaining field identifications are in principle straightforward, but some care is still needed.

The simple currents of G and $\hat{H}$ that are not projected out by the fixed-point currents, as well as the $SO(2d)$ and U_n currents, act in a well-defined and computable way on the fixed-point fields. They may take component i of a certain fixed point f to a different component j of another fixed point g . The remaining identification currents act therefore on the labels of the fixed points, the $SO(2d)$ conjugacy class and the U_n charge. Effectively this means that a field identification is occurring for the matrix $\hat{S}$, of which some rows and columns are removed and others are identified. By keeping track of this, one can compute the correct matrix $\hat{S}'$ that resolves the fixed points of the coset theory. The character corrections that are needed have to be done with character representations of this matrix $\hat{S}'$ (we will omit the prime in the following).

We will view the matrix $\hat{S}$ as the modular transformation matrix of a “coset theory”

$$\frac{\hat{G} \times SO(2d)}{\hat{H} \times U_n}, \quad (4.2)$$

where $\hat{G}$ and $\hat{H}$ denote the fixed-point conformal field theory of G and $\hat{H}$. Although we will refer to this coset-like object as the “fixed-point conformal field theory” of the coset theory, this is nothing more than a convenient name that should not be taken literally. As far as we know, nothing even guarantees that $\hat{H}$ can be embedded in $\hat{G}$. In fact, there are cases where the fixed-point CFT’s are not Kac-Moody algebras, and where the interpretation of (4.2) is not at all clear. Furthermore, even when (4.2) can be interpreted as a coset conformal field theory, the field identifications are often in conflict with that interpretation.

Such an interpretation is not needed, however: the only relevant information is the matrix $\hat{S}$ that resolves the fixed points. This matrix is in any case well-defined. Nevertheless, there is a *practical* advantage in identifying $\hat{S}$ with the matrix S of a known conformal field theory, since this makes it much easier to work out the character corrections.

The results for the various $N=2$ coset theories are as follows[★]

★ There may be more ways of embedding H into G , differing from each other by charge conjugation in one or more factors. The identification currents listed here correspond to one possible choice. This choice is in agreement with the rules we use to find the Ramond ground states, based on [3] (see section 5.1)

Cosets of type $A(n_1, n_2, k)$:

$$\frac{SU(n_1 + n_2)_k \times SO(2n_1 n_2)_1}{SU(n_1)_{k+n_2} \times SU(n_2)_{k+n_1} \times U_{n_1 n_2 (n_1 + n_2) (k + n_1 + n_2)}} .$$

The identification currents are

$$\begin{aligned} J_1 &= (J, J_v^{n_2}, J, \mathbf{1}, n_2(k + n_1 + n_2)) \\ J_2 &= (J, J_v^{n_1}, \mathbf{1}, J, -n_1(k + n_1 + n_2)) . \end{aligned} \tag{4.3}$$

These currents have order $n_1(n_1 + n_2)$ and $n_2(n_1 + n_2)$, but they are not independent, since $J_1^{n_1} J_2^{n_2} = \mathbf{1}$. Fixed points occur if and only if n_1, n_2 and k have a common factor p . Defining $r_i = \frac{n_i}{p}$, we can write the current that has fixed points as

$$J_{\text{FP}} = J_1^{r_1} J_2^{r_2} = (J^{r_1 + r_2}, \mathbf{1}, J^{r_1}, J^{r_2}, 0)$$

The “fixed-point CFT” is

$$\frac{SU(r_1 + r_2)_l \times SO(2n_1 n_2)_1}{SU(r_1)_{l+r_2} \times SU(r_2)_{l+r_1} \times U_{n_1 n_2 (n_1 + n_2) (k + n_1 + n_2)}} ,$$

where $k = pl$.

Note that if n_1 and n_2 have a common factor p the current

$$J_1^{n_1(n_1 + n_2)/p} = (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, n_1 n_2 (n_1 + n_2) (k + n_1 + n_2) / p)$$

acts only on the U_m factor, and identifies charges in groups of p . Furthermore all charges are then a multiple of p . As explained above, we may then replace U_m by U_{m/p^2} , and divide

all charges by p . If furthermore k is also a multiple of p so that there are fixed points, the charges of those fixed points can be divided once more by p . In this case the current

$$J_1^{r_1 r_2} J_2^{-r_1 r_2} = (\mathbf{1}, \mathbf{1}, J^{r_1 r_2}, J^{-r_1 r_2}, r_1 r_2 (n_1 + n_2) (k + n_1 + n_2)) ,$$

that changes only the U_m -charge when acting on fixed points, identifies charges in groups of p^2 , and disallows any charges that are not a multiple of p^2 (in the original normalization). Hence for the fixed-point fields separately we may replace U_m by U_{m/p^4} and divide all charges by p^2 . Then the fixed-point CFT can be written as

$$\frac{SU(r_1 + r_2)_l \times SO(2n_1 n_2)_1}{SU(r_1)_{l+r_2} \times SU(r_2)_{l+r_1} \times U_{r_1 r_2 (r_1 + r_2) (l + r_1 + r_2)}} . \quad (4.4)$$

The remaining identification currents (4.3) relate different fixed-point fields to each other. Since those fields are identified, their fixed-point resolution matrix elements are also identified. In this way the field identifications of the original theory impose a field identification on the fixed-point CFT. The corresponding identification currents are precisely those of the original Grassmannian manifold, but with the new charge normalization that we chose for the fixed points:

$$\begin{aligned} J_1 &= (J, J_v^{n_2}, J, \mathbf{1}, r_2(l + r_1 + r_2)) \\ J_2 &= (J, J_v^{n_1}, \mathbf{1}, J, -r_1(l + r_1 + r_2)) . \end{aligned} \quad (4.5)$$

On the fixed points, these currents have order $r_1(r_1 + r_2)$ and $r_2(r_1 + r_2)$ respectively, and they are not independent, since the combination $J_1^{r_1} J_2^{r_2}$ acts trivially on the fixed points. Comparing (4.4) and (4.5) with the original coset manifold and its identification currents, we see that the fixed-point CFT is again a Grassmannian coset manifold apart from three differences. The first difference is that the identification currents in $SO(2d)$ are $J_v^{n_i}$ instead of J_v^i . This difference is of course irrelevant if p is odd. The second difference is that the $SO(2d)$ factor is $SO(2n_1 n_2)$ instead of $SO(2r_1 r_2)$. The third difference is that the eigenvalues of T of the fixed points are in general not exactly those of the fixed-point CFT,

but differ from them by a field-independent factor $\exp(2\pi i \frac{s}{12})$, $s \in \mathbf{Z}$. This shift s can be computed using formulas given in [12], and the result is such that $h - c/24 = 0 \bmod 1$ for the chiral Ramond ground states. In other words, the extra contribution to the conformal weights due to the replacement of $SO(2r_1 r_2)$ by $SO(2n_1 n_2)$ is precisely cancelled by the shift s , but only in the Ramond sector.

