ABSTRACT

We construct the q -superoscillator algebra of n bosonic and m fermionic q -oscillators covariant under the coactions of quantum supergroup $SU_q(n|m)$. The $SU_q(n|m)$ -invariant form bilinear in q -superoscillator variables describes q -deformation of the Hamiltonian for supersymmetric oscillator in quantum mechanics. It is shown that such a Hamiltonian is also invariant under the action of $U_q(U(n|m))$ superalgebra, generated by the bilinears constructed from the q -superoscillators. The simplest case $n = m = 1$ is considered in detail. The formulae relating the supercovariant q -oscillators with the independent ones are given.

SUPERCOVARIANT SYSTEMS OF
 q -OSCILLATORS AND q -SUPERSYMMETRIC
 HAMILTONIANS

M. CHAICHIAN¹⁾ and P. KULISH²⁾

CERN, Theory Division, 1211 Geneva 23, Switzerland

J. LUKIERSKI³⁾*

Département de Physique Théorique
 Université de Genève

24, quai Ernest-Ansermet, 1211 Genève 4, Switzerland

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¹⁾ On leave of absence from the Department of High Energy Physics, University of Helsinki, Sitavuorenpenger 20C, SF-00170 Helsinki, Finland

²⁾ On leave of absence from Steklov Mathematical Institute of USSR Acad. Sciences, Leningrad Branch, 191011 Leningrad, USSR

³⁾ On leave of absence from Institute for Theoretical Physics, University of Wrocław, ul. Cybulskiego 36, P-50205 Wrocław, Poland

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1. INTRODUCTION

Quantum deformations (the so-called q -deformations) of Lie groups and Lie algebras (cf. 1-3) have found many applications in diverse domains of both theoretical physics and mathematics (see e.g. [4-6]). These mathematical objects which appeared as a byproduct of quantum inverse scattering method [7-8] have found applications in integrable models [9,6], in the formulation of the examples of noncommutative geometry [10-13] and in the description of rational conformal field theories [14]. It should be added that the q -deformations have been subsequently extended to supergroups and superalgebras [12,15-17].

Independently different deformations of the Heisenberg algebra (bosonic creation and annihilation operators) had been already studied in seventies (see e.g. [18]). Recently the system of independent q -oscillators (bosonic as well fermionic) was used for the q -oscillator realizations of different quantum Lie algebras [19-21] and quantum superalgebras [16,12]. It is known that the set of n creation or annihilation operators forms a basis for $U(n)$ fundamental representation. Using the equivalence between the first order differential calculus and second quantization procedure the algebra of n q -oscillators covariant under the coaction of $SU_q(n)$ was given firstly in [23,24]. Interestingly enough, it appeared that $SU_q(n)$ -covariance requires a particular coupling of different modes, which determines also the noncommutativity factors in $SU_q(n)$ -covariant differential calculus [10-13].

The aim of this note is to construct the algebra of bosonic and fermionic q -oscillators covariant under the quantum supergroup $SU_q(n|m)$ (q -superoscillators). The formulae obtained here provide the construction of the differential geometry on non-commutative superspaces. After recalling the covariant formulation for $n = 2, m = 0$ (bosonic, non-symmetric case) we shall discuss in detail the case $n = m = 1$. We shall show that the scheme can be made covariant under the discrete group of transformations, representing the permutations of the q -superoscillator labels.

Further, the $SU_q(n|m)$ -invariant form defining the supersymmetric Hamiltonian for the q -superoscillators will be given. It appears that such a Hamiltonian also commutes with the quantum algebra $\mathcal{U}_q(U(n|m))$ with generators described by mixed bilinear products (creation, annihilation) of q -superoscillators. We shall see that the quantum algebra $\mathcal{U}_q(SU(1|1))(n = m = 1)$ can be treated as q -deformation of the $N = 2$ supersymmetry in quantum mechanics. Subsequently, the realizations of the supercovariant system of q -oscillators in terms of free q -oscillators (commuting different modes) as well as in terms of usual ones will be given.

It was pointed out recently by one of the authors that the q -deformed oscillators can be described by the R-matrix formalism (see e.g. [3]), following the description of scattering amplitudes of integrable models by Zamolodchikov-Fateev algebra (see e.g. [25]).

2. $SU_q(2)$ -COVARIANT PAIR OF q -OSCILLATORS.

As a start before considering the algebra of q -superoscillators let us consider the $n = 2, m = 0$ case. In this section we present in more explicit way the known results.

