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Thermodynamics of $SU(3)$ Lattice Gauge Theory in $(2 + 1)$ Dimensions

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Abstract

We investigate the finite temperature behavior of $(2 + 1)$ -dimensional $SU(3)$ lattice gauge theory by Monte Carlo methods. A continuous and presumably second order deconfining phase transition is found at a critical temperature T_c . Estimates are made for various critical exponents. The behavior of higher representation Polyakov loops is studied near the phase transition. In the high temperature phase we compare the internal energy density with expectations based on a non-interacting gas of bosons.

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1. Introduction

The thermodynamics and critical behavior of three-dimensional lattice gauge theories have largely been studied only in the case of $SU(2)$, where the deconfinement phase transition has been found to be of 2nd order [1, 2]. According to the universality hypothesis [3] this has placed this particular theory in the 2-D Ising universality class, in general in excellent agreement with the observed critical behavior [1, 2, 4]. In the case of $SU(3)$, a preliminary Monte Carlo study has indicated that the phase transition also in this case is of 2nd order [5]. Since the center symmetry that gets broken in this case is $Z(3)$, this would place the $SU(3)$ gauge theory in the universality class of the 3-state Potts model in $D = 2$ dimensions. However, a first rough attempt at extracting the magnetization exponent β from the $SU(3)$ gauge theory data yielded an estimate of $\beta \simeq 0.8 \pm 0.3$ [5], very far from the expected Potts model value of $\beta = 1/9$. This apparent discrepancy is strong motivation for trying to pin down the critical behavior of this particular theory in a more quantitative manner.

Although the other critical exponents (for the precise definition, see section 2) are notoriously difficult to estimate from Monte Carlo simulations, a more serious test of universality should include at least one more critical exponent apart from β . If chosen correctly, several other exponents are then in principle given through scaling and hyperscaling relations.³⁾ The difficulty in obtaining most other critical exponents by Monte Carlo methods stems from the fact that almost invariably long-distance correlations need to be determined accurately. It should perhaps be added at this point that critical exponents are not the *only* universal quantities on which one can focus. For example, there exists a tower of certain amplitude ratios that are also universal. A study of these requires much more statistics than we have available at the present time, but we hope to return to this point in a future study.

The present work describes our attempts at extracting the critical behavior at finite temperature $SU(3)$ lattice gauge theory in $(2 + 1)$ dimensions directly from Monte Carlo simulations on lattices of sizes $N_\sigma \times 2$, with $N_\sigma = 20, 30, 40$ and 60 . In section 2 we introduce the notation, and present the critical behavior expected from universality. As we need to compare our results for the internal energy density far above the phase transition with the expected finite volume lattice analogue of the $(2 + 1)$ -dimensional Stefan-Boltzmann law, we present the general expected form for a massless bose gas in section 3. In this section we also compare some of our numerical results for the internal energy density ϵ with this more general prediction. Section 4 is devoted to an overall presentation of our numerical results around the deconfinement phase transition, with special emphasis on the extraction of critical exponents from our Monte Carlo measurements. Here we also analyze the behavior of the free energy of heavy sources in various higher representations of $SU(3)$ near the phase transition. Finite size scaling theory is used to analyze the results for the internal energy density close to the transition. Finally, section 5 contains our conclusions.

2. Critical Behavior and Universality

Since, as explained in the introduction, there are indications that the $SU(3)$ deconfinement phase transition in $(2 + 1)$ dimensions is of 2nd order [5], this should place this

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3) In more ambitious undertakings all exponents could of course be independently determined, yielding extra consistency checks on the procedure.

theory in the universality class of the 2-dimensional $Z(3)$ Potts model. We therefore begin with a brief description of the known critical behavior of this spin model.

Based on both numerical work and the connection to Baxter's 8-state vertex model, the Ashkin-Teller model [6], and, quite recently, also to conformal field theory⁴⁾, it is known that the q -state Potts model for $q \leq 4$ has a thermal index

$$y_t = 1/\nu = 3(1-u)/(2-u) \quad (1)$$

where u is given by

$$0 \leq u = \frac{2}{\pi} \cos^{-1} \left(\frac{\sqrt{q}}{2} \right) \leq 1 \quad (2)$$

Also the magnetic index y_h is given by an explicit expression,

$$y_h = \frac{2\delta}{1+\delta} = \frac{(3-u)(5-u)}{4(2-u)} \quad (3)$$

for the same solution u of eq. (2). Here we have already introduced the critical exponents ν and δ . The other exponents follow from well-known scaling relations, giving in total, for $q = 3$:

$$\nu = \frac{5}{6}, \quad \delta = 14, \quad \gamma = \frac{13}{9}, \quad \beta = \frac{1}{9}, \quad \eta = \frac{4}{15}, \quad \alpha = \frac{1}{3} \quad (4)$$

In addition, even the next-to-leading exponent is known exactly in this case, due to a calculation of Nienhuis [9]. In what by now appears to be a standard notation we denote the correction exponent by θ . For $q = 3$, Nienhuis found $\theta = 2/3$. It is important to keep in mind that while also this next-to-leading exponent is *universal*, its amplitudes A_i in expansions such as for the magnetization M ,

$$M \sim t^\beta (1 + A_1 t^\theta + \dots) \quad (5)$$

are *not* universal. In particular, nothing prevents these amplitudes from vanishing, or, equivalently from our numerical point of view, from being exceedingly small. If the amplitude A_1 vanishes, the *next* correction term may possibly arise from *regular* functions *i.e.* to lowest order $M \sim t^\beta (1 + A_2 t + \dots)$, and so forth. Clearly the introduction of such correction terms in numerical estimates should be made with a great deal of care, as has so convincingly been demonstrated in the case of finite size scaling [10]. (Past experience with $SU(2)$ in $(2+1)$ dimensions [4] in any case tells us that we should not expect to be able to give a very reliable estimate for the non-leading exponent θ , except, possibly, in the case of the magnetization discussed here).

