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Fitting the $e^+e^- \rightarrow e^+e^-$ Lineshape

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Abstract

The implications of different treatments of the $e^+e^- \rightarrow e^+e^-$ cross sections in the context of Z parameter fitting are discussed. We show that fitting to a complete description of the process might become important for an accurate determination of the Z parameters. A fitting formula describing the integrated cross section in terms of the Z parameters is presented. This formula agrees with the most accurate calculation in the Standard Model to within 1 per mil.

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1 Introduction

Accurate measurements of cross sections have become one of the main experimental achievements at LEP. The large statistics accumulated (and foreseen) due to the good performance of the LEP machine together with the impressive level of understanding of the detector systematics achieved by the LEP experiments, has transformed cross section measurements at LEP in one of the higher precision measurements ever made in collider machines.

This experimental achievement has benefitted from an important theoretical work on the development of precise predictions, Monte Carlo (M.C.) event generators and fitting formulae for extracting the relevant information from the measurements. This work has been "on the right time" for processes such as $e^+e^- \rightarrow f\bar{f}$ where $f \neq e$ but is still at present incomplete for the process $e^+e^- \rightarrow e^+e^-$. This process plays an special rôle in the cross sections measurements:

- first because it is used, when considered at small angles, to compute the luminosity provided by the machine, and therefore it is a fundamental piece of all the cross sections computed by the experiments. The reason for using Bhabha scattering for this purpose is twofold:
 - in one hand, at low angles this process has a large cross section, thus allowing a luminosity determination with a small statistical error,
 - in other hand, the process at low angles is clearly dominated by the t-channel photon exchange, which can be accurately described in the frame of QED. This fact allows a measurement of the luminosity basically independent of the Z parameters which we pretend to determine by studying the lineshapes.
- second because this process allows a direct study of the couplings of the Z to the electron, which have to be determined first to extract from the lineshapes the coupling of the Z to other fermions. Since the initial state is set at LEP to be e^+e^- there is no way, if universality is not assumed from the very beginning, of disentangling from $e^+e^- \rightarrow f\bar{f}$ the information about $Z \rightarrow f\bar{f}$ from the one of $Z \rightarrow e^+e^-$ but knowing before the information about $Z \rightarrow e^+e^-$. This is possible since in the process $e^+e^- \rightarrow e^+e^-$ this information is the only one present (the final state is identical to the initial one) and therefore it allows a direct measurement.

Careful studies of the implementation of the non-QED corrections existed since some time ago [1] and the check of calculations performed by using different approaches has allowed a good understanding of the fact that the precision attained in this part of the problem is more than enough [2]. Anyway, complete standard model semianalytical calculations for low and large angle Bhabha scattering have been just recently produced [3]. The main reason for this delay has been the problem of understanding the proper way of introducing the QED corrections with enough accuracy.

Here we would like to present a rather different approach: a model independent parametrization useful for fitting purposes. The use of model independent formulae to describe the main measurements at the Z resonance was suggested during the preparation for LEP-SLC physics [4]. When fitting to experimental data, a model independent formulation is a clean way of extracting the relevant physical information of the Z resonance. The reasons for adopting a model independent formulation are twofold:

- Tests of the Standard Model may be biased if one assumes its validity from the beginning.

- By assuming only very basic theoretical concepts, one is able to analyze the results in terms of new theoretical models.

The basic idea of the model independent formulae for final states other than electrons, is the fact that the Z line shape, before considering initial state radiation, is well described by a Breit-Wigner resonance, with s -dependent width. Near the peak (say ± 3 GeV around the Z pole), the Z resonance dominates the photon exchange and Z -photon interference contributions, though these smaller terms are also carefully included in the existing fitting formulae used to analyse the Z lineshapes. These formulae reach, when compared to full calculations, accuracies better than one part in a thousand [5].

The goal of this work is to show how a similar procedure can be drawn for the process $e^+e^- \rightarrow e^+e^-$ and to demonstrate that a proper parametrization in the same language used for the study of the Z -driven lineshapes can be made leading to a comparable degree of precision.

The outline of the paper is as follows. In section 2 we analyze the physical information contained in the $e^+e^- \rightarrow e^+e^-$ cross section near the Z pole and we compare different possible ways of trying to extract it. Section 3 is devoted to the discussion on how a model independent parametrization of the non-QED corrected cross section can be made. Section 4 deals with the way QED corrections can be introduced. A discussion on the use of this calculation as a fitting program is presented in section 5. Section 6 shows how the necessary corrections can be computed to ensure proper matching between the experimental cross sections (at present obtained by using an $O(\alpha)$ M.C. event generator for detector efficiency corrections) and our fitting formula (which includes leading-log up to $O(\alpha^2)$). In section 7 we explain some checks on the validity of our approach. Finally in section 8 we try to present the conclusions of this analysis.