Cosets of type $B(n, k)$, n odd:

$$\frac{SO(n+2)_k \times SO(2n)_1}{SO(n)_{k+2} \times U_{4(k+n)}} \quad (n \text{ odd}) .$$

Identification currents

$$\begin{aligned} J_1 &= (J_v, \mathbf{1}, J_v, 0) \\ J_2 &= (J_v, J_v, \mathbf{1}, 2(k+n)) . \end{aligned}$$

Fixed point current

$$J_{\text{FP}} = (J_v, \mathbf{1}, J_v, 0) .$$

The fixed-point CFT depends on the level.

$$k = 2m + 1 : \quad \frac{Sp(n-1)_m SO(2n)_1}{Sp(n-3)_{m+1} U_{4(k+n)}} .$$

$$k = 2m : \quad \frac{\mathcal{B}_{(n+1)/2, m} SO(2n)_1}{\mathcal{B}_{(n-1)/2, m+1} U_{4(k+n)}} .$$

Here $\mathcal{B}_{n, m}$ are the unidentified objects described in [12]. We know the matrix $\hat{S}$ only for $n = 1$ (it is equal, up to a phase, to the matrix S of non-unitary minimal Virasoro models with $q = 2, q' = 2m + 3$), for $m = 1$ ($\hat{S}$ is equal, up to a phase, to the complex conjugates of the S -matrices of the non-unitary minimal models), and for $n = m = 2$. Since in any case neither of these fixed-point CFT's is manifestly an $N=2$ coset model, we will not examine them further here.

Cosets of type $B(n, k)$, n even:

$$\frac{SO(n+2)_k \times SO(2n)_1}{SO(n)_{k+2} \times U_{16(k+n)}} \quad (n \text{ even}) .$$

Identification currents

$$\begin{aligned} J_1 &= (J_v, \mathbf{1}, J_v, 0) \\ J_2 &= (J_s, J_v^{n/2}, J_s, 2(k+n)) . \end{aligned}$$

Fixed-point current

$$J_{\text{FP}} = (J_v, \mathbf{1}, J_v, 0) .$$

This current has fixed points only for $k = 2m$, and the fixed-point CFT is:

$$\frac{Sp(n-2)_m SO(2n)_1}{Sp(n-4)_{m+1} U_{16(k+n)}} .$$

Cosets of type $C(n, k)$, n odd:

$$\frac{Sp(2n)_k \times SO(n(n+1))_1}{SU(n)_{2k+n+2} \times U_{2n(k+n+1)}} \quad (n \text{ odd}) .$$

Identification currents

$$\begin{aligned} J_1 &= (J, J_v^{\frac{n+1}{2}}, \mathbf{1}, n(k+n+1)) \\ J_2 &= (\mathbf{1}, \mathbf{1}, J, -2(k+n+1)) \end{aligned}$$

There are no field identification fixed points.

Cosets of type $C(n, k)$, n even:

$$\frac{Sp(2n)_k \times SO(n(n+1))_1}{SU(n)_{2k+n+2} \times U_{2n(k+n+1)}} \quad (n \text{ even}) .$$

Identification currents

$$\begin{aligned} J_1 &= (J, \mathbf{1}, J^{n/2}, 0) \\ J_2 &= (\mathbf{1}, J_v, J, -2(k+n+1)) . \end{aligned} \tag{4.6}$$

Fixed-point current:

$$J_{\text{FP}} = (J, \mathbf{1}, J^{n/2}, 0) .$$

The fixed-point CFT is different for odd and even k :

$$k = 2m + 1 : \quad \frac{Sp(n)_m SO(n(n+1))_1}{SU(\frac{n}{2})_{2m+n/2+2} U_{2n(k+n+1)}} .$$

$$k = 2m : \quad \frac{B_{n/2,m} SO(n(n+1))_1}{SU(\frac{n}{2})_{2m+n/2+1} U_{2n(k+n+1)}} .$$

The first of these theories resembles an $N=2$ coset theory, but there is a rather important difference. Unlike the Grassmannians discussed above, the identification currents (4.6) do not even approximately generate those of the fixed-point CFT. Indeed, there is no current that can take the place of J_1 (which vanishes on the fixed points) in the fixed-point CFT. Note that the current J_1 enforces a pairing between even $Sp(n)$ tensors with even $SU(n)$ tensors, and the same for odd tensors. But there is no such pairing for the fields of the fixed-point CFT, making it quite a different object than the $N=2$ coset model it resembles.

Another essential difference with the Grassmannian models is that $Sp(n)$ (unlike $SU(n)$) does not have a hierarchy of fixed points. If the fixed-point CFT would correspond to an $N=2$ coset model, one would expect it to have its own identification fixed points, which should correspond to fixed points of a current of order 4 of $Sp(n)$. This is exactly what happens, for example, for cosets of type $A(2, 2, 2l)$. However, there are no simple currents of order 4 in $Sp(n)$.

Cosets of type $D(n, k)$, n odd:

$$\frac{SO(2n)_k \times SO(n(n-1))_1}{SU(n)_{k+n-2} \times U_{4n(k+2n-2)}} \quad (n \text{ odd}) .$$

Identification currents

$$J_1 = (J_s, J_v^{\frac{n-1}{2}}, 1, -n(k+2n-2))$$

$$J_2 = (J_v, 1, J, -2(k+2n-2)) .$$

There are no field identification fixed points.

Cosets of type $D(n, k)$, n even:

$$\frac{SO(2n)_k \times SO(n(n-1))_1}{SU(n)_{k+n-2} \times U_{4n(k+2n-2)}} \quad (n \text{ even}, \geq 4) .$$

Identification currents

$$J_1 = (J_x, 1, J^{n/2}, 0)$$

$$J_2 = (J_v, J_v, J, -2(k+2n-2)) ,$$

where $x = s$ if $n = 2 \bmod 4$ and $x = c$ if $n = 0 \bmod 4$. Fixed points occur only if $k(\equiv 2m)$ is even, for the current

$$(J_s, 1, J^{n/2}, 0) .$$

The fixed-point CFT is

$$\frac{SO(n+1)_m \times SO(n(n-1))_1}{SU(\frac{n}{2})_{m+n/2-1} \times U_{4n(k+2n-2)}} ,$$

which is not an $N=2$ coset theory.

Cosets of type E_6 :

$$\frac{(E_6)_k \times SO(32)_1}{SO(10)_{k+12} U_{12(k+12)}} .$$

Identification current

$$(J, 1, J_s, (k+12)) .$$

There are no field identification fixed points.

Cosets of type E_7 :

$$\frac{(E_7)_k \times SO(54)_1}{(E_6)_{k+18} U_{6(k+18)}} .$$

Identification current

$$(J, J_v, J, (k+18)) .$$

There are no field identification fixed points.

5. Computations[★]

5.1 General $N=2$ coset models

The Ramond ground states of $N = 2$ coset models have been determined in [3]. A representation of the Kac-Moody algebra G is characterized by its ground state, that is in some representation of the Lie-algebra $\mathcal{G}$ belonging to G . This representation is given by a highest weight vector Λ . Ground states of the modules of a coset theory do not necessarily come from ground states of G and H . However, it was shown in [3] that to determine all Ramond ground states it is sufficient to examine only the G and H ground states. Excitations of the G ground state may yield Ramond ground states as well, but they are always related to those coming from G -ground states by a field identification that fixes the G -representation.

An explicit list of all the $\hat{H}$ -representations for which Ramond ground states exist in combination with a certain G -representation with highest weight Λ is obtained as follows. First of all one determines all weights λ defined as

$$\lambda = w'(\Lambda + \rho_G) - \rho_H , \tag{5.1}$$

where w' is a coset representative of the Weyl-group of $\mathcal{G}$ modulo that of $\mathcal{H}$, chosen in such a way that λ is a highest weight of $\mathcal{H}$. Here ρ_G and ρ_H are equal to half the sum of the positive roots of $\mathcal{G}$ and $\mathcal{H}$ respectively (only the non-Abelian part of $\mathcal{H}$ contributes to ρ_H).

★ Some of the results in this section have been obtained in collaboration with Nick Warner.

This procedure yields all the Ramond ground states coming from the ground state of the G -representation. If one wishes to get all Ramond ground states, one has to act on solutions of (5.1) with all identification currents that fix the G -representation.