As order parameter for the $SU(3)$ gauge theory we can use the Polyakov loop of time-like links $U_0(x, t)$, $TrW(x) \equiv Tr_F \prod_t U_0(x, t)$, here explicitly chosen in the *fundamental* representation of $SU(3)$. But we have also the possibility of studying the critical behavior of the order parameter $TrW(x)$ in *higher irreducible representations* of the group, and in particular checking whether they are associated with the ~~same~~ critical exponents or not [11]. For the case of $SU(2)$ the situation is far from conclusive, both in $(2+1)$ dimensions [4] and $(3+1)$ dimensions [11, 12], presumably due to the limited spatial volumes probed.⁵⁾

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4) See *e.g.* ref.[7]. For a rather comprehensive review of results before the application of conformal field theory techniques to this model, see *e.g.* ref.[8].

In view of these facts, we have found it interesting to test also the critical scaling behavior of some higher representation Polyakov loops in the present theory. We restrict ourselves to the following representations:

$$\begin{aligned} Tr_6 W(x) &= (Tr_F W(x))^2 - Tr_F W(x)^\dagger \\ Tr_8 W(x) &= Tr_F W(x) Tr_F W(x)^\dagger - 1 \\ Tr_{10} W(x) &= (Tr_F W(x))^3 - 2 Tr_F W(x) Tr_F W(x)^\dagger + 1 \end{aligned} \quad (6)$$

where we have labelled the representations by their dimensions. In this notation we thus have $Tr_F W(x) = Tr_3 W(x)$. Note that of the three representations in eq. (6), both the adjoint representation $Tr_8 W(x)$ and the “decuplet” $Tr_{10} W(x)$ transform trivially under $Z(3)$. This is a crucial point. Since the deconfinement phase transition is associated with the breaking of the $Z(3)$ symmetry, these last two irreducible representations of the Polyakov loop cannot be true order parameters of the phase transition. We include them as a check on how meaningful the whole scenario is for lattices of the spatial volumes we are limited to. For $SU(2)$ in the same number of dimensions and on similar lattices sizes to the present study, one could not reliably argue that any difference at all could be seen between the behavior of representations transforming trivially or non-trivially under (in that case) $Z(2)$ [4]. If standard lore about screening and confinement of “quark” sources in various representations of the gauge group is valid, this presumably tells us that we are often dealing with very large screening lengths for the sources transforming trivially under the center group. For distances smaller than the screening length all sources could then behave quite similarly, *i.e.* effectively appear to be “confined”. But at present this interpretation remains at the level of a conjecture.

From the Polyakov loop we can also form the connected correlation function

$$\Gamma(r, T) \equiv \langle Tr_F W(r) Tr_F W(0)^\dagger \rangle - |\langle Tr_F W(0) \rangle|^2 \quad (7)$$

here for definiteness chosen for the fundamental representation. Precisely at the critical temperature T_c this correlation function is expected to decay algebraically:

$$\Gamma(r, t_c) = \text{const} \cdot r^{2-d-\eta} = \text{const} \cdot r^{-\eta}, \quad (8)$$

thus defining the critical exponent η . Away from the critical point we can extract the correlation length ξ from the exponential fall-off: $\Gamma(r, T) \sim \exp[-r/\xi(T)]$, where the correlation length diverges like $\xi(T) \sim |T - T_c|^{-\nu}$ close to T_c . This defines the critical exponent ν . The summed correlator $\chi = \sum_{x_i} \Gamma(x_i, T)$ over all lattice points yields a susceptibility, which diverges as $\chi \sim |T - T_c|^{-\gamma}$ near T_c . These quantities all refer to the fundamental representation of $SU(3)$. One could conceive different critical exponents for different irreducible representations of the gauge group, but we shall not pursue this question in the present work, except in the case of the magnetizations, as discussed above.

To all of the above expressions should in principle be added non-leading terms, as already mentioned in the case of the magnetization, eq. (5).

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5) Higher order representations have also been studied around the deconfinement phase transition of the $(3 + 1)$ -dimensional $SU(3)$ lattice gauge theory. There the transition is most likely 1st order, and the question of critical behavior of these representations can not be addressed. See also ref. [13].

In order to extract a reliable estimate of the correlation length, one needs to know not just the exponential fall-off of the two-point correlation function, but also the detailed algebraic prefactors. These are not universal in the same sense as the critical exponents, and certainly do not relate in any obvious way to the corresponding two-dimensional spin model. Still, within the framework of finite- T lattice gauge theory one can surmise a certain functional form. In particular, a string-like description of the confined side of the transition gives, in $(2 + 1)$ dimensions [4]:

$$\Gamma(r, T) = \text{const} \cdot r^{-1/2} e^{-\sigma(T)r/T} \quad (9)$$

from which one infers $\xi(T) = T/\sigma(T)$. Here $\sigma(T)$ is the finite temperature effective string tension between sources in the fundamental representation. It vanishes at the critical point..

