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W-symmetries of a class of $d=2$ $N=1$ supersymmetric sigma models

P.S. Howe,
Dept. of Mathematics,
King's College, London

and

G. Papadopoulos*,
CERN, Geneva

Abstract

Two-dimensional $N = 1$ supersymmetric sigma models with certain target spaces admit additional symmetries at the classical level. The target spaces are non-symmetric spaces with irreducible holonomy groups. It is shown that the algebraic structure of these symmetries is of W-type.

* Current address: Dept. of Physics, Queen Mary College, London

In a recent note the symmetries of two-dimensional $N = 1$ supersymmetric sigma models with a certain type of target space were discussed [1]. The target spaces were required to be non-symmetric (i.e. $\nabla R \neq 0$), and to have irreducible holonomy. The list of possible irreducible holonomy groups is: $SO(n)$, $U(m)$ ($n = 2m$), $SU(m)$, $Sp(k)$ ($n = 4k$), $Sp(k)Sp(1)$, $Spin(7)$ ($n = 8$) and G_2 ($n = 7$), where n is the dimension of the target space M [2]. Each group on this list is associated with one or more covariantly constant forms on M , and these forms can be used to define symmetries of the corresponding $N = 1$ supersymmetric sigma models [1].* In the case of $U(m)$ the additional symmetry is a second supersymmetry, while in the case of $Sp(k)$ there are three additional symmetries which are all supersymmetries. Thus $N = 2$ and $N = 4$ sigma models [4,5] are special cases of sigma models on target spaces with irreducible holonomy. In this note the algebraic properties of the symmetries associated with irreducible holonomy are discussed. Each sigma model is associated with a set of derivations of the algebra of differential forms on M and the symmetry transformations constructed using these derivations close in a field-dependent sense modulo the equations of motion; in other words, the symmetries of sigma models with irreducible holonomy are of W-type.

The $N = 1$ sigma model action (without torsion) is

$$S = \int d^2y d^2\theta g_{ij}(X) D_+ X^i D_- X^j \quad (1)$$

where $(y^\pm, \theta^\pm)$ are the coordinates of flat $d = 2, N = 1$ superspace, Σ , D_+, D_- are the usual covariant derivatives with $D_+^2 = i\partial_+$, $D_-^2 = i\partial_-$, $\{D_+, D_-\} = 0$, and $X^i(y, \theta)$, $i = 1 \dots n$, is the sigma model field in local coordinates x^i on M . A vector-valued l -form on M , $L = L_{l_1 \dots l_l}^i dx^{l_1} \wedge \dots \wedge dx^{l_l} \otimes \frac{\partial}{\partial x^i} := L_L^i dx^L \otimes \frac{\partial}{\partial x^i}$, defines a derivation ι_L of Ω , the algebra of differential forms on M which is a generalisation of the interior product of a vector with a form. If ω is a p -form, this derivation is given by

$$\omega \mapsto \iota_L \omega = p \omega_{i j_1 \dots j_{p-1}} L_L^i dx^L \wedge dx^{j_1 \dots j_{p-1}} \quad (2)$$

The derivation ι_L vanishes on functions and is determined by its action on one-forms. For the basis forms dx^i one has

$$\iota_L dx^i = L^i := L_L^i dx^L \quad (3)$$

A transformation of the sigma model field X can be defined by essentially replacing dx^i by $D_+ X^i$ in (3),

$$\delta_L X^i = a_l L_{l_1 \dots l_l}^i D_+ X^{l_1} \dots D_+ X^{l_l} := a_l L_L^i D_+ X^L \quad (4)$$

where the parameter a_{-l} has Lorentz weight $\frac{-l}{2}$. The transformation (4) is a symmetry of (1) provided that $L_{ij_1 \dots j_l}$ is totally antisymmetric and covariantly constant, and the parameter a_{-l} is semi-local, i.e. $D_- a_{-l} = 0$. We shall denote the $(l+1)$ -form associated with L by ω_L . (For each L we can also define a right transformation with D_+ replaced by

* Similar symmetries also occur in $N = 2$ models with almost complex target spaces [3].

D_- ; left and right transformations commute up to the field equations). The unit tensor 1 also defines a derivation of Ω , but the symmetry associated with it ($N = 1$ supersymmetry) has a slightly different form to (4),

$$\delta_1 X^i = -\frac{i}{2}(D_+ a_+) D_+ X^i + a_+ \partial_{\mp} X^i \quad (5)$$

where the parameter a_+ also satisfies $D_- a_+ = 0$. Corresponding to each symmetry of the form (4) there is a conserved current J_L which is essentially the pull-back of ω_L onto superspace,

$$J_L = L_{i_1 \dots i_{l+1}} D_+ X^{i_1} \dots D_+ X^{i_{l+1}} \quad (6)$$

The current corresponding to (5) is the supercurrent T ,

$$T = g_{ij} D_+ X^i \partial_{\mp} X^j \quad (7)$$

Each of these currents is annihilated by D_- when the equations of motion are satisfied.