It is important to note that this prescription is a property of branching functions only. For each G -representation Λ it simply lists all H -representations λ so that the branching function b_λ^Λ in the Ramond sector (*i.e.* with conjugacy class (s) or (c) in $SO(2d)$) has a lowest state with $h = \frac{c}{24}$. In particular it does not depend on field identifications and fixed points. However, in order to obtain the correct count of the number of chiral Ramond ground states one has to make sure to take only one such state into account per field identification orbit.

In practice, this can be achieved as follows. First one uses the field identifications to bring the G -representation to a standard form. For example one can choose a fixed irrational vector η in weight space, and minimize the value of $\eta \cdot J\Lambda$, for all simple currents J of G . Here $J\Lambda$ denotes the highest weight obtained by acting with J on some representation Λ used as the base point of the orbit. Note that for all coset models discussed in section 4 the identification currents cover the entire set of simple currents of G .

For each Λ there is a subset of identification currents that fixes it. This subset may be used to bring the non-Abelian part of H to a standard form in much the same way. Finally one may be left with a set of identification currents that fixes both the G - and the non-Abelian H representations. These currents, if they act non-trivially on the $U(1)$ part of H , impose identifications on the $U(1)$ charges, and may be used to bring the charge within some fundamental range (for example, by minimizing its absolute value, and choosing the positive sign if both signs are identified). In this way one can define for each identification orbit a canonical representative, and by bringing every state into this canonical form one can immediately determine to which G/H -representation it belongs. This is true for any representation of the coset theory, not just the Ramond ground states.

Each combination of G and $\hat{H}$ representations acquires in this way its own value of Δ_q , where Δ_q is the smallest increase in the charge of a field that leads to a field identified with it. Of course Δ_q is a divisor of m , if the original coset theory had a denominator factor U_m .

We have not mentioned the $SO(2d)$ factor of the coset theory since it plays a fairly trivial rôle. The identification currents may act like an $SO(2d)$ vector, and in performing the identifications one simply has to keep track of the resulting changes of the $SO(2d)$ conjugacy class. In particular a shift in the U_m -charge by Δ_q may lead to an identified field

only if the $SO(2d)$ conjugacy class is changed simultaneously. Coset representations that differ only in the $SO(2d)$ conjugacy class and are identical otherwise cannot be identified, since any identification current that would achieve this must have zero U_m -charge. However, one may check explicitly that any zero-charge current acts trivially on $SO(2d)$.

The orbit of the spinor current can be determined as follows. A representative of the field identification orbit of this current is, for a coset $G \times SO(2d)/\hat{H} \times U(1)$,

$$S = (\mathbf{1}, (s), \mathbf{1}, q_s) .$$

This representative has the obvious advantage that it does not change the G and $\hat{H}$ representations. Acting on any Ramond ground state it only alters the charge and the $SO(2d)$ conjugacy class. To determine the intersections of the S -orbit and the chiral ring we only have to consider the set of Ramond ground states that all have the same G -representation and the same $\hat{H}$ representations. To each of these states one adds all even multiples of q_s , and checks whether the resulting charge occurs in the set of R -ground states, modulo Δ_q . If it is, one determines its $SO(2d)$ conjugacy class. The latter is equal to the $SO(2d)$ conjugacy class of the starting point of the orbit, flipped once if one has acted m times with S and $m = 2 \bmod 4$, and flipped once more if an odd number of shifts Δ_q is needed and if such a shift changes the $SO(2d)$ conjugacy class (here “flipped” means that (s) and (c) are interchanged). This information can then be converted to the x -dependence of the extended Poincaré polynomial as discussed in section 3.

If there are field identification fixed points there is a complication with this argument. Suppose one starts with a Ramond ground state R_f that is a fixed point. It is split into p components R_{fi} . Only one of them can be a Ramond ground state, and without loss of generality we may choose this to be R_{f1} . The fusion rules of the original theory may tell us that S^m maps this field to a component R_{gi} of a second Ramond ground states R_g , but they do not tell us which value of i we will obtain. Nevertheless, this is very important, since only for one value of i we get a Ramond ground state.

If $p > 2$ there is no complication, since first of all the simple currents always act diagonally (see section 2.1) and secondly the character modifications are the same for each fixed point, for a given i (this will be explained below). Therefore the S orbit of a Ramond ground state R_{f1} contains only first components of fixed-point fields, and if the branching

function R_g contains a Ramond ground state, that ground state will appear in the character R_{g1} .

Even if $p = 2$ the current S still acts diagonally, because it is equal to the identity in G and $\hat{H}$. However, it is not *a priori* clear anymore whether R_{g1} or R_{g2} are the actual Ramond ground states. To resolve this ambiguity, we have to know the characters $\mathcal{X}$ of the fixed-point conformal field theory.

The chiral Ramond (“PP”) characters are defined as the difference between the characters of fields whose quantum number differ only in the $SO(2d)$ factor,

$$\mathcal{X}_{\text{PP}}^a = \mathcal{X}_s^a - \mathcal{X}_c^a ,$$

where a denotes all other quantum numbers. These characters are constant, and they can take the values 1, 0 or -1. Since they are constant, they are eigenvectors of S_{PP} and T_{PP} with eigenvalue 1.

If $a = f$ is a fixed point, each character $\mathcal{X}_{\text{PP}}^f$ has an associated fixed-point correction $\mathcal{X}_f$, which is a (constant) character of the PP-sector of the fixed-point CFT. The modified characters are, for $p = 2$

$$\begin{aligned} (\mathcal{X}_{\text{PP}}^f)_1 &= \frac{1}{2}(\mathcal{X}_{\text{PP}}^f + \mathcal{X}_{\text{PP}}^{\mathfrak{g}f}) \\ (\mathcal{X}_{\text{PP}}^f)_2 &= \frac{1}{2}(\mathcal{X}_{\text{PP}}^f - \mathcal{X}_{\text{PP}}^{\mathfrak{g}f}) \end{aligned} \tag{5.2}$$

and for $p > 2$

$$\begin{aligned} (\mathcal{X}_{\text{PP}}^f)_1 &= \frac{1}{p}(\mathcal{X}_{\text{PP}}^f + (p-1)\mathcal{X}_{\text{PP}}^{\mathfrak{g}f}) \\ (\mathcal{X}_{\text{PP}}^f)_i &= \frac{1}{p}(\mathcal{X}_{\text{PP}}^f - \mathcal{X}_{\text{PP}}^{\mathfrak{g}f}) \quad (i = 2, \dots, p-1) \end{aligned} \tag{5.3}$$

In both cases the choice of the label that is corrected by a positive multiple of the fixed-point character is optional, but must be the same for all fixed points.

Since the PP-characters are not necessarily positive, we may change their sign if necessary. However, since they must be an eigenvector of S and T one must then change the signs of all characters simultaneously. In particular one may have to change the signs of

all $\hat{\mathcal{X}}_{\text{PP}}^f$ in order to get integer coefficients in (5.3). Clearly for $p > 2$ the only acceptable solution (apart from this overall sign flip) is

$$\hat{\mathcal{X}}_{\text{PP}}^f = \mathcal{X}_{\text{PP}}^f, \quad (5.4)$$

For $p = 2$ there is more freedom: one may change the signs of any character of the fixed-point CFT, and compensate this by changing the matrix $\hat{S}$, as explained in section 2.2. This is also clear from (5.2), which does not restrict the sign of $\hat{\mathcal{X}}_{\text{PP}}^f$ needed to get integers. Hence for $p = 2$ one can allow extra $-$ signs in (5.4), that may depend of f . The true Ramond ground state is always the first component if $p > 2$, but if $p = 2$ it is the second component for those fields f for which the equality holds with a $-$ sign. One can choose signs in such a way that the Ramond ground state is always the first component, but this will change the fusion rules, because $\hat{S}$ changes.