2 Physics from the Bhabha Cross Section

Bhabha scattering differs from $e^+e^- \rightarrow \mu^+\mu^-$ because of the fact that, since the final state is identical to the initial one, elastic scattering through the exchange of either a photon or a Z is also possible (see fig. 1).

Out of the four amplitudes describing the process at tree level, just the s -channel Z exchange is resonant and hence dominant near the Z peak. Anyway, depending on the angular range under consideration, the contribution due to the t -channel photon exchange may become also very important. The contribution due to the rest of diagrams (s -channel photon exchange and t -channel Z exchange) is expected to be very small. Figures 2 and 3 show respectively the total and relative contribution to the cross section coming from the different terms for an angular range of $|\cos\theta| < 0.9$ expected from the tree level calculation. We see that, in contrast with what happens in the process $e^+e^- \rightarrow \mu^+\mu^-$, two more terms besides the one due to the s -channel Z exchange, must be taken into consideration.

To understand the physics that can be extracted from the Bhabha scattering, it is important to isolate the dominant terms from the differential cross section. At tree level we can write

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (S_{Z,Z} + S_{\gamma,\gamma} + S_{Z,\gamma} + \text{small}) \quad (1)$$

where

$$S_{Z,Z} = \frac{s^2}{(s - M_Z^2)^2 + (M_Z \Gamma_Z)^2} \left((1 + \cos^2\theta)(v^2 + a^2)^2 + 8 \cos\theta v^2 a^2 \right)$$

$$\begin{aligned}
S_{\gamma\gamma} &= 2 \frac{(1 + \cos \theta)^2 + 4}{(1 - \cos \theta)^2} \\
S_{Z,\gamma} &= 2 \frac{s(s - M_Z^2)}{(s - M_Z^2)^2 + (M_Z \Gamma_Z)^2} \frac{(1 + \cos \theta)^2}{(\cos \theta - 1)} (v^2 + a^2)
\end{aligned} \tag{2}$$

In these equations, v^2 and a^2 are the squares of the vector and axial-vector couplings of the electrons with the Z . Now, using the following tree level relation:

$$v^2 + a^2 = \frac{3\Gamma_e}{\alpha M_Z} \tag{3}$$

we can write the integrated cross section as:¹

$$\sigma = \int_{-\cos \theta}^{\cos \theta} \frac{d\sigma}{d\cos \theta} = A(s, \cos \theta) \Gamma_e^2 + B(s, \cos \theta) \Gamma_e + C(s, \cos \theta) \tag{4}$$

where the terms A , B and C come from the contributions $S_{Z,Z}$, $S_{Z,\gamma}$ and $S_{\gamma\gamma}$ respectively.

It is clear that the interference between the s-channel Z exchange and the t-channel photon exchange contains also physical information about the Γ_e parameter. This information is completely new with respect to other processes. To understand quantitatively how much information can one extract from this new term, we will use the approach of the sensitivity functions.

2.1 Sensitivity function for the integrated Bhabha Cross Section

The use of sensitivity functions provides a clear way of understanding how much information about a parameter such as Γ_e can be extracted from the study of any given function of it².

To build the sensitivity function, we will use the fact that the cross section is, as we have shown, a function $\sigma = \sigma(\Gamma_e)$. Therefore, we have

$$\Delta \Gamma_e = \frac{1}{\frac{d\sigma}{d\Gamma_e}} \Delta \sigma \tag{5}$$

The error in the cross section can be written as

$$\Delta \sigma = \sigma \sqrt{\frac{1}{N}} = \sqrt{\frac{\sigma}{L}} \tag{6}$$

where N denotes the number of observed events and L denotes the luminosity accumulated during the study of the process. From these two equations we can write

$$\Delta \Gamma_e = \frac{1}{\sqrt{L}} \frac{1}{\frac{d\sigma}{d\Gamma_e}} \sqrt{\sigma} = \frac{1}{\sqrt{L}} \frac{1}{S_\sigma(\Gamma_e)} \tag{7}$$

¹In this discussion we are assuming a symmetric $\cos \theta$ region. If the integration region is not symmetric then an additional term of the form $D(s, \cos \theta) v^2 a^2$ shows also up, coming from the $S_{Z,Z}$ contribution but, since this term is very small (v^2 is very small), we can always consider it among the rest of small terms which we are disregarding for the present discussion.

²In order to simplify the mathematics involved in the discussion we will follow the approach of considering the ideal case in which only statistical errors will be taken into account. The quality of the conclusions will be valid for a real case in spite that the actual numerical values might change according to the size of the systematic errors involved in the measurement.

so that, we have disentangled in the error that we can obtain in the determination of Γ_e the dependence on the amount of data collected ($1/\sqrt{L}$) from the dependence on the functionality of the cross section with respect to Γ_e , which is given by the inverse of the **sensitivity function** $S_\sigma(\Gamma_e)$, defined as:

$$S_\sigma(\Gamma_e) \equiv \frac{1}{\sqrt{\sigma}} \frac{d\sigma}{d\Gamma_e} \quad (8)$$

Clearly, for a fixed collected luminosity, the larger the sensitivity is, the smaller the error is in Γ_e .