Similarly, in the deconfined phase $T > T_c$, the connected correlation function is expected to behave as [4]

$$\Gamma(r, T) \sim |\langle Tr_F W(x) \rangle|^2 \cdot \frac{e^{-\mu r}}{r^{1/2}} \quad (10)$$

where μ^{-1} is the Debye screening radius, evidently identifiable as the correlation length in this phase. Note that the behavior as a function of r turns out to be the same above and below the transition.

3. The Internal Energy Density

From $(3 + 1)$ -dimensional studies the finite size corrections to the internal energy density far from the critical region in the high temperature phase are known to be substantial [14].

In our $(2 + 1)$ -dimensional case we have only a few modifications. First, what is the continuum version of the generalized Stefan-Boltzmann law in $(d + 1)$ dimensions? A simple way to get this is to use the method of Appendix A in ref. [15]. Write the partition function of a non-interacting bose gas as

$$Z = \text{Tr}\{e^{-H/T}\} \quad (11)$$

or, explicitly [15]:

$$Z = \exp \left[-V \int \frac{d^d k}{(2\pi)^d} \ln(1 - \exp[-\sqrt{k^2 + m^2}/T]) \right] \quad (12)$$

which in the massless case yields

$$Z = \exp [AVT^d] \quad (13)$$

with

$$A \equiv \frac{2}{d\Gamma(d/2)(4\pi)^{d/2}} \Gamma(d+1)\zeta(d+1) \quad (14)$$

Next, the definition of the internal energy density,

$$\epsilon \equiv -\frac{1}{V} \left. \frac{\partial \ln Z}{\partial (1/T)} \right|_V = \frac{T^2}{V} Z^{-1} \left. \frac{\partial Z}{\partial T} \right|_V, \quad (15)$$

gives

$$\epsilon = \frac{2T^{d+1}}{\Gamma(d/2)(4\pi)^{d/2}}\Gamma(d+1)\zeta(d+1) \quad (16)$$

For $d = 3$ this reduces to the well-known Stefan-Boltzmann law $\epsilon_{d=3} = \pi^2 T^4/30$. For the case $d = 2$ we have

$$\epsilon_{d=2} = \frac{\zeta(3)}{\pi} T^3 \quad (17)$$

This is for one degree of freedom per particle. For a (weakly interacting) $SU(N)$ gauge theory in $(d+1)$ dimensions the number of degrees of freedom per gluon equals $(d-1)(N^2-1)$, which should multiply the above result. Specifically, for $SU(3)$ in $(2+1)$ dimensions we find

$$\epsilon_{d=2}^{SU(3)} = \frac{8\zeta(3)}{\pi} T^3 \quad (18)$$

Except where we obviously refer to eq.(17), we shall from now on, and in particular on all plots with Monte Carlo results, denote this by ϵ_{SB} .

In order to determine the finite-size lattice corrections to the above, we can borrow directly from the corresponding calculation in $d = 3$ dimensions [14], making the minor changes from $d = 3$ to $d = 2$. Using periodic boundary conditions, the explicit formula for ϵ then becomes in our case:

$$\epsilon = -\frac{1}{N_\sigma^2 N_\tau} \sum_j' \frac{\sin^2(\pi j_0/N_\tau)}{b(j)^2 + \sin^2(\pi j_0/N_\tau)} + \frac{1}{N_\sigma^2} \sum_j \frac{1}{b(j)\sqrt{1+b(j)^2} + 1 + b(j)^2} \quad (19)$$

where

$$b(j)^2 \equiv \sum_{\mu=1}^2 \sin^2(\pi j_\mu/N_\sigma) \quad (20)$$

and the index j is summed over

$$\begin{aligned} j_0 &= 0, \pm 1, \dots, \pm(\frac{1}{2}N_\tau - 1), \frac{1}{2}N_\tau \\ j_\mu &= 0, \pm 1, \dots, \pm(\frac{1}{2}N_\sigma - 1), \frac{1}{2}N_\sigma, \quad \mu = 1, 2 \end{aligned} \quad (21)$$

We have here restricted ourselves to isotropic lattices, and set the lattice spacing $a = 1$. The prime on the summation symbol in eq. (19) indicates that the zero mode corresponding to all $j = 0$ is not to be included (the effect of this approximation is a small correction, vanishing in the limit of large volumes). In fig. 1 we have plotted what is probably the most useful way of representing the finite size corrections, ϵ/ϵ_{SB} as a function of N_{tau} , for various values of N_σ . The resulting finite lattice corrections are very similar to those in $(3+1)$ dimensions [14], but of course differ in detail.

As can be seen from fig. 1, the finite size corrections to ϵ are almost completely independent of N_σ for $N_\tau = 2$, the case at hand. We compare our Monte Carlo measurements for $SU(3)$ with the $T \gg T_c$ prediction above in fig. 2, here for definiteness choosing $N_\sigma = 30$.