It is by no means obvious that (5.4) will hold with *any* choice of sign. However, at least for the Grassmannian manifolds we can check whether it does since we know how to compute the chiral Ramond characters of the fixed-point CFT, which is another Grassmannian manifold (at least in the chiral Ramond sector). In other cases with $p > 2$ one can check (5.4) by “brute force” if the matrices $\hat{S}$ and $\hat{T}$ are known explicitly. The chiral Ramond fixed-point characters $\mathcal{X}_{\text{PP}}$ must be eigenvectors of these matrices (in the PP-sector) with eigenvalue 1. This becomes much harder for $p = 2$ because of the unknown signs. Unfortunately, all non-Grassmannian cosets have only fixed points with $p = 2$.

5.2 Grassmannian manifolds

Now we will discuss in some detail how the extended Poincaré polynomial can be computed for Grassmannian manifolds. To represent the $SU(N)$ Weyl transformation of (5.1) it is convenient to work in a basis of orthonormal vectors $\vec{e}_1, \dots, \vec{e}_N$. In this basis one can choose for the simple roots the vectors $\vec{\alpha}_i = \vec{e}_i - \vec{e}_{i+1}$, $i = 1, \dots, N - 1$. All weights can be written as $\sum_i w_i \vec{e}_i$, where the coefficients w_i must satisfy $\sum_i w_i = 0$ in order to lie in the $N - 1$ dimensional hyperplane of the $SU(N)$ weight space.

Highest weight vectors have coefficients so that the Dynkin labels $a_i = w_i - w_{i+1}$ are positive integers. Another well-known description of irreducible $SU(N)$ representations is

by Young tableaux, with N rows of lengths $l_1, \dots, l_N$, $l_1 \geq l_2, \dots, \geq l_N$. The Dynkin labels are related to this description as $a_i = l_i - l_{i+1}$. It follows that apart from an overall constant the lengths of the rows of the Young tableaux are equal to the cartesian coefficients w_i .

In cartesian basis the basic Weyl reflection with respect to the plane orthogonal to the simple root $\vec{\alpha}_i$ becomes simply the interchange of the coefficients w_i and w_{i+1} , or, equivalently, the interchange of the corresponding rows of the Young tableau. Together these reflections generate the permutation group of N objects.

Consider now the embedding $SU(m+n) \supset SU(m) \times SU(n)$. If we choose $N = m+n$, and assign the first m vectors e_i to $SU(m)$ and the rest to $SU(n)$, the rule (5.1) has the following effect (we ignore the vectors ρ_G and ρ_H for the moment). Any Weyl transformation takes the set l_i to a permutation $l_{\pi(i)}$. The result should be a highest weight of $\mathcal{H}$, which means that it should respect the ordering

$$l_{\pi(1)} \geq l_{\pi(2)} \dots \geq l_{\pi(m)} \quad \text{and} \quad l_{\pi(m+1)} \geq l_{\pi(m+2)} \dots \geq l_{\pi(n)} .$$

This can be achieved by selecting any m rows of the $SU(m+n)$ Young tableau and permuting them to the first m positions without changing their order. The remaining n rows automatically satisfy the ordering condition to become a valid $SU(n)$ Young tableau.

An important quantity is the *length* of the Weyl transformation. The length is the smallest number of simple root reflections (*i.e.* nearest neighbor permutations) out of which the Weyl transformation can be built. It can be computed as follows. Suppose the m rows selected for $SU(m)$ are $\pi(1), \dots, \pi(m)$. Since they should be correctly ordered, we must have $\pi(1) < \pi(2) \dots < \pi(m)$. The smallest number of nearest neighbor permutations need to put row $\pi(1)$ in position 1 is clearly $\pi(1) - 1$. Similarly, to put $\pi(2)$ in position 2 we need $\pi(2) - 2$ nearest neighbor permutations, etc. The $SU(n)$ rows are then automatically in the correct order. The length L of the Weyl transformation w is thus

$$L(w) = \sum_{i=1}^m (\pi(i) - i) . \tag{5.5}$$

In the $N=2$ coset models the length of the Weyl rotation used in (5.1) determines the $SO(2d)$ conjugacy class to which a Ramond ground state belongs: even length corresponds to the class (s) and odd length to (c) (up to a conventional overall chirality choice).

Now consider field identification fixed points. In $SU(N)_k$, a fixed point of a current $J^{\frac{N}{p}}$ of order p is characterized by a set of extended Dynkin labels with periodicity p . Translated in terms of Young tableaux, this yields a set of rows of lengths

$$\begin{aligned} &((p-1)l + m_1, (p-1)l + m_2, \dots, (p-1)l + m_t, \\ &(p-2)l + m_1, (p-2)l + m_2, \dots, (p-2)l + m_t, \\ &\dots, \\ &m_1, m_2, \dots, m_t) , \end{aligned} \tag{5.6}$$

where $k = lp$ and $N = tp$. Now consider again the embedding $SU(n+m) \supset SU(m) \times SU(n)$, with level $k = lp$ for all algebras, and $m = rp$, $n = sp$, $r + s = t$. If we want the weight of $\mathcal{H}$ to be a fixed point of the $SU(m)$ current J^r and the $SU(n)$ current J^s we must make the selection $\pi(i)$ in a special way. Suppose $\pi(1)$ has been chosen, and the corresponding row has length $l_{\pi(1)}$. Then the length of row $\pi(r+1)$ must be equal to $l_{\pi(1)} - l$ in order to get an $SU(n)$ fixed point. Unfortunately this does not necessarily determine $\pi(r+1)$ since there may be more than one row with that length.

At this point we have to go to the slightly more complicated embedding which we are really interested in, and include the vectors ρ_G and ρ_H in the discussion. On the one hand, adding the vector ρ_G to the highest weight Λ solves the problem we just encountered, since after adding ρ_G (which has a Young tableau $(N, N-1, N-2, \dots, 3, 2, 1)$) all rows have different lengths. On the other hand, ρ_G is not a fixed point of the current J^t (except when $k = p$), which complicates the argument somewhat. Furthermore we should take the subtraction of ρ_H into account, as well as the level of $SU(m)$ and $SU(n)$, which is $k + n$ and $k + m$ respectively.

After adding ρ_G to a fixed-point highest weight Λ one obtains the following Young tableau

$$\begin{aligned} &((p-1)l + m_1 + tp, (p-1)l + m_2 + tp - 1, \dots, (p-1)l + m_t + tp - t + 1, \\ &(p-2)l + m_1 + t(p-1), (p-2)l + m_2 + t(p-1) - 1, \dots, (p-2)l + m_t + t(p-1) - t + 1, \\ &\dots, \\ &m_1 + t, m_2 + t - 1, \dots, m_t + 1) , \end{aligned} \tag{5.7}$$

Suppose we select the first row of $SU(m)$ from the first set,

$$l_{\pi(1)} = (p-1)l + m_{\pi(1)} + tp - \pi(1) + 1 ,$$

After subtracting the first entry of ρ_H (that equals rp) this becomes $(p-1)(l+s) + m_{\pi(1)} + s - \pi(1) + 1$. If this is to be a fixed point, we can compute the lengths of all rows $jr+1$, $j=0, \dots, p-1$ by subtracting $j(l+s)$. Then we can compute the length of the rows of (5.7) from which these rows must originate by adding the corresponding entries of ρ_H , $r(p-j)$. The result is

$$l_{\pi(jr+1)} = (p-j-1)l + m_{\pi(1)} + t(p-j) - \pi(1) + 1 .$$

Since the length of the rows of (5.7) is monotonically decreasing, this completely fixes $\pi(jr+1)$:

$$\pi(jr+1) = \pi(1) + jt . \quad (5.8)$$

Note that this can only work consistently if we choose the first row of $SU(m)$ out of the first $r+s$ rows of $SU(n+m)$. The argument works in precisely the same way for the second row of $SU(m)$, and the last row to be chosen is the r^{th} one. All further ones are completely determined by the required periodicity for a fixed point.