In our case, we can write $\sigma = \sigma_{Z,Z} + \sigma_{\gamma,\gamma} + \sigma_{Z,\gamma}$, so that the sensitivity is

$$S_\sigma(\Gamma_e) \equiv \frac{1}{\sqrt{\sigma}} \left(\frac{d\sigma_{Z,Z}}{d\Gamma_e} + \frac{d\sigma_{Z,\gamma}}{d\Gamma_e} \right) \quad (9)$$

where the fact that $d\sigma_{\gamma,\gamma}/d\Gamma_e = 0$ has been used. To understand which additional sensitivity comes from the interference term, let's compute also the sensitivity which we would obtain by freezing the dependence on Γ_e in the interference term (that is, by assuming $d\sigma_{Z,\gamma}/d\Gamma_e = 0$). This is, in fact, what is done when the contribution of the t-channel and its interference are treated as a background to the Z exchange term and then subtracted from the observed cross section ("t-channel subtraction" method [6]). Therefore

$$S_{\sigma(\text{t-subtracted})}(\Gamma_e) \equiv \frac{1}{\sqrt{\sigma}} \left(\frac{d\sigma_{Z,Z}}{d\Gamma_e} \right) \quad (10)$$

Finally, in order to have a known scale for comparison, let's compute the sensitivity that the process would have if just the Z exchange diagram would be present, that is, $\sigma = \sigma_{Z,Z}$,

$$S_{\sigma(\text{pure Z})}(\Gamma_e) \equiv \frac{1}{\sqrt{\sigma_{Z,Z}}} \left(\frac{d\sigma_{Z,Z}}{d\Gamma_e} \right) \quad (11)$$

It is important to point out the difference between these two latter sensitivities. The sensitivity when performing t-channel subtraction is not the same as the one you would have if the process would be purely Z driven because of the fact that the fluctuation in the "background" (t channel plus interference) degrades the sensitivity of the measurement. This fact corresponds to having $\sqrt{\sigma}$ instead of $\sqrt{\sigma_{Z,Z}}$ in the denominator.

These three sensitivities as a function of the maximum $\cos\theta$ can be seen in figures 4, 5 and 6 for $\sqrt{s} = M_Z - 1, M_Z, M_Z + 1$ which corresponds basically to the energies for which the interference is maximal, vanishes or is minimal.

One trend that these figures show is the fact that unlike in the case of pure Z processes, for which maximal sensitivity would be reached for full solid angle, for Bhabha, maximal sensitivity is reached for a specific value of $\cos(\theta_{min})$ which depends on $\sqrt{s}$. This is due to the compromise between increasing the angle to accept "more Z events" and decreasing it to have smaller t-channel "background".

Also the fact that the sensitivity is smaller than in pure Z processes is clear from these plots. This is only reversed in the case that we are above the Z peak (where the interference is negative) and we use a rather small $\cos(\theta_{min})$. The reason is that, then, the "background" is reduced considerably and, since the interference is negative, the total cross section showing up in the denominator of the sensitivity (which accounts for the statistical fluctuations) is even smaller than the pure Z one such that the statistical fluctuation is smaller keeping basically the same information in the numerator.

These figures show also that, below the peak, the sensitivity obtained in the t-channel subtraction method is sizably underestimative with respect to the real one. At the peak both are equal and above the peak the effect is roughly the opposite. This effect is due to the sensitivity of the interference, which is positive below the peak, zero at the peak and negative above it. Figure 7 shows the three sensitivities as a function of $\sqrt{s}$ for an angular range of $|\cos \theta| < 0.9$. This bias in the sensitivity is very important on the wings.

Nevertheless, figure 7 shows that in a scanning of the resonance, if the luminosity is distributed symmetrically with respect to the Z peak, there is an approximated cancellation of the underestimation and overestimation of the sensitivities in the t-channel subtraction method which results then in roughly the same overall sensitivity as the complete treatment. This cancellation justifies the correctness of using the t-channel subtraction method (with the symmetric scanning scheme used in LEP so far) as a good approach to the analysis of the physics in Bhabha scattering.

In any case, since this cancellation is accidental and the proper use of t-channel subtraction makes the rigorous treatment of errors and the calculation of subtractions a rather complicated process, it would be better trying to use an approach in which the full Bhabha cross section is fitted to the data. Such an approach is what we will present next.

3 Improved Born Formula

In the spirit of what was done in ref. [5] we will obtain an improved Born formula which will constitute the kernel for our Bhabha cross section. In the next section we will explain how it can be dressed with QED corrections to render the final result.

We want our formula to include the main electroweak corrections and at the same time to be as model independent as possible. However, we already know from the start that some pieces (such as the interference between the s-channel Z and photon exchanges) will have to be built within the framework of the Standard Model ³.