4. Numerical Results

We now turn to a more systematic presentation of our numerical results. We have used the standard $SU(3)$ lattice action⁶⁾

$$S = \beta \sum_P (1 - \frac{1}{3} \text{ReTr} U_P) \quad (22)$$

where the sum runs over all fundamental plaquettes on the (periodic) lattice of size $N_\sigma^2 \times N_\tau$. For the Monte Carlo procedure we have used the Cabibbo-Marinari heat bath algorithm [16]. A major part of the runs were performed on an Amdahl VP1100 vector processor. For reasons of computer time economy, we have restricted ourselves to lattices with $N_\tau = 2$. As far as the test of universality is concerned, it is as non-trivial on lattices with $N_\tau = 2$ as on lattices with larger N_τ . We have therefore opted for rather large *spatial* volumes instead. In total we are going to probe $N_\sigma = 20, 30, 40$ and 60 .

It is natural to begin with the magnetization(s). Consider first the trace of the Polyakov loop in the *fundamental* representation. Since this character of $SU(3)$ is *complex*, we have some ambiguity in how this precisely should be defined. Physically, it is clear that the magnetization should be defined as the *real part* of the above trace. Indeed, if one were to couple the $SU(3)$ gauge theory to an external “magnetic field”, it would be of the form

$$\frac{1}{2}h \sum_x \left\{ Tr_F W(x) + Tr_F W(x)^\dagger \right\} = h \sum_x Re Tr_F W(x) \quad (23)$$

This also explains why the expectation value of the Polyakov line,

$$e^{-(F_q - F_0)/T} = \frac{1}{3} \langle Tr_F W(x) \rangle \quad (24)$$

(where F_q is the free energy of a single heavy quark source, and F_0 is the free energy of the vacuum), could appear to yield a complex free energy whenever the $Z(3)$ symmetry would be broken into one of the complex directions. The spontaneous breaking of the $Z(3)$ symmetry should always be achieved through the addition of the (real) “magnetic” term of eq. (23) above, which then smoothly should be sent to zero. This guarantees that the Polyakov loop expectation value always has zero phase.

In practice we cannot go through the above procedure of first introducing a magnetic field term in the action, and then slowly sending it to zero. (This obviously would be extremely costly in terms of CPU time, much exceeding our limitations). Fortunately there is no need to either, once the problem has been recognized and understood. We will in any case always be working with a finite spatial volume, which cannot be tuned to yield a true thermodynamic limit. In the case of $SU(2)$, the most common way of treating this problem is to work instead with the *absolute value* $\langle |Tr_F W| \rangle$. In the infinite volume limit this will clearly coincide with the usual definition. On a finite spatial volume the absolute value has the disadvantage of being always non-zero, in both phases. But in practice it has been found that the non-zero tail in the confined phase is of little significance in determining the critical behavior. The expectation value of the absolute value above is defined by the expectation value of the absolute value of a single lattice average, *i.e.* $|Tr_F W| \equiv |\frac{1}{N_\sigma^2} \sum_{x_i} Tr_F W(x_i)|$. In the case of $SU(3)$ we have even more freedom. We can choose to measure the *modulus* of $Tr_F W(x)$, or we can form the third root of the averaged quantity $Tr_F W(x)^3$, which in the case of a pure $Z(3)$ phase always maps onto the positive real axis. In practice we have found that these two definitions of the order parameter differ only at a completely insignificant level of accuracy, - a comforting observation.

The averages are defined over single lattice averages, as explained above.

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6) We use the same symbol β for both the magnetization exponent and the lattice gauge theory coupling. For a given lattice size the “critical” coupling will be denoted by β_c . No confusion of the two different meanings of β should be possible.

The slight ambiguity in defining the appropriate order parameter of course propagates itself to the definition of what we mean by the two-point correlation function. Although eq. (7) perhaps is the most natural definition, we could equally well have chosen $\tilde{\Gamma}(r, T) = \langle \text{ReTr}_F W(r) \text{ReTr}_F W(0) \rangle - \langle \text{ReTr}_F W \rangle^2$. The correlation length extracted from these two definitions should be identical. Similarly we could have defined a susceptibility $\tilde{\chi}$ by $\tilde{\chi} = \sum_{x_i} \tilde{\Gamma}(r, T)$. In fact, it is this choice which corresponds to the definition of the susceptibility as $\partial M / \partial h|_{h=0}$, where by M we mean $\langle \text{ReTr}_F W \rangle$, and where the magnetic field h is introduced as above. Again, the critical behavior of χ and $\tilde{\chi}$ should be identical.

4.1 POLYAKOV LOOPS: MAGNETIZATION EXPONENTS

In fig. 3 we show a typical set of results for the magnetizations of all the irreducible representations we have included, here chosen for $N_\sigma = 60$. The results look very similar for all other spatial volumes we have considered. What appears to be a smooth phase transition occurs around $\beta \simeq 8.1$ (just estimated by eye; a more accurate determination follows later). As in the case of earlier studies of different theories, all irreducible representations appear to deconfine at exactly the same critical coupling β_c . At this level of accuracy we see no difference between representations transforming trivially or non-trivially under $Z(3)$.