Verification of the consistency of the character corrections amounts to checking that the following diagram commutes

$$\begin{array}{ccc} \Lambda[SU(n+m)] & \xrightarrow{1} & \Xi_{\Lambda}[SU(r+s)] \\ \downarrow 2 & & \downarrow 4 \\ \lambda[SU(m) \times SU(n)] & \xrightarrow{3} & \xi_{\lambda}[SU(r) \times SU(s)] \end{array} \quad (5.9)$$

Here Λ , Ξ , λ and ξ are highest weight vectors of the groups in square brackets. The representation Λ is a fixed point of the current J^{r+s} and λ is a fixed point of $(J^r, 1)$ and $(1, J^s)$. The representations Ξ and ξ are the representations of the fixed-point CFT associated to these fixed points according to [12].

The relation between Λ and λ is as discussed above, for some choice of the Weyl transformation π . This choice fixes λ and then also ξ . What has to be proved is that there is a one-to-one correspondence between permutations π relating fixed-point fields in $SU(n+m)/SU(m) \times SU(n)$ and the complete set of permutations π' in $SU(r+s)/SU(r) \times SU(s)$ that relate highest weights Ξ to highest weights ξ .

The foregoing discussion establishes this map. As we have seen, the map π is completely determined by its action on the first $r+s$ rows if we require λ to be a fixed point. These first $r+s$ rows are mapped to the first r rows of $SU(m)$ and the first s rows of $SU(n)$. Hence the restriction of π to the first $r+s$ rows can be related in a canonical way to a permutation π' in $SU(r+s)$, that respects the ordering of the rows of the subgroup. Clearly this relation works also in the other direction, so that there is indeed a one-to-one correspondence.

The four maps in (5.9) yield the following results if we start with the fixed-point representation (5.6). Maps 1 and 3 can be read off directly from the results of [12], and maps 2 and 4 are an application of eq. (5.1). Explicitly, this yields

1. Yields the $SU(r+s)$ highest weight

$$(m_1, m_2, \dots, m_{r+s}) \tag{5.10}$$

2. Yields the $SU(m)$ highest weight

$$\begin{aligned} & [(p-1)(l+s) + m_{\pi(1)} + s - \pi(1) + 1, \dots, (p-1)(l+s) + m_{\pi(r)} + s - \pi(r) + 1, \\ & (p-2)(l+s) + m_{\pi(1)} + s - \pi(1) + 1, \dots, (p-2)(l+s) + m_{\pi(r)} + s - \pi(r) + 1, \\ & \dots, \\ & m_{\pi(1)} + s - \pi(1) + 1, \dots, m_{\pi(r)} + s - \pi(r) + 1 \quad] , \end{aligned} \tag{5.11}$$

combined with the $SU(n)$ highest weight

$$\begin{aligned}
 & [(p-1)(l+r) + m_{\pi(m+1)} + r - \pi(m+1) + 1, \\
 & \quad \dots, (p-1)(l+r) + m_{\pi(m+s)} + r - \pi(m+s) + 1, \\
 & (p-2)(l+r) + m_{\pi(m+1)} + r - \pi(m+1) + 1, \\
 & \quad \dots, (p-2)(l+r) + m_{\pi(m+s)} + r - \pi(m+s) + 1, \\
 & \quad \dots, \\
 & m_{\pi(m+1)} + r - \pi(m+1) + 1, \dots, m_{\pi(m+s)} + r - \pi(m+s) + 1 \quad] .
 \end{aligned} \tag{5.12}$$

3. Yields the $SU(r)$ highest weight

$$m_{\pi(1)} + s - \pi(1) + 1, \dots, m_{\pi(r)} + s - \pi(r) + 1 , \tag{5.13}$$

combined with the $SU(s)$ highest weight

$$m_{\pi(m+1)} + r - \pi(m+1) + 1, \dots, m_{\pi(m+s)} + r - \pi(m+s) + 1) . \tag{5.14}$$

4. Is simply (5.1) applied to the $SU(r+s)$ -representation (5.10). After adding $\rho_{SU(r+s)}$, performing the permutation π' and subtracting $\rho_{SU(r)}$ or $\rho_{SU(s)}$ one immediately gets (5.13) and (5.14).

This proves that for every Ramond ground state which is a fixed point, the corresponding state in the fixed-point CFT is a Ramond ground state as well, and vice versa. Thus the character corrections are available exactly where they are needed. What remains to be done is compute the relative signs of the PP-characters between the fixed points and the fixed-point CFT.

As discussed above, this sign is determined by the length of the Weyl transformation.

For a transformation π that takes fixed points into fixed points, we get, starting from (5.5):

$$\begin{aligned} L(\pi) &= \sum_{i=1}^{pr} (\pi(i) - i) \\ &= \sum_{j=0}^{p-1} \sum_{i=1}^r (\pi(i + jr) - i - jr) . \end{aligned}$$

Using (5.8) we can rewrite this as

$$\begin{aligned} L(\pi) &= \sum_{j=0}^{p-1} \sum_{i=1}^r (\pi(i) + js - i) \\ &= pL(\pi') + \frac{1}{2}p(p-1)s . \end{aligned}$$

The sign of the PP-characters of the coset theory is $(-1)^{L(\pi)}$, and the sign of the corresponding character of the fixed-point CFT is $(-1)^{L(\pi')}$. The factor $\frac{1}{2}p(p-1)s$ is not of much importance, since its only possible effect is an overall sign change of all the PP-characters of the fixed-point CFT. This is merely a matter of convention, since a simultaneous sign change of all PP-characters obviously yields a correct set of characters as well. Apart from this overall sign, we see that for p odd the coset characters and those of the fixed-point CFT have the same signs. This is another check that the character corrections work as conjectured, since we have seen earlier in this section that this was a necessary condition. For p even there is *a priori* no constraint on the signs of the character corrections. It is therefore not too surprising that both relative signs occur: the fixed points of the coset theory always have the same chirality ($L(\pi) \bmod 2$ is independent of π) but the corresponding fields of the fixed-point CFT can have both chiralities. Fortunately, we know now exactly which sign appears for which fixed point: if the value of $L(\pi)$ modulo 4 is different for two fixed points, the corresponding fields of the fixed-point CFT have opposite chirality. It is tempting to conjecture that the same rule will hold also for all other cosets.

5.3 Examples of extended Poincaré polynomials

To illustrate the extended Poincaré polynomials introduced in section 3 it may be useful to begin with the minimal models, in which fixed-point problems are absent. The polynomials of these theories are

$$\mathcal{P}_k(t, x) = \sum_{l=0}^k t^{2l-k} \frac{1 + (-1)^l x^{2l+2}}{1 - (-1)^k x^{2k+4}}.$$

For a less trivial example, consider the coset

$$\frac{SU(4)_2 \times SO(8)_1}{SU(2)_4 \times SU(2)_4 \times U_{96}}.$$

This coset has field identification fixed points, that occur for the highest weights $(0, 1, 0)$ of $SU(2)_4$ and (2) in both of the $SU(2)$ factors. There are in fact two such fields that are Ramond ground states, and they have U_{96} charges 4 and -4 respectively. Because of the field identification these charges are defined modulo 24 and not 96. Both fields, denoted R_4 and R_{-4} , have the same $SO(8)$ -chirality.