The fact that in Bhabha scattering at lowest order we have four diagrams (depicted in fig. 1) means that there will be ten different terms (including interferences) in the full matrix element squared expression. Those ten terms, after the inclusion of the phase space factors to get the differential cross section, can be written in the following way:

$$\begin{aligned} \left(\frac{d\sigma^0}{d\cos\theta} \right)_{Z,Z} &= \frac{\pi\alpha^2}{2s} \frac{s^2}{(s - M_Z^2)^2 + (M_Z\Gamma_Z)^2} \left((1 + \cos^2\theta)(v^2 + a^2)^2 + 8\cos\theta v^2 a^2 \right) \\ \left(\frac{d\sigma^0}{d\cos\theta} \right)_{\gamma,\gamma} &= \frac{\pi\alpha^2}{2s} 2 \frac{4 + (1 + \cos\theta)^2}{(\cos\theta - 1)^2} \\ \left(\frac{d\sigma^0}{d\cos\theta} \right)_{Z,\gamma} &= \frac{\pi\alpha^2}{2s} \frac{s(s - M_Z^2)}{(s - M_Z^2)^2 + (M_Z\Gamma_Z)^2} 2(v^2 + a^2) \frac{(1 + \cos\theta)^2}{(\cos\theta - 1)} \\ \left(\frac{d\sigma^0}{d\cos\theta} \right)_{\gamma,Z} &= \frac{\pi\alpha^2}{2s} (1 + \cos^2\theta) \end{aligned}$$

³This can be avoided by introducing additional fitting parameters which account for combinations of the couplings other than the ones appearing in partial widths.

$$\begin{aligned}
\left(\frac{d\sigma^0}{d\cos\theta}\right)_{\gamma,\gamma_i} &= \frac{\pi\alpha^2}{2s} 2 \frac{(1+\cos\theta)^2}{(\cos\theta-1)} \\
\left(\frac{d\sigma^0}{d\cos\theta}\right)_{Z,\gamma_i} &= \frac{\pi\alpha^2}{2s} \frac{s(s-M_Z^2)}{(s-M_Z^2)^2+(M_Z\Gamma_Z)^2} 2 \left((1+\cos\theta^2)v^2+2\cos\theta a^2\right) \\
\left(\frac{d\sigma^0}{d\cos\theta}\right)_{Z_i Z_i} &= \frac{\pi\alpha^2}{2s} \frac{s^2}{(t-M_Z^2)^2} \frac{1}{2} \left((1+\cos\theta)^2((v^2+a^2)^2+4v^2a^2)+4((v^2+a^2)^2-4v^2a^2)\right) \\
\left(\frac{d\sigma^0}{d\cos\theta}\right)_{Z,\gamma_i} &= \frac{\pi\alpha^2}{2s} \frac{s(s-M_Z^2)}{(s-M_Z^2)^2+(M_Z\Gamma_Z)^2} \frac{s}{t-M_Z^2} (1+\cos\theta)^2((v^2+a^2)^2+4v^2a^2) \\
\left(\frac{d\sigma^0}{d\cos\theta}\right)_{\gamma_i Z_i} &= \frac{\pi\alpha^2}{2s} \frac{s}{t-M_Z^2} \frac{2}{\cos\theta-1} ((1+\cos\theta)^2(v^2+a^2)+4(v^2-a^2)) \\
\left(\frac{d\sigma^0}{d\cos\theta}\right)_{\gamma_i Z_i} &= \frac{\pi\alpha^2}{2s} \frac{s}{t-M_Z^2} (1+\cos\theta)^2(v^2+a^2)
\end{aligned} \tag{12}$$

Let's recall that v^2 and a^2 are the squares of the vector and axial-vector couplings of the electrons with the Z :

$$\begin{aligned}
a^2 &= \frac{1}{16\sin^2\theta_W\cos^2\theta_W} \\
v^2 &= a^2(1-4\sin^2\theta_W)^2
\end{aligned} \tag{13}$$

Equations (12) can be easily integrated to render the ten contributions to the integrated cross section between the angle θ_m and θ_M :

$$\sigma^0(\theta_m, \theta_M) = \sum \left(\sigma_{ij}^0(\cos\theta_m) - \sigma_{ij}^0(\cos\theta_M) \right) \tag{14}$$