Shown in table 1 is a compilation of least squares fits to the functional form $\langle \text{Tr}_n W \rangle = f(y) = A(y - \beta_c)^\beta$, where both the overall normalization A , the finite volume critical coupling β_c , and the magnetization exponent β are determined. The upper cutoff in the range of couplings y is introduced by hand. Our upper cutoffs in the couplings are 8.20, for all lattices except $N_\sigma = 60$, for which the upper cutoff was 8.22. The fits are weighted by the errors in the individual measurements. The uncertainties in the fits could then be determined in the following way. Values for A, β_c and the exponent β were chosen such that $\chi^2_1 = \sum_y \left(\frac{L(y) - f(y)}{\epsilon(y)} \right)^2$ is at a minimum. Here $\epsilon(y)$ denotes the error of each individual measurement $L(y)$. We then assume that χ^2_1 follows a χ^2 -distribution with $n - r$ degrees of freedom, where n is the number of data for the given sample, and r is the number of parameters in the fit (in this particular case 3). Errors indicating a 95% confidence level can then be estimated from $\chi^2_1 = \chi^2_{95\%}(n - r)$. All errors listed are of this kind. It sometimes occurs that $\chi^2_1 > \chi^2_{95\%}(n - r)$, in which case the data are simply too poor to allow for a 95% confidence level fit. No errors are indicated in this case.

It will be noticed that at any given finite volume, we find (slightly) different critical couplings β_c for different representation Polyakov loops. This is simply because we decided to let the least squares fit determine a critical coupling independently for each representation, thus giving us an extra consistency check on the results obtained for the fundamental representation. This can be used to get yet another idea of the similar uncertainties in the critical coupling. The smallest volume considered, $N_\sigma = 20$, has been treated in two different ways. The first, indicated in table 1 by A, is a true 3-parameter fit of the form discussed above. This fit was difficult because of few data points in the relevant critical region (and thereby greater sensitivity to any single point), and it yielded a critical coupling β_c quite far from, for example, the peak in the susceptibility on the same lattice size. Although we include these “blind fits” in the table, we believe one should not attach too much importance to the numbers they give (and which are consistently deviating from the other lattice sizes). To get an idea of the uncertainties involved, we have also included one other fit (B) on the same data, now determining β_c by extrapolation from the

larger size lattices, and performing a 2-parameter fit to get the overall normalization and the magnetization exponent. These turn out to be in far greater agreement with results obtained on the other lattice sizes.

To give a more visual idea of how good the critical scaling fits were for the fundamental representation, we show in fig. 4 the collection of results on a logarithmic scale in which power law scaling should appear as straight lines. The lines shown are the fits to data from table 1. Finally, just to make it completely clear that on these lattices, and with our range of statistics, one gets very similar straight line fits also for the higher representation Polyakov loops, we show in fig. 5 an example from our data on the $N_\sigma = 60$ lattice. The straight lines are again the fits listed in tabel 1.

Indicated in table 1 are also numerical fits for the correction-to-scaling exponent θ already discussed in section 2. These numbers were *not* obtained by true 5-parameter fits (as would be required for the function (5)), but rather by *keeping fixed all 3 parameters determined in the procedure explained above*, and then fitting the function (5) on a larger coupling constant interval. The upper cutoffs used were: 8.40, 9.00, 9.00, 9.00, for lattices of size 20, 30, 40 and 60, respectively. In all cases the fits for this correction exponent were too poor to allow for a 95% confidence interval. So the inclusion of these results should not be taken very seriously, as is also obvious from the spread in the values obtained. There are, though, some indications that a correction exponent of between, roughly, 0.6 and 1.4 will emerge. This clearly does not even distinguish between a true correction exponent, and an analytic contribution, as discussed in section 2. We have also tried to fix the critical coupling and the normalization at their values obtained from the first fit, and then, in the case of the fundamental representation, used the exact 3-state Potts model exponent $1/9$ to find the correction exponent. Not surprisingly (since the leading exponent came out quite close to the Potts model exponent in the first fit), this yielded numbers very close to those shown in the table.

4.2 THE STRING TENSION

Having thus determined quite accurately at what coupling β_c the deconfinement transition occurs on a lattice with $N_\tau = 2$, it is amusing that we can turn this into a prediction for the string tension in the *zero temperature* $SU(3)$ lattice gauge theory in the same number of dimensions. The only essential ingredient is the assumption of an effective string-like description of the confined phase in terms of, say, the Nambu-Goto string action. Then one can solve explicitly for the deconfinement phase transition temperature T_c (the point at which the effective, temperature dependent, string tension $\sigma(T)$ vanishes) in any number of spatial dimensions d , with the result [17]⁷⁾

$$T_c = \sqrt{\frac{3\sigma(0)}{\pi(d-1)}} \quad (25)$$

As far as just the magnitude of T_c is concerned, this has been shown to be a very accurate prediction in all cases where the transition is of 2nd order [19]. (A 2nd order phase transition is universally predicted from the string model). From our present study we do not know T_c , but only β_c . However assuming 3-dimensional scaling we must have $\sigma(0) = C/a^2\beta^2$, where C is some numerical constant, and where we have reintroduced the lattice

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7) To leading order in $1/d$ the same result holds in a much larger class of bosonic string models [18].

spacing a . Also, the temperature on our $N_\tau = 2$ lattice is given by $T = 1/N_\tau a$. Finally, from eq. (25) above we get $T_c = \sqrt{3\sigma(0)/\pi}$. With $\beta_c \simeq 8.2$ this gives $C \simeq \pi\beta_c^2/12 \simeq 17.6$. In other words, from our finite temperature measurements we predict

$$\sigma(0) \simeq 17.6/\beta^2 \quad (26)$$

in units where $a = 1$.