The quantum group $GL_q(2)$ is an associative Hopf algebra [1-3] generated by four elements satisfying the relations

$$T^{2,0} \in GL_q(2): \quad \begin{aligned} ab &= qba & bc &= cb \\ ac &= qca & bd &= qdb \\ cd &= qdc & ad - da &= rbc \end{aligned} \quad (2.1)$$

where $r = q - q^{-1}$. The algebra (2.1) is restricted to $SL_q(2)$ if the element of the center is put equal to one:

$$SL_q(2): \quad \det_q T^{2,0} \equiv ad - qbc = 1 \quad (2.2)$$

Further restrictions are imposed by the $*$ -involution:

$$U_q(2): \quad T^{2,0} \in GL_q(2); \quad c = -\frac{1}{q} b^*, \quad d = a^* \quad (2.3)$$

and subsequently

$$SU_q(2): \quad T^{2,0} \in U_q(2); \quad \det T^{2,0} = aa^* + bb^* = 1 \quad (2.4)$$

We introduce a pair of bosonic q -oscillators (A_1, A_2) transforming linearly under $GL_q(2)$, and satisfying the relations

$$A_1 A_2 = q A_2 A_1 = q A_1 A_2 \quad (2.5a)$$

$$A_1^\dagger A_2^\dagger = \frac{1}{q} A_2^\dagger A_1^\dagger \quad (2.5b)$$

Further

$$A_1 A_1^+ - q^2 A_1^+ A_1 = 1, \quad (2.6a)$$

$$A_2 A_2^+ - q^2 A_2^+ A_2 = 1 + (q^2 - 1) A_1^+ A_1 \quad (2.6b)$$

and $(i, j = 1, 2)$

$$A_i A_j^+ - q A_j^+ A_i = 0, \quad (i \neq j) \quad (2.7)$$

The coaction of $GL_q(2)$

$$\begin{aligned} A_1' &= a A_1 + b A_2, & A_2' &= c A_1 + d A_2, \\ A_1^{+'} &= a^* A_1^+ + b^* A_2^+, & A_2^{+'} &= c^* A_1^+ + d^* A_2^+, \end{aligned} \quad (2.8)$$

where the elements of quantum group $GL_q(2)$ commute with the q-oscillators (A_i, A_i^+) , leaves invariant (2.5a-b), preserves (2.7) if $T^{2,0} \in U_q(2)$, and leaves invariant (2.6a-b) if $T^{2,0} \in SU_q(2)$. Therefore, the algebra of q-oscillators (2.5-7) is $SU_q(2)$ -covariant.

The choice of the relations (2.6a-b) introduces preferred set of indices. If we replace the indices $i = 1 \leftrightarrow i = 2$, in order to obtain the $SU_q(2)$ -covariance we have to replace $q \rightarrow q^{-1}$ in the q-oscillator algebra. We obtain

$$A_1 A_2 - q A_2 A_1 = 0, \quad (2.9a)$$

$$A_1 A_1^+ - \frac{q}{q^2} A_1^+ A_1 = 1 + \left(\frac{1}{q^2} - 1 \right) A_2^+ A_2, \quad (2.9b)$$

$$A_2 A_2^+ - \frac{1}{q^2} A_2^+ A_2 = 1, \quad (2.9c)$$

$$A_i A_j^+ - \frac{1}{q} A_j^+ A_i = 0. \quad (2.9d)$$

The $SU_q(2)$ -invariant Hamiltonian is described by the following bilinear form:

$$H^{2,0} = \sum_{i=1}^2 A_i^+ A_i. \quad (2.10)$$

It should be added that :

i) The invariant form (2.10) describes after the symmetrization $A^+ A \rightarrow \frac{1}{2}(A^+ A + A A^+)$ the $SU_q(2)$ -covariant Laplace-Beltrami operator (see [23] for details):

$$\Delta^{2,0} = \frac{1}{2}(d\delta + \delta d) \quad (2.11)$$

where d is an exterior differential of degree one ($d^2 = 0$), and $\delta = d^+$ is its dual (lowering the degree of differential form by one).

ii) Following the construction of q -deformation of Lie algebra in terms of independent q-oscillators [19-21] one can also introduce a quantum Lie algebra $U_q(U(2))$ generated by the bilinear products $I_{ij}^{2,0} = A_i^+ A_j$ (see also [26]). It appears that the Hamiltonian (2.10) is $U_q(2)$ -invariant, i.e.

$$[H^{2,0}, I_{ij}^{2,0}] = 0 \quad (2.12)$$

It is easy to see that Hamiltonian (2.10) describes the central extension of the quantum algebra

$$U_q(SU(2)) \xrightarrow{H \neq 0} U_q(U(2)) \quad (2.13)$$

because the formula $U(2) = SU(2) \oplus U(1)$ cease to be valid if $q \neq 1$.