where we have

$$\begin{aligned}
\sigma_{Z,Z_i}^0(c) &= \frac{\pi\alpha^2}{2s} \frac{s^2}{(s-M_Z^2)^2+(M_Z\Gamma_Z)^2} \left(\left(c+\frac{c^3}{3}\right)(v^2+a^2)^2+4c^2v^2a^2 \right) \\
\sigma_{\gamma_i\gamma_i}^0(c) &= \frac{\pi\alpha^2}{2s} 2 \frac{4c\ln(1-c)-4\ln(1-c)+c^2-9c}{c-1} \\
\sigma_{Z,\gamma_i}^0(c) &= \frac{\pi\alpha^2}{2s} \frac{s(s-M_Z^2)}{(s-M_Z^2)^2+(M_Z\Gamma_Z)^2} (v^2+a^2)(8\ln(1-c)+c^2+6c) \\
\sigma_{\gamma_i\gamma_i}^0(c) &= \frac{\pi\alpha^2}{2s} \left(c+\frac{c^3}{3} \right) \\
\sigma_{\gamma_i\gamma_i}^0(c) &= \frac{\pi\alpha^2}{2s} 2 \left(4\ln(1-c)+3c+\frac{c^2}{2} \right) \\
\sigma_{Z,\gamma_i}^0(c) &= \frac{\pi\alpha^2}{2s} \frac{s(s-M_Z^2)}{(s-M_Z^2)^2+(M_Z\Gamma_Z)^2} 2 \left(\left(c+\frac{c^3}{3}\right)v^2+c^2a^2 \right) \\
\sigma_{Z_i Z_i}^0(c) &= \frac{\pi\alpha^2}{2s} 2F \left(c+2(D+1)\ln(D-c)-\frac{D^2+2D+BG}{c-D} \right) \\
\sigma_{Z_i Z_i}^0(c) &= \frac{\pi\alpha^2}{2s} \frac{s(s-M_Z^2)}{(s-M_Z^2)^2+(M_Z\Gamma_Z)^2} 2F \left(\frac{c^2}{2}+(D+2)c+(D+1)^2\ln(D-c) \right)
\end{aligned}$$

$$\begin{aligned}
\sigma_{\gamma, Z_1}^0(c) &= \frac{\pi\alpha^2}{2s} 4(v^2 + a^2) \left(c - \frac{P+3}{D-1} \ln(1-c) + \frac{D^2+2D+P}{D-1} \ln(D-c) \right) \\
\sigma_{\gamma, Z_1}^0(c) &= \frac{\pi\alpha^2}{2s} 2(v^2 + a^2) \left(\frac{c^2}{2} + (D+2)c + (D+1)^2 \ln(D-c) \right)
\end{aligned} \tag{15}$$

where we have defined

$$\begin{aligned}
F &= (v^2 + a^2)^2 + 4v^2 a^2 \\
G &= (v^2 + a^2)^2 - 4v^2 a^2 \\
D &= 1 + 2 \frac{M_Z^2}{s} \\
BG &= 1 + 4 \frac{G}{F} \\
P &= 1 + 4 \frac{v^2 - a^2}{v^2 + a^2}
\end{aligned} \tag{16}$$

The ordering followed in eqs. (12) and (15) turns out to be one of decreasing numerical importance at the Z region for angular cuts around 30° .

We will try now to include the most important weak corrections into those Born formulae. We will do it for the most important terms while writing them in a model independent way, that is, in terms of physical, directly measurable observables. The first five pieces of, for instance, the integrated cross section can be written as:

$$\begin{aligned}
\sigma_{Z, Z_1}^w(c) &= \frac{\pi\alpha^2(M_Z^2)}{2s} \frac{s^2}{(s - M_Z^2)^2 + (s\Gamma_Z/M_Z^2)^2} \left(\left(c + \frac{c^3}{3} \right) \left(\frac{3\Gamma_e}{M_Z\alpha(M_Z)} \right)^2 + 4c^2 v^2 a^2 \right) \\
\sigma_{\gamma, \gamma_1}^w(c) &= \frac{\pi\alpha^2(\bar{t})}{2s} 2 \frac{4c \ln(1-c) - 4 \ln(1-c) + c^2 - 9c}{c-1} \\
\sigma_{Z, \gamma_1}^w(c) &= \frac{\pi\alpha(M_Z^2)\alpha(\bar{t})}{2s} \frac{s(s - M_Z^2)}{(s - M_Z^2)^2 + (s\Gamma_Z/M_Z^2)^2} \frac{3\Gamma_e}{M_Z\alpha(M_Z^2)} (8 \ln(1-c) + c^2 + 6c) \\
\sigma_{\gamma, \gamma_2}^w(c) &= \frac{\pi\alpha^2(s)}{2s} \left(c + \frac{c^3}{3} \right) \\
\sigma_{\gamma, \gamma_1}^w(c) &= \frac{\pi\alpha(s)\alpha(\bar{t})}{2s} 2 \left(4 \ln(1-c) + 3c + \frac{c^2}{2} \right)
\end{aligned} \tag{17}$$

Here, Γ_e is the physical partial width to electrons, including all sort of corrections. The scale $\bar{t}$ is somewhat arbitrary. We have chosen it as the mean value of t in the integration range. Final results do not depend on the precise choice of $\bar{t}$. The changes with respect to the tree level come from inserting the physical partial width where possible, running the QED coupling constant as appropriate and including the running Z^0 width in the propagator. Note that in the first term of the last equation, there is a piece depending on the value of the Standard Model parameters v^2 and a^2 . However, since v^2 is very small, this term is almost negligible. Furthermore, it would completely vanish upon symmetrical integration.