4.3 THE SUSCEPTIBILITY AND THE CORRELATION LENGTH

With our present limited statistics we cannot expect to be able to investigate the detailed critical behavior of either the susceptibility or the correlation length to any reasonable accuracy. (Both are extracted from the two-point correlation function, which, being completely non-local, obviously requires very high statistics). Still, since we have the data available, we can see what they give. In fig. 6 we show as an example the susceptibility on the $N_\sigma = 60$ lattice. It is obvious that much more statistics should be accumulated before any serious fits can be made. If we blindly make a 2-parameter fit on all our data (assuming the β_c values found above), we get $\gamma \simeq 0.6 - 0.8$ for all lattice sizes. This is substantially smaller than the expected Potts model exponent (4), but it is clear that the rounded peak in the data have pulled the value down. If we had more data available, we should extract a small region both around β_c , and the upper cutoff. Only then can we hope to get more realistic fits.

Apart from the question of statistics, we have found that the formation of isolated $Z(3)$ domains during the Monte Carlo simulation is the single most difficult problem which has prevented us from obtaining good results for the susceptibility and the correlation function. (The two problems are of course related; If we wait long enough, all effects of intermediate domains will have averaged out). Although the correlation function (7) is a real quantity on any translationally invariant lattice (of course in steps of the lattice spacing), this is no longer the case if long-lived domains of different $Z(3)$ phase form within the volume. In practice this fortunately affects only results very close to and below β_c , - in the broken phase such domains only form with very small probability.

For the correlation length we have extracted from the formulas (9) and (10). We have also made some simple-minded 2-parameter fits in which we have used the β_c -values obtained from subsection 4.1, and then found the overall normalization and critical exponent. This gives an exponent ν about a factor of two different from the Potts model values, but with undetermined errors, and of course obtained in a completely unrealistic fit.

4.4 THE INTERNAL ENERGY DENSITY: FINITE SIZE SCALING

We have already in section 3 presented some data for the internal energy density ϵ in the case of $N_\sigma = 40$, there with an emphasis on its approach to the finite volume lattice Stefan-Boltzmann law at very high temperatures. What about its critical behavior? Near the phase transition we should expect a power law for the *singular* part of the energy density:

$$\epsilon_{sing} \sim |T - T_c|^{1-\alpha} = |T - T_c|^{2/3} \quad (27)$$

where we have substituted the expected value $\alpha = 1/3$ from the two-dimensional 3-state Potts model. However, ϵ is known to contain also large *regular* terms [20, 4], which can totally mask the singular behavior, even very close to T_c . We shall now show how finite

size scaling can be used to eliminate the regular term, and simultaneously test the power law behavior of the singular term.

First, we have in eq. (15) given the definition of the internal energy density. Blindly using this definition on our lattice action (22) gives

$$\epsilon = \beta(\langle P_\sigma \rangle - 2\langle P_\tau \rangle) \quad (28)$$

where $P_{\sigma,\tau}$ are spatial and timelike “plaquettes” defined by $P_{\sigma,\tau} = 1 - 1/3 \text{ReTr} U_{\sigma,\tau}$. But from the above should be subtracted the (same volume) zero temperature result

$$\epsilon_0 = -\beta\langle P \rangle \quad (29)$$

giving in total for the *physical* energy density, in the limit in which we approximate the symmetric plaquette expectation value by the spatial expectation value on our asymmetric lattice:

$$\epsilon/T^3 = 2N_\tau^3\beta(\langle P_\sigma \rangle - \langle P_\tau \rangle) \quad (30)$$

Here we have already anticipated the T^3 -behavior, and divided this factor out.

Now, from fig. 1 we see that in the high temperature phase, far from the critical region, there is essentially no finite size dependence for the internal energy density on a lattice with $N_\tau = 2$. It is tempting to see if this property holds for the *regular* part of ϵ all the way into the critical regime. (In ref. [4] this was indeed found to be the case for $SU(2)$). With this assumption, which we will justify *a posteriori*, we can extract the critical behavior of ϵ quite accurately. First introduce the reduced “temperature” variable

$$x \equiv \frac{\beta - \beta_c}{\beta_c} \quad (31)$$

Then critical scaling is expressed as

$$\epsilon_{sing} \sim x^{1-\alpha}, \quad (32)$$

and *finite size scaling* should be of the form [21]

$$\epsilon_{sing} = N_\sigma^{(\alpha-1)/\nu} Q_\epsilon(x N_\sigma^{1/\nu}) \quad (33)$$

where Q_ϵ is a *universal scaling function* for the internal energy. With the above remarks in mind, we now replace the full free energy by

$$\epsilon(N_\sigma) = \epsilon_{reg} + N_\sigma^{(\alpha-1)/\nu} Q_\epsilon(x N_\sigma^{1/\nu}) \quad (34)$$