It is now very interesting to consider the action of the spinor current S , with U_{96} charge 8. To get another Ramond state it must act an even number of times. If we ignore the fact that these fields are fixed points and therefore consist of two fields each, we would conclude that

$$S^2 R_4 = R_{-4} \quad \text{and} \quad S^4 R_{-4} = R_4 \tag{5.15}$$

However, we should pay attention to the proper fixed-point resolution.

The fixed-point CFT associated with this coset is essentially the $k = 1$ minimal model. The fields of that model associated with the fixed points have $l = 0$ and $q = +1, -1$ for R_4 and R_{-4} respectively (note that to get the standard charge normalization of the fixed-point CFT we have to divide all fixed-point charges by p^2 , as explained in section 4). These two fields are indeed Ramond ground states of the $k = 1$ minimal model, but with *opposite* chirality. Therefore the actual Ramond ground states originating from R_4 and R_{-4} belong

to different components of these two fields. If we choose these Ramond states to be $(R_4)_1$ and $(R_{-4})_2$, the correct version of (5.15) is

$$S^2(R_4)_1 = (R_{-4})_1 \quad \text{and} \quad S^4(R_{-4})_2 = (R_4)_2 .$$

However, neither one of the right-hand sides is a Ramond ground state or a T_F excitation of such a state. The states $(R_4)_2$ and $(R_{-4})_1$ are ordinary Ramond states, with $h - \frac{c}{24} = 1$, and the fact that they lie on the orbits of the fields on the left-hand side is of no interest whatsoever. The correct extended Poincaré polynomial of this theory is thus

$$\mathcal{P}(t, x) = t^{-4} \frac{1+x^2}{1-x^6} + [4t^{-2} + 6 + 4t^2] \frac{1}{1-x^6} + t^4 \frac{1+x^4}{1-x^6} . \quad (5.16)$$

The two fixed-point fields contribute to the two coefficients 4 in this polynomial. Without a careful analysis of the chirality of the fixed-point CFT one would have concluded that their coefficient functions are $(1+x^2)/(1-x^6)$ and $(1+x^4)/(1-x^6)$ respectively.

The polynomial (5.16) is equal to

$$P_1(t, x) * P_1(t, x) * P_1(t, x) * P_1(t, x) ,$$

which describes the tensor product of 4 $k = 1$ minimal models. This observation seems to agree with [20], where the $A(2, 2, 2)$ theory was argued to be a twisted version of this tensor product. Nevertheless, it is somewhat mysterious that these twists do not affect the Poincaré polynomial.

Our last explicit example is the $N=2$ coset theory $SU(6)_3/SU(3)_6 \times SU(3)_6 \times U(1)$. The fixed-point CFT of this theory is again the $k = 1$ minimal model, but since the common factor is odd the subtleties encountered above do not occur. For the extended Poincaré polynomial we find

$$\begin{aligned} \mathcal{P}(t, x) = & \left[\frac{1+x^2}{1+x^6} \right] (t^{-9} + t^{-3} + t^3) + \left[\frac{1-x^4}{1+x^6} \right] (t^{-3} + t^3 + t^9) \\ & + \frac{1}{1+x^2} (t^{-3} + t^3) + \frac{3}{1+x^6} (t^{-7} + 4t^{-5} + 9t^{-3} + 16t^{-1} + 16t + 9t^3 + 4t^5 + t^7) . \end{aligned}$$

The third term originates from the two field identification fixed points. Clearly these fields

are also fixed points of S^2 (up to a world-sheet supersymmetry transformation), but this sort of fixed point is harmless in comparison to field identification fixed points.

A rather interesting fact is that for all extended polynomials we have obtained for coset theories the coefficient polynomials for each power of t have at most two terms in their numerator. In other words, the orbit of S intersects the chiral ring at most twice. This property does not hold, in general, for tensor products of several coset theories. Usually the two intersection points have opposite $N = 2$ charges, but there are exceptions.

5.4 String Spectra

To conclude we present some results on the number of generations and anti-generations for four-dimensional string theories. We will only study $(2, 2)$ models and only the diagonal invariant. Extra simple current twists can be added without great difficulty, but a systematic listing is not practical.

We do not limit ourselves to cosets with fixed points here, since many cosets without fixed point problems can be included without any extra efforts. However, we did not consider $N = 2$ cosets based on exceptional algebras, since none of them has field identification fixed points. We will omit classes that have already been discussed in the literature, and with which we are in complete agreement. This includes in particular the minimal model tensor products [17], [21], and more generally tensor products including only cosets of the form $A(n, 1, k)$, already completely discussed in [22]. We do not consider $N = 2$ cosets based on exceptional algebras, since none of them has field identification fixed points.

Checks

The necessary computations are straightforward, but somewhat complicated, and it is therefore important to have as many checks as possible. We have verified the following points

- We select G and H representations on an identification orbit by maximizing the inner product of their weight vector with a fixed random vector in weight space. The representation selected by this criterium serves as a representative for the orbit. Obviously the final result, *i.e.* the extended Poincaré polynomial should be independent of the choice of the random vector (*cf.* gauge choices in Feynman diagram computations).

- The known relation $A(n, m, k) = A(n, k, m)$ [1], and the relations $B(3, k) = C(2, k)$, $B(4, k) = A(2, 2, k)$, $B(6, k) = D(4, k)$, $C(1, k) = A(1, 1, k)$ and $D(3, k) = A(1, 3, k)$ that follow from Kac-Moody isomorphisms should be satisfied.
- The values of N_g and N_a obtained from the x -dependence of the extended Poincaré polynomial should match the value of $|N_g - N_a|$ obtained from the t -dependence of the (ordinary) Poincaré polynomial using the formula of [14].

Non-Grassmannian cosets

The last point can only be used as a check for Grassmannians, for which we can compute the extended Poincaré polynomial directly. The two ways of computing $|N_g - N_a|$ are completely independent, and provide an extremely powerful check. It becomes even more powerful once one realizes that it should not just be satisfied for $c = 9$, but for all $c = 3 \bmod 6$. Thus by considering $c = 15$ tensor products one obtains a huge number of consistency conditions on the x -dependence of extended Poincaré polynomials. It is even possible that this set of conditions (which can be augmented indefinitely by considering $c = 21$, $c = 27$, etc.) is powerful enough to fix the x -dependence of the polynomial completely, once the t -dependence is known. This is certainly not a practical method for computing the extended polynomial, but it would be quite interesting to know whether in principle the x -dependence is completely determined by the t -dependence.

In coset theories with field identification fixed points we know the complete t -dependence of the polynomial as well as most of the x -dependence. Only on the fixed-point orbits there is some remaining uncertainty. We know which terms *may* appear in the polynomial, but we do not know whether they actually *do* appear. To resolve this ambiguity one can determine the chiralities of the Ramond ground states of the fixed-point CFT, as we were able to do analytically for Grassmannians. For other coset theories it should at the very least be possible to solve this problem numerically, using an explicit expression for the matrix $\hat{S}$.

Instead of doing that one can try to make the $|N_g - N_a|$ check work, by removing all combinations of uncertain terms from the polynomial, until a correct polynomial is found. One starts with the “naive” polynomial, in which all terms are kept regardless of whether they are upper or lower components of a resolved fixed point. For each fixed point one expands the corresponding term in the polynomial in x up to powers equal to the order N_S of the current. All terms x^m , $1 < m < N_S$ are ambiguous. They are actually present in the

polynomial only if their chirality in the fixed-point CFT is the same as that of the starting point of the orbit. If there are in total N_A ambiguous terms, there are thus 2^{N_A} candidates for the correct Poincaré polynomial and one can try to find the correct one by computing $|N_g - N_a|$ and comparing with (3.5).