The other five terms cannot be written in terms of physical parameters. Then, for them we will keep their expressions in (15) after changing the Z propagator as above and evaluating the coupling constants as follows:

$$\sigma_{Z, \gamma_2}^w \longrightarrow \alpha(M_Z^2)\alpha(s)$$

$$\begin{aligned}
\sigma_{Z,Z_i}^w &\longrightarrow \alpha^2(\bar{t}) \\
\sigma_{Z,Z_i}^w &\longrightarrow \alpha(M_Z^2)\alpha(\bar{t}) \\
\sigma_{\gamma,Z_i}^w &\longrightarrow \alpha^2(\bar{t}) \\
\sigma_{\gamma,Z_i}^w &\longrightarrow \alpha(s)\alpha(\bar{t})
\end{aligned} \tag{18}$$

This, together with the previous equation, constitutes our final expression for the Born-like kernel. We have compared it with the most accurate calculation of Bhabha scattering currently available [3]. Figures 8 and 9 show the results of such a comparison for $|\cos\theta| < 0.9$. The agreement between the two programs is better than one mil across the resonance and also at lower center of mass energies (which will be effectively seen when including initial state radiation).

The advantage of our expression is that the bulk of it is expressed in terms of parameters that can be directly fitted and that the assumptions on the Standard Model are very few and only enter as subleading effects. The most important one comes from the fact that we have to assume a value for the weak mixing angle in the model-dependent part of the calculation. However, even this is very small: changing $\sin^2\theta_W$ in the range 0.231 ± 0.002 (as suggested by the latest experimental results [7]), only produces a change in the total cross section of ± 0.7 per mil at the peak (± 1.2 at the wings).

4 QED Corrections

After explaining in the previous section the Born-like kernel that we will use to include the most important weak corrections, we turn now our attention to the much more cumbersome discussion of the QED corrections. The point is that for Bhabha scattering we cannot use any of the many elegant approaches which have been used for s-channel processes [8], the reason being that we cannot have a totally inclusive cross section, which would diverge. Then we are faced with the problem of doing the QED calculations within some cuts. Our choice has been to follow closely the ideas of ref. [3] because this is the most complete approach to the problem existing so far, and has become a sort of standard like in the case of the s-channel processes was the formulation of reference [8]. In there an $O(\alpha^2)$ leading log calculation within a number of cuts (acollinearity, laboratory angles and energies for both outgoing electrons) is presented. Furthermore, the $O(\alpha)$ non leading log terms are also included such that the QED treatment is exact up to $O(\alpha)$.

In spite that for a detailed discussion of the QED treatment we refer to [3], we would like to sketch the basic idea of the approach to emphasize, later, where our treatment deviates from the one applied therein. Basically the QED corrected cross section is obtained by adding four different terms, namely:

$$\begin{aligned}
\sigma(s) = & \int dz I(z) \sigma^w(z, s) F(z) \\
& + \int dx_1 \int dx_2 I_2(x_1, x_2) \sigma^w(x_1 x_2, s) \\
& + F_2(s) \sigma^w(s) \\
& + N(s)
\end{aligned} \tag{19}$$

In these equations σ^w is the kernel described in the previous section and the meaning of the different terms is the following:

- The first term accounts for the part of the LL corrections which can be reduced to a one-dimensional integration. This corresponds basically to the emission of initial state soft plus hard photonic energy effectively assignable to one hard photon convoluted with the radiation in the final state of zero or one photon. In this term:
 - $I(z)$ is a radiator function which contains initial state LL QED corrections up to $O(\alpha^2)$ plus soft photon exponentiation.
 - $F(z)$ is a radiator function which contains final state LL QED corrections up to $O(\alpha)$.
- The second term accounts for the part of LL corrections non-reducible to a one-dimensional integration. Since we consider hard photons in LL only up to $O(\alpha^2)$ (two hard photons), that means that this part needs a two-dimensional integration. Unlike in the case of s-channel processes, these integrations do not reduce to a one-dimensional one and this is entirely due to the fact that detection cuts have to be applied to keep the cross section meaningful. In this term:
 - $I_2(x_1, x_2)$ is a radiator function which contains initial state LL QED corrections at $O(\alpha^2)$ due to two hard photons.
- The third term accounts for the radiation of two hard photons in the final state. Since this radiation does not imply any change in the effective center of mass energy of the Bhabha scattering, no convolution integral is needed.
- The fourth term accounts for non-LL $O(\alpha)$ corrections. In order to introduce the detection cuts, the way it is computed in practice is

$$\text{non-LL } O(\alpha) = \text{complete } O(\alpha) - \text{LL } O(\alpha)$$

and therefore, in order to compute the complete $O(\alpha)$ contribution, a five-dimensional integration is unavoidable in this part.