close to the infinite volume critical point β_c ($x = 0$). Having data available from different lattice sizes, we can now eliminate the unwanted regular contribution. First we Taylor expand the quantity $\epsilon N_\sigma^{(1-\alpha)/\nu}$ as a function of $x N_\sigma^{1/\nu}$ close to $x = 0$:

$$\epsilon N_\sigma^{(1-\alpha)/\nu} = \epsilon_{reg} N_\sigma^{(1-\alpha)/\nu} + A_\epsilon + B_\epsilon x N_\sigma^{1/\nu} + \dots \quad (35)$$

which allows us to compute $\epsilon_{reg}(N_\sigma^{(1-\alpha)/\nu} - \tilde{N}_\sigma^{(1-\alpha)/\nu})$ for two different lattice sizes N_σ and $\tilde{N}_\sigma$ at $x = 0$. These differences already tell us whether the above assumption of almost N_σ independent regular contributions is justifiable. Taking the average value from different lattice differences, we can now subtract this term from all data, and plot only the remaining *singular* contribution to the finite size scaling function.

Note that β_c in the above formalism refers to the *infinite volume critical coupling*. We have no accurate information available from e.g. the fourth order cumulant to determine this number. However we feel that the error introduced by replacing the infinite volume critical coupling with the β_c we have found from our largest lattice size, $N_\sigma = 60$ will be below the uncertainties already present in the raw data. We thus use $\beta_c = 8.136$.

Shown in fig. 8 are the values of ϵ_{sing} extracted from the full internal energy ϵ in the manner described above, and plotted on graph giving $(\epsilon_{sing}/\epsilon_{Boltzmann})N_\sigma^{(1-\alpha)/\nu}$ versus $xN_\sigma^{1/\nu}$. It follows from eq. (33) that this should describe a single unique finite size scaling function $Q_\epsilon(xN_\sigma^{1/\nu})$. Our data collected in this way in fig. 8 is made with the use of 3-state Potts model critical exponents. The results are clearly in very good agreement with this choice of exponents.

5. Conclusion

This is the first detailed study of the critical behavior of a clearly continuous deconfinement phase transition arising from a gauge group as involved (in terms of Monte Carlo simulations) as $SU(3)$. We have been faced both with the problem of needing almost one order of magnitude more CPU time to acquire the same statistics as for $SU(2)$, and the problem of the symmetry breaking pattern in $SU(3)$ being more complicated than in $SU(2)$. The first difficulty has forced us to focus almost exclusively on local observables such as the magnetization and the internal energy density. The second problem, which is simply a consequence of a $Z(3)$ spin theory having three broken vacua instead just two for $Z(2)$, has made the question of precise determinations of, for example, correlation functions even more difficult. In the broken phase, and close to the transition, domains of different complex $Z(3)$ phase can now easily form within the lattice. The geometry of our lattice (that of a very large thin slab) has made this even more likely to happen during our simulations. There are certainly methods to overcome this problem (such as the “cold wall” technique [22]), and possible future studies should consider such alternatives.

From the point of view of a test of universality at deconfinement phase transitions, what have we learned? First of all, we have now quite substantial evidence that the $(2+1)$ -dimensional finite- T phase transition of $SU(3)$ lattice gauge theory is indeed continuous, and presumably of 2nd order. A universality test is then indeed possible. When comparing our measured values for the magnetization exponent β of table 1 with the expected 3-state Potts model exponent $\beta = 1/9$, the agreement is quite remarkable. Although there is obviously no way to distinguish, say, between 2-d Ising behavior ($\beta = 1/8$) and 3-state Potts model behavior, just the fact that the numbers turn out to be so close to expectations is in itself highly non-trivial. As in the case of $SU(2)$, the rôle of higher representation sources is still not well understood. We see equally convincing “critical behavior” for also the 8 and 10-dimensional representations of $SU(3)$, despite the fact that these transform trivially under the center group $Z(3)$. This is a sobering observation, which unavoidably casts a certain shadow on also on the measurement of the magnetization exponent for the fundamental representation. Do we really need to go to immensely large lattices to see deviations from almost-critical scaling for the trivially transforming representations? Do those representations that transform non-trivially then show the same scaling behavior as the fundamental representation? Or can we just try to move much closer to the finite volume β_c in order to find the expected behavior? These are just some of the questions that remain to be answered.

The non-local objects such as the susceptibility χ and the two-point correlation

function $\Gamma(r, T)$ can only with great difficulty be studied with our available statistics. These objects show the expected peak structure close to β_c , but it is difficult to attach much significance to the least squares fits we have been able to perform for the critical exponents. Also here is definitely room for future study.

The strength of finite size scaling analyses becomes apparent when we consider a quantity such as the internal energy density ϵ . Although ϵ contains a “singular” (but of course non-divergent) piece ϵ_{sing} , it is on any given lattice size totally dominated by the regular (analytic) term ϵ_{reg} . When results from several different lattice sizes are combined, we can in one step eliminate the unwanted regular term, and test for finite size critical scaling of the remaining singular quantity. These results, described in section 4.4, beautifully confirm the expected scaling behavior, despite the fact that our input for the infinite volume critical coupling constant β_c to a first approximation was taken to equal that of the $N_\sigma = 60$ lattice. With higher statistics, a more accurate determination of the infinite volume critical coupling constant, and perhaps also with the inclusion of a few more lattice sizes, this method can be turned into a very powerful tool also for the determination of other critical exponents.