The commutator of two transformations of the type (4) is given by

$$[\delta_L, \delta_M] X^i = \delta_{LM}^{(1)} X^i + \delta_{LM}^{(2)} X^i \quad (8)$$

where

$$\delta_{LM}^{(1)} X^i = a_{-m} D_+ a_{-l} (M \bar{\wedge} L)_{LM_2}^i D_+ X^L D_+ X^{M_2} - (L \leftrightarrow M) \quad (9)$$

and

$$\delta_{LM}^{(2)} X^i = i l m (-)^l (\mathcal{L}^i_j \wedge \mathcal{M}^j_k)_{L_2 M_2} D_+ X^{L_2} D_+ X^{M_2} - (L \leftrightarrow M) \quad (10)$$

and where $D_+ X^{L_2} := D_+ X^{l_2} \dots D_+ X^{l_l}$. In (9) we have used the notation

$$L \bar{\wedge} M := l L_{j L_2}^i M_M^j dx^M \wedge dx^{L_2} \quad (11)$$

and in (10) we have introduced the Lie $O(n)$ -valued $(l-1)$ -form $\mathcal{L}^i_j := L_{j L_2}^i dx^{L_2}$. The commutator of a transformation of the form (4) with an $N = 1$ supersymmetry transformation (5) is

$$[\delta_1, \delta_L] X^i = \left(\frac{l}{2} a_{-l} \partial_{\mp} a_+ - a_+ \partial_{\mp} a_{-l} + \frac{i}{2} D_+ a_+ D_+ a_{-l} \right) L_L^i D_+ X^L \quad (12)$$

The main result of this paper is that the algebra (8) always closes in a field-dependent sense for the symmetries of sigma models with irreducible target spaces. For this to hold it is necessary to include $N = 1$ supersymmetry transformations. In this article we shall not give a general proof of this statement; rather, we shall focus on the Calabi-Yau case where the holonomy group is $SU(m)$. The forms characterising this holonomy group are ω_I , the Kähler form for the complex structure I , and a nowhere vanishing $(m, 0)$ -form λ with complex conjugate $\bar{\lambda}$. We set $\omega_L = \lambda + \bar{\lambda}$ and $\omega_{\bar{L}} = i(\lambda - \bar{\lambda})$. The vector-valued forms of the theory are thus 1, I , L and $\hat{L}$ with $l+1 = m = \frac{n}{2}$. The corresponding currents are

T, J_I, J_L and $J_{\hat{L}}$. Since the algebra generated by 1 and I is just the $N = 2$ supersymmetry algebra we need only compute the commutators which involve L and/or $\hat{L}$.

The result can be established using the following identities:

$$(L\bar{\wedge}L)^i = (\hat{L}\bar{\wedge}\hat{L})^i = -\frac{l!l!}{2^{l-1}} \begin{cases} I^i_j dx^j \wedge (\omega_I)^{l-1} & l \text{ even} \\ dx^i \wedge (\omega_I)^{l-1} & l \text{ odd} \end{cases} \quad (13)$$

$$(\hat{L}\bar{\wedge}L)^i = \frac{l!l!}{2^{l-1}} \begin{cases} dx^i \wedge (\omega_I)^{l-1} & l \text{ even} \\ -I^i_j dx^j \wedge (\omega_I)^{l-1} & l \text{ odd} \end{cases} \quad (14)$$

$$\mathcal{L}^i_j \wedge \mathcal{L}^j_k = -\frac{(l-1)!}{2^{l-1}} \delta^i_k (\omega_I)^{l-1} + \frac{(l-1)(l-1)!}{2^{l-1}} (g_{kl} I^i_m + \delta^i_l I_{km}) dx^l \wedge dx^m \wedge (\omega_I)^{l-2}, \quad l \text{ odd} \quad (15)$$

and

$$\hat{\mathcal{L}}^i_j \wedge \mathcal{L}^j_k = -\frac{(l-1)!}{2^{l-1}} \delta^i_k (\omega_I)^{l-1} - \frac{(l-1)(l-1)!}{2^{l-1}} (g_{kl} I^i_m + \delta^i_l I_{km}) dx^l \wedge dx^m \wedge (\omega_I)^{l-2}, \quad l \text{ even} \quad (16)$$