The explicit expression for the different radiators can be found in reference [3]. The one-dimensional integration is performed with an standard adaptive gaussian algorithm which provides a very fast and accurate estimation. For the two and five-dimensional integration, the fact that the experimental cuts define very complicated regions in phase space and that sharp peaks show up in any choice of the integration variables, forces to use an adaptive MC integration algorithm [10] to cope with the required accuracy of around a per mil. Needless to say that this is very much CPU time consuming and very specially in the case of the five-dimensional integration.

Although following essentially ref. [3], we allow for a wider choice of cuts. Then, besides the cuts mention above, in our approach also cuts in the electron pair invariant mass, the pseudorapidity and the polar angle in the electron pair center of mass frame $\cos \theta^*$ are possible. This treatment makes easier for us to accommodate to the experimental cuts used by the LEP experiments.

Comparisons have made been with the calculation of ref. [3] after the inclusion of the QED dressing and (not surprisingly, since we use almost the same treatment and the Born-like kernels agree over a wide range of effective center of mass energies) the agreement has been found to be better than one per mil in the Z region (fig. 10).

The accuracy of the QED part of the formula will determine its overall accuracy. Reference [3] quotes a 0.5% error coming mainly from the QED part. This will be, then also, our overall accuracy.

5 The Fitting Program

As we have pointed out, the need for multi-dimensional integration of complicated functions in the implementation of the QED corrections makes the whole cross section evaluation extremely CPU time consuming. This fact prevents the direct use of the QED radiation scheme implemented in [3] in a fitting formula (unless you can afford many hours of a supercomputer to perform a simple fit). To avoid the CPU problem without sacrificing the requested accuracy we have applied a couple of calculational tricks that, as we shall see, are rather specific to the use of our program as a fitting formula and not as a general purpose semianalytical program to compute Bhabha cross sections.

5.1 Two initial state hard photon integral

The use of adaptive MC integration in the case of the two-dimensional integration, though it is the best way of reaching the required accuracy, is incompatible with the use of the cross section in a fitting procedure. The reason is the following: in the MC estimation of the integrals there is an inherent statistical fluctuation which can be as big as the tiny changes obtained in the cross sections when exploring the fitting parameters to obtain the best fitting values. As a consequence, the convergence of the fitting procedure is seriously compromised.

Anyway, these adaptive algorithms use the MC approach to estimate the value of the integral but also to explore the function behaviour in an unbiased way to build a numerical mapping of it. We can use only this second feature by proceeding in the following way:

- at initialization time, the adaptive MC integration routine is used to compute the grid which maps the behaviour of the function. The distribution of points obtained when the statistical accuracy of the integral is well below the required one is then frozen.
- during fitting time, the frozen point distribution produced for each energy is used to evaluate the integral.

The accuracy of this procedure is good provided that the first step is done requiring good statistical precision and also, just to make sure, absorbing as much as possible of the peaking structure of the integrand in an analytical mapping of the integration variables. This last step is done by just using a simple change of integration variables

$$dy_1 dy_2 = \frac{dx_1}{x_1(1-x_1)} \frac{dx_2}{x_2(1-x_2)} \quad (20)$$

where $1-x$ denotes fractional photon energy and, therefore, the mapping accounts for the soft peak of the radiator and for the Coulomb peak of the reduced cross section. This simple mapping, although absorbs the bulk of the peaks, is insufficient to describe completely the function behaviour and, therefore, must be complemented by the adaptation performed by the program of ref. [10].

This approach, besides needed, reduces (in the conditions of our study), the CPU time request by a factor of around 10.

Table 1: The non leading-log contribution (in per cent) at LEP energies for two different values of the minimum polar angle.

$\sqrt{s}(\text{GeV})$	88.22	89.22	90.22	91.22	92.22	93.22	94.22
$\cos(\theta_{\min}) = 0.7$	2.05	1.67	1.17	0.77	0.65	0.80	1.08
$\cos(\theta_{\min}) = 0.9$	2.17	1.89	1.54	1.30	1.41	1.88	2.23

5.2 Non Leading Log Terms

The most time-consuming part of the QED implementation is the five-dimensional integration needed for the calculation of the non-LL $O(\alpha)$ corrections. Anyway, it is well known that the size of the non-LL part can be made small with the use of the proper scale for the LL calculation (which up to some extend is a free parameter of the QED corrections).

Of course, the choice of a given value for this scale is just arbitrary if a complete calculation of each order is made, but might be dangerous if just the LL part is used (as it happens in fact in our case at $O(\alpha^2)$). Therefore, it should be clarified that the scale of the LL shouldn't be chosen to minimize the non-LL contribution at a given order but rather using physical arguments. On the other hand, the fact that the non-LL corrections become important, usually indicates that a wrong scale has been chosen for the LL part. We have studied the effect in the non-LL contribution of fixing the LL scale to a given value within the range suggested by physical arguments and what we will discuss next remains unchanged.