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Figure Captions

Fig. 1. The finite lattice effects on the internal energy density ϵ of a free Bose gas. Here ϵ_{SB} denotes the $(2 + 1)$ -dimensional analogue of the infinite volume Stefan-Boltzmann law in the continuum. Note that all curves happen to converge at $N_\tau = 2$, the point of interest for the present study.

Fig. 2. An example of how well the internal energy density of $(2 + 1)$ -dimensional $SU(3)$ lattice gauge theory compares with that of a free Bose gas with the same number of degrees of freedom, and on the same volume. Below the phase transition only massive glueball states can be excited. These evidently contribute insignificantly to the internal energy density.

Fig. 3. The four irreducible representations of the Polyakov loop that we consider in the present work. Here data are shown for $N_\sigma = 60$. Note that superficially all of these irreducible representations display the same behavior, apparently becoming “deconfined” at the same critical temperature. See the main text for a discussion of this point.

Fig. 4. A compilation of data close to β_c for $N_\sigma = 20, 30, 40$ and 60 . The straight lines shown are best fits, corresponding to table 1.

Fig. 5. Higher representation Polyakov loops, here as an example shown for $N_\sigma = 60$. Note that to this level of accuracy these representations also seem to display critical scaling. Compare again with table 1., and see also the main text.

Fig. 6. An example of the susceptibility χ for one of the lattices, $N_\sigma = 60$. A peak is clearly indicated around the β_c -value extracted from fits to the Polyakov loops. The shown curve is a 2-parameter fit to the data, giving an exponent $\gamma \simeq 0.7$, a result which clearly cannot be taken seriously on account of the poor data.

Fig. 7. The correlation length extracted on the $N_\sigma = 60$ lattice for $\beta > \beta_c$. A simple 2-parameter fit gives $\nu \simeq 0.4$, which, as in the case of the susceptibility (fig. 6) cannot be taken very seriously. Far more statistics is needed to extract both of these exponents.

Fig. 8. Finite size scaling analysis of the singular part of the internal energy density ϵ_{sing} , using the method described in the main text. Shown is the emergence of a universal scaling function $Q_\epsilon(xN^{1/\nu})$, if we substitute 3-state Potts model exponents. Of course only the region in the immediate vicinity of $x = 0$ should be considered.

Tables

Polyakov loop representation	Lattice size N_σ	Critical coupling β_c	Magnetization exponent β	Correction exponent θ
3	20 A	$8.002^{+0.017}_{-0.021}$	$0.2296^{+0.0179}_{-0.0179}$	1.35
3	20 B	8.054	$0.1083^{+0.0083}_{-0.0106}$	1.32
3	30	$8.105^{+0.005}_{-0.012}$	$0.1096^{+0.0097}_{-0.0096}$	0.78
3	40	8.118	0.1255	0.75
3	60	$8.137^{+0.003}_{-0.004}$	$0.1221^{+0.004}_{-0.004}$	1.00
6	20 A	$7.963^{+0.011}_{-0.012}$	$0.4809^{+0.0254}_{-0.0241}$	1.82
6	20 B	8.050	$0.1697^{+0.0175}_{-0.0167}$	1.70
6	30	$8.102^{+0.008}_{-0.012}$	$0.1815^{+0.0152}_{-0.0145}$	0.97
6	40	8.116	0.2095	1.08
6	60	$8.132^{+0.003}_{-0.004}$	$0.1879^{+0.0047}_{-0.0048}$	1.11
8	20 A	$7.955^{+0.011}_{-0.012}$	$0.4606^{+0.0236}_{-0.0222}$	1.91
8	20 B	8.052	$0.1477^{+0.0152}_{-0.0167}$	1.69
8	30	$8.098^{+0.018}_{-0.011}$	$0.1738^{+0.0113}_{-0.0128}$	0.97
8	40	8.113	0.1874	1.11
8	60	$8.126^{+0.004}_{-0.004}$	$0.1838^{+0.0047}_{-0.0047}$	1.13
10	20 A	$7.891^{+0.012}_{-0.013}$	$0.5972^{+0.0252}_{-0.0243}$	3.10
10	20 B	8.008	$0.1976^{+0.0122}_{-0.0121}$	2.32
10	30	$8.080^{+0.009}_{-0.011}$	$0.2941^{+0.0193}_{-0.0183}$	1.22
10	40	8.108	0.2828	1.34
10	60	$8.124^{+0.004}_{-0.005}$	$0.2663^{+0.0082}_{-0.0081}$	1.33

Table 1 A compilation of least squares fits for the critical exponent β , here defined for any of the irreducible representations shown (despite the fact that neither the 8-representation nor the 10-representation should show true critical behavior). For the fundamental representation results are in excellent agreement with the 3-state Potts model value $\beta = 1/9 = 0.1111\dots$. Two fits are shown for the $N_\sigma = 20$ lattice. One (labelled by A) is a blind 3-parameter fit as described in the main text. Unfortunately not very many points were available to give the shown fit. The other (B) is made as a 2-parameter fit, where the critical coupling β_c was obtained by extrapolation from the $N_\sigma = 30, 40$ and 60 data. See the main text for an interpretation of these results.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8