It should be mentioned that using the two-parameter R-matrix given in [27] defining the 2-parameter quantum group $GL_{q,\gamma}(2)$, one can introduce into the formulae (2.5) and (2.7) an additional complex deformation parameter γ .

3. $SU_q(1/1)$ -COVARIANT PAIR OF q-OSCILLATORS.

The quantum supergroup $GL_q(1|1)$ is a graded associative Hopf algebra with two even (a,d) and two odd (β, γ) generators satisfying the following algebraic relations (see e.g. [28])

$$\begin{aligned} \alpha\beta &= q\beta a & d\beta &= q\beta d \\ \alpha\gamma &= q\gamma a & d\gamma &= q\gamma d \\ \Gamma^{1,1} \in GL_q(1|1) : & \beta\gamma &= -\gamma\beta & ad - da &= -\tau\beta\gamma \\ & \beta^2 &= 0 & \gamma^2 &= 0 \quad (q \neq -1) \end{aligned} \quad (3.1)$$

Introducing q-deformed superdeterminant one defines

$$SL_q(1|1) : \quad s \det_q T^{1,1} \equiv ad^{-1} - \beta d^{-1} \gamma d^{-1} = 1 \quad (3.2)$$

Further the $\ast$ -involution restricts consistently the algebra (3.1) by imposing the relations

$$U_q(1|1) : \quad T^{1,1} \in GL_q(1|1) : \quad c = -\frac{1}{aa^\ast} b^\ast \frac{1}{a^\ast}, \quad d = (a^\ast)^{-1} \quad (3.3)$$

and

$$SU_q(1|1) : \quad T^{1,1} \in U_q(1|1) : \quad s \det_q T^{1,1} = a^\ast a + \beta\beta^\ast = 1 \quad (3.4)$$

We introduce the supersymmetric pair of q-oscillators (A, B) by postulating the following relations

$$\begin{aligned} AB &= qBa \\ A^+ B^+ &= -\frac{1}{q} B^+ A^+ & B^2 &= B^{+2} = 0 \end{aligned} \quad (3.5a)$$

$$AA^+ - q^2 A^+ A = 1 \quad (3.5b)$$

$$BB^+ + B^+ B = 1 + (q^2 - 1) A^+ A \quad (3.5c)$$

$$AB^+ = qB^+ A \quad BA^+ = qA^+ B \quad (3.5d)$$

The coaction of $GL_q(1|1)$

$$\begin{aligned} A' &= aA + \beta B & B' &= \gamma A + dB \\ A^+ &= a^\ast A^+ - \beta^\ast B^+ & B^+ &= \gamma^\ast A^+ + d^\ast B^+ \end{aligned}$$

with the generators of quantum group $GL_q(1|1)$ graded-commuting with the q-oscillators (A, B, A^+, B^+) leave invariant (3.5a), the relations (3.5d) (if $T^{1,1} \in U_q(1|1)$), and the relations (3.5b-c) (if $T^{1,1} \in SU_q(1|1)$). The relations (3.5a-d) distinguish the deformation of bosonic and fermionic sector (no operator tail on the rhs of (3.5b)). One can show however that the following "permuted" set of relations

$$AB = qBA \quad B^2 = 0 \quad (3.6a)$$

$$AA^+ - \frac{1}{q^2} A^+ A = 1 + \left(\frac{1}{q^2} - 1 \right) B^+ B \quad (3.6b)$$

$$BB^+ + B^+ B = 1 \quad (3.6c)$$

$$AB^+ = \frac{1}{q} B^+ A \quad (3.6d)$$

is covariant with respect to the same choices of quantum supergroups ((3.6a) for $GL(1|1)$, (3.6d) for $U(1|1)$, (3.6b-c) for $SU_q(1|1)$). The $U_q(1|1)$ -invariant Hamiltonian is described by the following bilinear form

$$H^{1,1} = A^+ A + B^+ B \quad (3.7)$$

Introducing $d = dz \nabla_z + d\theta \nabla_\theta$ as an exterior differential on $R_q^{[1]}$ quantum superplane (z, θ) , and δ as dual operator intertwining k -th order and $(k-1)$ -th order differential forms on $R_q^{[1]}$, after symmetrization one can interpret (compare with (2.11)) the operator (3.7) as the Laplace-Beltrami operator on $R_q^{[1]}$.