Remarkably this works much better than one might have expected. Note first of all that the changes in N_g and N_a resulting from modifying just one polynomial in a tensor product are linear. Therefore instead of trying all 2^{N_A} possibilities it suffices to consider just all N_A changes of one term only. Secondly, it turns out that in most cases there is only one way of getting the correct value of $|N_g - N_a|$, because all corrections with the correct sign precisely add up to the deficit. Then all terms whose removal yields a correction with the wrong sign should stay in the polynomial. There may also be terms whose removal does not affect $|N_g - N_a|$ at all, but usually they do not affect N_g and N_a separately either. Hence there may be some remaining uncertainty in the form of the polynomial, but N_g and N_a are completely determined.

When necessary, one can remove the remaining ambiguities by considering $c = 15$ theories. We have used this trick to verify equivalences like $B(3, 15) \sim B(15, 3)$, discussed below.

As a simple illustration, consider $B(3, 1)$. The naive extended Poincaré polynomial is

$$\frac{1}{1 - x^{16}} \left[t^{-6}(1 + x^2) + t^{-4}(1 - ax^4 - bx^8 + cx^{12}) + t^{-2}(1 + x^6) + (1 - x^8) \right. \\ \left. + t^6(1 + x^{14}) + t^4(1 + c'x^4 - b'x^8 - a'x^{12}) + t^2(1 + x^{10}) \right] ,$$

where a, b, c, a', b', c' are the unknown parameters which are either equal to zero or one. Actually Poincaré duality relates these parameters to each other, so that $a' = a$, $b' = b$ and $c' = c$, but it is not necessary to impose this as a condition. There are several ways of tensoring $B(3, 1)$ with other theories to obtain a combination with $c = 9$. For a sufficiently large set of such combinations we have computed N_g and N_a with all coefficients equal to 1, and with all but one equal to 1, and the sixth one equal to zero. The result is listed in Table 1.

From the tensor product with $(6, 6, 6)$ we learn that $c' = 0$; from the tensor product with $(4, 6, 10)$ that $b = 0$; from the tensor product with $(4, 4, 22)$ that $c = 0$ and that a and

Table 1. *Effect of modifications of the undetermined parameters of the extended Poincaré polynomial of $B(3,1)$. In column 1, “1” indicates that all parameters are equal to 1; in the last six rows all parameters are equal to 1 except the one indicated in column 1. In the first row we list the combination of minimal models tensored with $B(3,1)$ and between square brackets the correct value of $\frac{1}{2}\chi = N_g - N_a$.*

	(6,6,6) [148]			(4,6,10) [60]			(4,4,22) [84]			(2,8,38) [84]		
parm.	N_g	N_a	$\frac{1}{2}\chi$	N_g	N_a	$\frac{1}{2}\chi$	N_g	N_a	$\frac{1}{2}\chi$	N_g	N_a	$\frac{1}{2}\chi$
1	149	2	147	71	8	63	90	13	77	97	20	77
$a = 0$	149	2	147	71	7	64	88	13	75	97	19	78
$b = 0$	149	2	147	67	8	59	90	13	77	96	18	78
$c = 0$	149	2	147	71	7	64	90	7	83	97	17	80
$c' = 0$	149	1	148	71	7	64	90	12	78	97	19	78
$b' = 0$	149	2	147	70	8	62	90	13	77	97	18	79
$a' = 0$	149	2	147	71	8	63	88	13	75	97	20	77

a' must be equal to 1. Going back to (4,6,10) we see then that $c = 0$ implies that b' must be zero as well. Now all parameters are determined, Poincaré duality is satisfied, and the tensor product with (2,8,38) serves as an additional check. The correct polynomial is thus

$$\frac{1}{1-x^{16}} [t^{-6}(1+x^2) + t^{-4}(1-x^4) + t^{-2}(1+x^6) + (1-x^8) + t^6(1+x^{14}) + t^4(1-x^{12}) + t^2(1+x^{10})] .$$

This is precisely the extended Poincaré polynomial of $A(1,1,6)$, an example of the relation $B(n,1) \sim B(1,n) \equiv A(1,1,2n)$ for n odd.

Relations

As the last example shows, the extended Poincaré polynomials are quite useful for establishing relations between different coset models. Of course equality of polynomials is not sufficient for proving equivalence of coset models, but in any case two such coset models will yield the same values for N_g and N_a , even when the spinor currents are used to build

non-diagonal invariants. It would be quite interesting to compare the rest of the spectrum as well.

We have already mentioned in the previous sub-section that the coset $A(2, 2, 2)$ has exactly the same extended Poincaré polynomial as four copies of $A(1, 1, 1)$. There have been speculations, and some indications, about a relation $B(k, n) \sim B(n, k)$. We have found empirically that this relation holds exactly (for the extended Poincaré polynomials) for k and n both odd. For n even and k odd one has $B(k, n) = B(n, k)_D$, where on the right-hand side one must take the D -invariant, generated by the integer spin current J_v of $SO(n)$ (or, equivalently, J_v of $SO(n+2)$, which are both identified with the spinor current S to the power $\frac{1}{2}N_S$). For k and n both even, the relation (if any exists) appears to be more complicated, but cosets of this type do not appear in the list of $c = 9$ theories.

In addition we have found the polynomial equalities $C(3, 1) = A(1, 1, 3)^2$ and $D(5, 1) = A(1, 1, 1)A(1, 1, 7)$.

Results

The results for all remaining cosets, up to known equivalences, are listed in Table 2. The spectra of the irreducible cosets have also been computed in [23]. We disagree with the results of this paper. For some of these theories this is not surprising, since the treatment given to field identification fixed points in [23] is different from ours. However, the coset $A(2, 3, 5)$ does not have any fixed points, and nevertheless we disagree with [23] also for this theory.

It is interesting to compare this table with the one obtained in [24]. These authors have used the Poincaré polynomials constructed by Lerche, Vafa and Warner [3], and then compute the corresponding deformation class of the Landau-Ginzburg polynomial. Then Vafa’s method [25] can be used to compute N_g and N_a . This procedure may yield consistent string theories, but not necessarily the ones obtained from the coset CFT. The problem is that the polynomials of [3] are only guaranteed to be those of the coset for level 1 simply-laced groups. Indeed, our results agree with those of [24] for $D(n, 1)$, the only simply-laced level 1 models on the list. In most other cases our results are different, and indeed one may check that the Poincaré polynomials of the coset theory are not those of [3] (this difference has nothing to do with field identification fixed points, and has already been observed in [3]).

There are a few interesting cases of agreement with [24], most notably for all tensor products involving $C(4, 1)$. Here indeed we find that the Poincaré polynomial of [3] agrees with the one of the coset model. This is especially interesting since this coset model has field identification fixed points. The agreement of N_g and N_a separately provides another non-trivial check of our computations (as well as those of [24], of course). The other spectra that are in agreement occur for $A(2, 2, 4)$, $B(3, 3)$, $B(3, 6)$, and $B(6, 3)$. However, in all four cases the Poincaré polynomials are different, and the agreement with [24] does not hold for all tensor products involving these coset models.

The values for N_g and N_a are not spectacularly different from those obtained for other conformal field theories. Most of them appear already in existing lists of simple current invariants of minimal model tensor products ([26], [27]).

6. Conclusions

The results obtained in this paper strengthen our belief that a sensible conformal field theory exists for every coset theory, even if the field identification has fixed points. We have shown that the method for resolving fixed points introduced in [6] works for the chiral Ramond sector of Grassmannian $N=2$ coset theories with fixed points of prime order. For these theories we can compute all fusion coefficients and all chiral characters. For all other cosets with fixed points of prime order we are able to obtain all the fusion rules, but more work is needed to obtain the chiral characters, since we do not know an efficient way to compute these characters for the fixed-point CFT.