With the aid of these identities it is straightforward to compute the commutators in (8). The commutators of L and $\hat{L}$ with I are

$$[\delta_L, \delta_I] X^i = \delta_{\hat{L}} X^i \quad (17)$$

with parameter

$$\hat{a}_{-l} = -(a_{-l} D_+ a_{-1} + l a_{-1} D_+ a_{-l}) \quad (18)$$

and

$$[\delta_{\hat{L}}, \delta_I] = \delta_L X^i \quad (19)$$

with parameter

$$a_{-l} = \hat{a}_{-l} D_+ a_{-1} + l a_{-1} D_+ \hat{a}_{-l} \quad (20)$$

For l even the commutator of an L transformation with another is

$$[\delta_L, \delta_L] X^i = \delta_I X^i \quad (21)$$

with parameter

$$a_{-1} = -\frac{l!l!}{2^{l-1}} (a'_{-l} D_+ a_{-l} - a_{-l} D_+ a'_{-l}) (J_I)^{l-1} \quad (22)$$

while for l odd one finds

$$[\delta_L, \delta_L] X^i = \delta_1 X^i + \delta_I X^i \quad (23)$$

where the parameters are

$$a_{\pm} = i \frac{l!l!}{2^{l-1}} a'_{-l} a_{-l} (J_I)^{l-1} \quad (24)$$

and

$$a_{-1} = -\frac{l(l-1)!l!}{2^{l-2}} a'_{-l} a_{-l} T(J_I)^{l-2} \quad (25)$$

The same results hold for the commutator of two $\hat{L}$ transformations with the parameters a_{-l} , a'_{-l} replaced by $\hat{a}_{-l}$ and $\hat{a}'_{-l}$. Finally, the commutator of an L -transformation with an $\hat{L}$ -transformation is, for l even,

$$[\delta_L, \delta_{\hat{L}}]X^i = \delta_1 X^i + \delta_I X^i \quad (26)$$

with parameters

$$a_{-} = i \frac{l!}{2^{l-2}} \hat{a}_{-l} a_{-l} (J_I)^{l-1} \quad (27)$$

and

$$a_{-1} = i \frac{l(l-1)!}{2^{l-2}} \hat{a}_{-l} a_{-l} T (J_I)^{l-2} \quad (28)$$

and for l odd

$$[\delta_L, \delta_{\hat{L}}]X^i = \delta_I X^i \quad (29)$$

with parameter

$$a_{-1} = -\frac{l!}{2^{l-1}} (\hat{a}_{-} D_{+} a_{-l} + a_{-l} D_{+} \hat{a}_{-l}) (J_I)^{l-1} \quad (30)$$

The field-dependent parameters in these equations satisfy $D_{-}(\text{parameter}) = 0$ when the field equations are satisfied since the currents are conserved, $D_{-}T = D_{-}J_L = D_{-}J_{\hat{L}} = 0$, on-shell.

The analysis for the other holonomy groups can be carried out in a similar fashion; the details will be given in [6]. Since all the models are $N = 1$ supersymmetric they all have a supercurrent T which has component spins $(3/2, 2)$. The spin content of the additional component currents for each model is given by:

$$\begin{aligned} SO(n) &: \left(\frac{n}{2}, \frac{n+1}{2}\right) \\ U(m) &: \left(1, \frac{3}{2}\right) \\ SU(m) &: \left(1, \frac{3}{2}\right) \text{ and } 2 \times \left(\frac{m}{2}, \frac{m+1}{2}\right) \\ Sp(k) &: 3 \times \left(1, \frac{3}{2}\right) \\ Sp(k)Sp(1) &: \left(2, \frac{5}{2}\right) \\ Spin(7) &: \left(2, \frac{5}{2}\right) \\ G_2 &: \left(\frac{3}{2}, 2\right) \text{ and } \left(2, \frac{5}{2}\right) \end{aligned} \quad (31)$$

The $U(m)$ and $Sp(k)$ cases correspond to $N = 2$ and $N = 4$ supersymmetry respectively. $N = 1$ superconformal algebras with one additional $(s, s + 1/2)$ current multiplet have been considered for $\frac{1}{2} \leq s \leq \frac{7}{2}$ in ref. [7]. The $Sp(k)Sp(1)$, $Spin(7)$ and $SO(n)$ ($n \leq 7$) cases are examples of (classical) realisations of these algebras. Algebras

with the spin contents of the remaining cases do not seem to have been considered previously in the literature, although some similar currents to (6) have been written down for free models in reference [8]. The situation for models with torsion seems to be more complicated and will be discussed elsewhere [6].

To conclude, we have shown that classical $N = 1$ $d = 2$ supersymmetric sigma models with irreducible non-symmetric target spaces admit additional symmetries which are of W-type. It does seem likely, however, that these symmetries could be anomalous at the quantum level. Indeed, for the $SU(m)$ example, if the additional symmetries were to be preserved quantum mechanically it would imply the finiteness of $N = 1$ Calabi-Yau sigma models [1]. It is well-known that there are divergences in such models starting at four loops [9], so there is presumably an anomaly in the “Calabi-Yau” symmetry at the three loop level. Quantum versions of these algebras can probably be constructed using appropriate numbers of free $N = 1$ scalar superfields.

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