Now we shall calculate the quantum superalgebra $U_q(U(1|1))$ with the generators

$$\begin{aligned} Q &= A^+ B & Q^+ &= B^+ A \\ X &= A^+ A - B^+ B & Z &\equiv H^{1,1} \end{aligned} \quad (3.8)$$

One obtains :

$$Q^2 = Q^{+2} = 0 \quad (3.9)$$

$$\begin{aligned} q^2 Q^+ Q^+ + q^{-2} Q^+ Q = q^{-2} Z + \\ + \frac{1}{2} \left(1 - \frac{1}{q^2} \right) (X + Z) + \frac{1}{4} (q^2 - 1) (Z + X)^2 \end{aligned} \quad (3.10a)$$

$$Q^+ Y - q^2 Z Q^+ = -2Q^+ + (1 - q^2) Z Q^+ \quad (3.10b)$$

$$QY - q^2 YQ = 2Q - (1 - q^2) ZQ \quad (3.10c)$$

$$[Q, Z] = [Q^+, Z] = [X, Z] = 0 \quad (3.10d)$$

Due to the relation (3.10d) we see that the Hamiltonian (3.7) describes the central extension of quantum algebra with basic generators Q, Q^+, Y , obtained by putting $Z = 0$ in the relations (3.10a-c). The formulae (3.10) describe the q -extension of the $N=2$ supersymmetry in quantum mechanics ($\{Q, Q^+\} = H^{1,1}$, $Q^2 = Q^{+2} = 0$), with the supersymmetric doublets ($|E >, Q|E >$) having the same energy values (see (3.10d)).

It should be mentioned that for quantum $GL(1|1)$ one can introduce also a two-parameter R -matrix (compare with [27])

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & s^{-1} & 0 & 0 \\ 0 & r & s & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \quad \begin{cases} r = q - q^{-1}, \\ s = \text{complex} \end{cases} \quad (3.11)$$

defining the quantum group $GL_{q,s}(1|1)$ by means of the relation $R(T \otimes T) = (I \otimes T)(T \otimes I)R$ as also follows :

$$\begin{aligned} GL_{q,s}(1|1) : \quad & a\beta = qs^{-1}\beta a & d\beta &= qs^{-1}\beta d \\ & a\gamma = qs\gamma a & d\gamma &= qs\gamma d \\ & \beta\gamma = -s^2\gamma\beta & ad - da &= rs\gamma\beta \\ & \beta^2 = 0 & \gamma^2 &= 0 \end{aligned} \quad (3.12)$$

It would be interesting to generalize the q -superoscillator algebra (3.5) or (3.6) which is covariant under the coaction of the quantum supergroup (3.12) restricted by suitable unitarity conditions.

4. GENERAL CASE.

For arbitrary values of n, m we propose the following set of relations for q -deformed superoscillators variables (q real)

$$\begin{aligned} A_i A_j &= q A_j A_i, & i < j \\ A_i^+ A_j^+ &= \frac{1}{q} A_j^+ A_i^+, & i < j \\ A_i A_j^+ &= q A_j^+ A_i, & i \neq j, \end{aligned} \quad (4.1)$$

$$A_i A_i^+ - q^2 A_i^+ A_i = 1 + (q^2 - 1) \sum_{k < i} A_k^+ A_k \quad (4.2)$$

$$\begin{aligned} A_i B_r &= q B_r A_i, \\ A_i^+ B_r^+ &= \frac{1}{q} B_r^+ A_i^+, \\ A_i B_r^+ &= q B_r^+ A_i, \end{aligned} \quad \begin{cases} 1 \leq i \leq n \\ 1 \leq r \leq m \end{cases} \quad (4.3)$$

$$\begin{aligned}
B_r B_s &= -q B_s B_r, & r < s \\
B_r^\dagger B_s^\dagger &= -\frac{1}{q} B_s^\dagger B_r^\dagger, & r < s \\
B_r B_s^\dagger &= q B_s^\dagger B_r, & r \neq s, \\
B_r B_r^\dagger + B_r^\dagger B_r &= 1 + (q^2 - 1) \sum_{k=1}^n A_k^\dagger A_k \\
&\quad + (q^2 - 1) \sum_{s < r} B_r^\dagger B_r
\end{aligned} \tag{4.4}$$

The supercovariant Hamiltonian is given by the formula

$$H^{n,m} = \sum_{i=1}^n A_i^\dagger A_i + \sum_{r=1}^m B_r^\dagger B_r \tag{4.5}$$

which is commuting with the $U_q(U(n|m))$ quantum algebra with the q -deformed Cartan-Weyl basis obtained by the bilinear Hermitian products of the superoscillators (4.1-4).