The Poincaré polynomial of the chiral ring of any $N=2$ coset theory can be computed as easily as for theories without field identification fixed points. This computation is nearly independent of the details of field identification fixed points. Each identification orbit, no matter how long or short, yields either exactly one Ramond ground state, or none at all. The extended Poincaré polynomial governing the spinor current/chiral ring intersections is harder to obtain. We have presented a direct computation only for Grassmannians, and a “trial and error” method for the other cosets.

This work can be extended in many directions. It is certainly possible to compute the character corrections in the chiral Ramond sector also for other coset theories that are not Grassmannians. Also of interest, although much harder, would be an extension of these

results to the Neveu-Schwarz and non-chiral Ramond sector, where character corrections at higher excitation levels become relevant. This would also make possible a complete computation (including E_6 -singlets) of massless string spectra. The extended Poincaré polynomial can be extended further to include information about other $\mathbf{Z}_N$ -factors in the center of the coset theory, by including extra variables analogous to t and x . This situation arises precisely in coset theories with field identification fixed points, which have an extra factor $\mathbf{Z}_p$, where p is the order of the fixed-point current. It arises more trivially in tensor products of $N=2$ models, where only one linear combination of the spinor currents is used as the spinor current of the product. This particular combination is singled out by the “ $*$ ” product (3.3), but one can keep all information by choosing different variables t_i and x_i in each factor, and simply multiplying the polynomials.

This information can be used to construct other simple current invariants for a given tensor product. This will yield far more modular invariants than the one used in Table 2, and not only $(2,2)$ but also $(2,1)$ spectra (a complete classification of all simple current invariants is available if the center does not contain any factors $\mathbf{Z}_{p^n}$, $n \geq 2$ [15]). Finally, it is of utmost importance to develop fixed point resolution methods for correlation functions, so that one can work out the tree-level couplings among fixed-point fields of interesting four-dimensional string theories, if any are found.

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Table 2. *Spectra for tensor products of $N=2$ models. Equalities in column 1 indicate that the extended Poincaré polynomials of two theories are identical, not that they are identical as conformal field theories.*

tensor product	N_g	N_a	χ
$A(2, 2, 12)$	43	11	64
$A(2, 3, 5)$	15	3	24
$A(3, 3, 3)$	32	2	60
$A(2, 3, 4)A(1, 1, 1)$	17	5	24
$A(2, 3, 3)A(1, 1, 6)$	31	3	56
$A(2, 2, 8)A(1, 1, 1)$	43	7	72
$A(2, 2, 6)A(1, 1, 3)$	49	5	88
$A(2, 2, 5)A(1, 1, 7)$	35	11	48
$A(2, 2, 4)A(1, 2, 3)$	51	3	96
$A(2, 2, 4)A(1, 1, 1)^3$	$K_3 \times T_2$		0
$A(2, 2, 4)A(1, 1, 1)A(1, 1, 4)$	51	3	96
$A(2, 2, 4)A(1, 1, 2)^2$	61	1	120
$A(2, 2, 3)A(1, 3, 3)$	41	5	72
$A(2, 2, 3)A(1, 1, 1)A(1, 1, 40)$	29	17	24
$A(2, 2, 2) = A(1, 1, 1)^4$			
$B(6, 6)$	41	5	72
$B(3, 3)B(3, 3)$	59	3	112
$B(3, 3)A(1, 2, 9)$	55	7	96
$B(3, 3)A(1, 2, 3)A(1, 1, 2)$	51	3	96
$B(3, 3)A(1, 3, 4)$	59	3	112
$B(3, 3)A(1, 1, 1)^3A(1, 1, 2)$	$K_3 \times T_2$		0
$B(3, 3)A(1, 1, 1)^2A(1, 1, 10)$	37	13	48
$B(3, 3)A(1, 1, 1)A(1, 1, 2)A(1, 1, 4)$	51	3	96
$B(3, 3)A(1, 1, 2)^3$	73	1	144
$B(3, 3)A(1, 1, 3)A(1, 1, 18)$	51	11	80
$B(3, 3)A(1, 1, 6)A(1, 1, 6)$	69	5	128

Table 1. (*Cont'd*)

tensor product	N_g	N_a	χ
$B(3,4)A(1,3,3)$	41	5	72
$B(4,3)A(1,3,3)$	41	5	72
$B(3,6)A(1,2,3)$	53	5	96
$B(6,3)A(1,2,3)$	62	2	120
$B(3,6)A(1,1,1)^3$	62	2	120
$B(6,3)A(1,1,1)^3$	62	2	120
$B(3,6)A(1,1,1)A(1,1,4)$	53	5	96
$B(6,3)A(1,1,1)A(1,1,4)$	62	2	120
$B(3,6)A(1,1,2)^2$	23	23	0
$B(6,3)A(1,1,2)^2$	$K_3 \times T_2$		0
$B(5,5)A(1,1,2)$	41	5	72
$B(3,24)A(1,1,1)$	41	35	12
$B(24,3)A(1,1,1)$	41	35	12
$B(3,15)A(1,1,2)$	31	31	0
$B(15,3) = B(3,15)$			
$B(3,12)A(1,1,3)$	47	11	72
$B(12,3)A(1,1,3)$	47	11	72
$B(3,9)A(1,1,6)$	49	9	80
$B(9,3) = B(3,9)$			
$B(3,8)A(1,1,9)$	29	19	20
$B(8,3)A(1,1,9)$	29	19	20
$B(3,7)A(1,1,18)$	35	19	32
$B(7,3) = B(3,7)$			

Table 1. (*Cont'd*)

tensor product	N_g	N_a	χ
$C(3, 4)$	19	3	32
$C(5, 1)A(1, 1, 12)$	53	5	96
$C(4, 1)A(1, 2, 6)$	44	8	72
$C(4, 1)A(1, 2, 3)A(1, 1, 1)$	62	2	120
$C(4, 1)A(1, 1, 1)^4$	62	2	120
$C(4, 1)A(1, 1, 1)^2A(1, 1, 4)$	62	2	120
$C(4, 1)A(1, 1, 1)A(1, 1, 2)^2$	40	4	72
$C(4, 1)A(1, 1, 2)A(1, 1, 10)$	61	1	120
$C(4, 1)A(1, 1, 4)^2$	74	2	144
$C(3, 2)A(1, 2, 3)$	51	3	96
$C(3, 2)A(1, 1, 1)^3$	51	3	96
$C(3, 2)A(1, 1, 1)A(1, 1, 4)$	51	3	96
$C(3, 2)A(1, 1, 2)^2$	$K_3 \times T_2$		0
$C(3, 1) = A(1, 1, 3)^2$			
$D(7, 2)$	16	8	16
$D(9, 1)A(1, 1, 15)$	119	7	224
$D(8, 1)A(1, 2, 2)A(1, 1, 1)$	55	7	96
$D(8, 1)A(1, 1, 1)A(1, 1, 8)$	55	7	96
$D(6, 2)A(1, 1, 2)$	18	6	24
$D(6, 1)A(1, 1, 1)^2A(1, 1, 64)$	67	19	96
$D(6, 1)A(1, 1, 4)A(1, 1, 64)$	84	24	120
$D(6, 1)A(1, 1, 9)^2$	144	4	280
$D(5, 2)A(1, 2, 3)$	$K_3 \times T_2$		0
$D(5, 2)A(1, 1, 1)^3$	$K_3 \times T_2$		0
$D(5, 2)A(1, 1, 1)A(1, 1, 4)$	$K_3 \times T_2$		0
$D(5, 2)A(1, 1, 2)^2$	$K_3 \times T_2$		0
$D(5, 1) = A(1, 1, 1)A(1, 1, 7)$			

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