In order to obtain the formulae (4.1-4) let us introduce n bosonic and m fermionic independent q -oscillators [19-21]

$$\begin{aligned}
a_i a_j^\dagger - q^{\delta_{ij}} a_j^\dagger a_i &= q^{-N_i} \delta_{ij} \\
[a_i, a_j] &= [a_i^\dagger, a_j^\dagger] = 0
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
b_r b_s^\dagger + q^{-\delta_{rs}} b_s^\dagger b_r &= q^{-M_r} \delta_{rs} \\
\{b_r, b_s\} &= \{b_r^\dagger, b_s^\dagger\} = 0
\end{aligned} \tag{4.7}$$

and (no summation)

$$a_i^\dagger a_i = [N_i]_q, \quad b_r^\dagger b_r = [M_r]_{q^{-1}} \tag{4.8}$$

One can also introduce another type of q -independent oscillators

$$\begin{aligned}
\alpha_i &= q^{\frac{N_i}{2}} a_i, & \beta_r &= q^{\frac{M_r}{2}} b_r \\
\alpha_i^\dagger &= a_i^\dagger q^{\frac{N_i}{2}}, & \beta_r^\dagger &= b_r^\dagger q^{\frac{M_r}{2}}
\end{aligned} \tag{4.9}$$

satisfying the algebra

$$A_i = q^{i < k} \alpha_i = q^{\left(\sum_{i < k}^{N_k + \frac{N_i}{2}}\right)} a_i, \tag{4.14a}$$

$$\begin{aligned}
\alpha_i \alpha_j^\dagger - q^{\delta_{ij}} \alpha_j^\dagger \alpha_i &= \delta_{ij}, \\
[\alpha_i, \alpha_j] &= [\alpha_i^\dagger, \alpha_j^\dagger] = 0,
\end{aligned} \tag{4.10a}$$

$$\begin{aligned}
\beta_r \beta_s^\dagger + \beta_s^\dagger \beta_r &= \delta_{rs}, \\
\{\beta_s, \beta_r\} &= \{\beta_s^\dagger, \beta_r^\dagger\} = 0.
\end{aligned} \tag{4.10b}$$

The relations (4.8) are replaced by the following ones

$$\alpha_i^\dagger \alpha_i = [N_i]_{q^2}, \quad \beta_r^\dagger \beta_r = [M_r]_{q^2} \tag{4.11}$$

where $[a]_q^c = \frac{q^a - 1}{q - 1}$, or

$$q^{2N_i} = 1 + (q^2 - 1) \alpha_i^\dagger \alpha_i, \tag{4.12a}$$

$$q^{2M_r} = 1 + (q^2 - 1) \beta_r^\dagger \beta_r. \tag{4.12b}$$

The formulae (4.12a-b) can be generalized as follows :

$$2 \sum_{q^{i < k}}^{N_k} = 1 + (q^2 - 1) \sum_{i < k} A_k^\dagger A_k, \tag{4.13a}$$

$$2 \sum_{q^{r < s}}^{M_r} = 1 + (q^2 - 1) \sum_{r < s} \tilde{B}_s^\dagger \tilde{B}_s, \tag{4.13b}$$

where A_k satisfy the relations (4.1) and B_r the relation (4.4) with $n = 0$. In order to obtain the set of all formulae (4.1-4) we should introduce

$$B_r = q^{\left(\sum_{i=1}^n N_i + \sum_{r < s} M_s\right)} \beta_i = q^{\left(\sum_{i=1}^n N_i + \sum_{r < s} M_s + \frac{M_r}{2}\right)} b_r. \quad (4.14b)$$

In order to obtain the representations of the algebra (4.1-4) the relations (4.14a-b) are very useful. One can construct the representations of the relations (4.6-7) or (4.9-10) in the Fock spanned by the normalized eigenstate $|n_{\alpha_i}, m_{\beta_r}\rangle$ of the number operators $N_i, M_r (i = 1..n; r = 1..m)$. In particular the spectrum of the Hamiltonian (3.5) is given by the following formula.

$$\begin{aligned} H^{(n|m)} |n_{\alpha_1} \dots n_{\alpha_n}; m_{\beta_1} \dots m_{\beta_m}\rangle & \\ = [N + M]_q^{c} |n_{\alpha_1} \dots n_{\alpha_n}; m_{\beta_1} \dots m_{\beta_m}\rangle, & \end{aligned} \quad (4.15)$$

where $N + M = \sum_{i=1}^n n_{\alpha_i} + \sum_{r=1}^m m_{\beta_r}$.

We have chosen in the formulae (4.1-4) the one-parameter family of the q-super-oscillators corresponding to the standard $GL_q(n|m)$ R-matrix given in [16].

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