Since the non-LL part is rather small (see table 1 for our particular case) it can be treated as a correction, namely:

$$\delta_N(s) \equiv \frac{\sigma(s) - \sigma_{LL}(s)}{\sigma_{LL}(s)} \quad (21)$$

The dependence of $\sigma(s)$ on the fitting parameters (M_Z , Γ_Z and Γ_e) is, up to a good approximation, very well described by $\sigma_{LL}(s)$ so that, the procedure followed is:

- at initialization time, fixing physical measured values for the fitting parameters, compute δ_N at every energy point for the known detection cuts used in the cross section. Since for that we have to use the five-dimensional integration, this is rather time consuming but must be done just once for ever (it is not even necessary to repeat it every time that a new fit is started).
- during fitting time, assume that $\delta_N(s)$ is just an energy-dependent correction which does not change when you explore the fitting parameters and compute the cross section using the values for $\delta_N(s)$ already calculated in the first step.

This approximation turns out to be very good for realistic conditions and, in our case, reduces the CPU time request by a further factor of around 30.

6 Efficiency Corrections

In the previous sections we have presented a formula to fit the Bhabha cross section integrated within some kinematical cuts. However, the actual experimental cuts are, of course, more

complicated and their effects can only be estimated using a Monte Carlo method. This involves an event generator and a detector simulation program. The problem is that no $O(\alpha^2)$ event generator exists at the moment. The more accurate Bhabha event generator [9] is just a complete $O(\alpha)$ electroweak calculation.

In this section we will deal, then, with this problem: the calculation of the efficiency correction using the program of ref. [9] and correcting for the fact that it is not a complete calculation. Although the way to solve this problem depends strongly on the actual experimental cuts that are applied to the data, we will try to give some general ideas about it.

Let's call σ_{obs} the observed (measured) cross section and σ_c the one corrected by efficiency. For efficiency corrections we mean the ones due to detector effects, experimental cuts, etc. but not the kinematical cuts which are dealt with at the level of the fitting function. Then, we will have

$$\sigma_c = \sigma_{obs} \cdot E \quad (22)$$

where

$$E = \frac{\sigma_T(KC)}{\sigma_T(EC)} \quad (23)$$

Here σ_T means the cross section computed at the level of the fitting formula, that is, $O(\alpha^2)$ plus soft photon exponentiation. KC means within the kinematical cuts (angular range, invariant mass, acollinearity,...). EC means within the experimental cuts (the previous ones plus quality of tracks, energy in calorimeters, etc). Clearly, $\sigma_T(EC)$ has to be computed taking into account the detector effects.

Let's split σ_T into two pieces: the $O(\alpha)$ part (the one included in [9]), σ_B , and the rest, σ_2 . Then, we have

$$E = \frac{\sigma_B(KC) + \sigma_2(KC)}{\sigma_B(EC) + \sigma_2(EC)} = \frac{\sigma_T(KC)}{\frac{\sigma_B(KC)}{R} + \sigma_2(EC)} \quad (24)$$

where $R = \frac{\sigma_B(KC)}{\sigma_B(EC)}$ is the efficiency correction obtained from ref. [9] (taking into account detector effects). There is no problem, in principle, then, in computing R . We will use eq. (24) to compute (an approximation of) E :

- To compute $\sigma_T(KC)$ we will use our program, which allows us to cut in all the previously mentioned kinematical variables.
- $\sigma_B(KC)$ comes from the same program but with the matrix elements evaluated to pure one-loop level.
- R depends on the characteristics of the detector and of the cuts applied.
- $\sigma_2(EC)$ will be computed as $\sigma_2(EC) = \sigma_T(EC) - \sigma_B(EC)$.

The only difficulties are in the last piece, since to compute it we would need a complete second order Bhabha Monte Carlo. We will be forced, then, to make two approximations:

1. First of all, we will approximate the experimental cuts by their kinematical definitions. That is, for instance, calorimetric energies will be approximated by actual energies. We will apply this approximation both to $\sigma_T(EC)$ and $\sigma_B(EC)$, since in this way we minimize any bias due to neglecting the detector effects.

Table 2: Three parameter fit to the electron lineshape.

Parameter	M_Z (MeV)	Γ_Z (MeV)	Γ_e (KeV)
ALIBABA input	91170	2490.8	83630
fit result	91170	2491.0	83621

2. Sometimes, however, it can be impossible even to implement the kinematical definition of the experimental cuts in our calculation. If this is the case, instead of this, some suitable approximation will have to be performed.

Assuming that this programme has been fulfilled, we have still to worry about the problem of considering, one way or another, the detector effects to the missing cross section in [9]. This can be done by assuming that these detector effects will be the same as those affecting the $O(\alpha)$ cross section, that is, R . Then we would have,

$$E = \frac{\sigma_T(KC)}{\frac{\sigma_B(KC)}{R} + \frac{\sigma_2(EC)}{R}} \quad (25)$$

However, this is not obvious, since, for instance, some of these effects (for instance the tracking efficiency) could be different for this part of the cross section which includes events with two hard photons radiated and, hence, lower momentum tracks.

From what has been said above, it is clear that to do a rigorous, thorough, job a complete $O(\alpha^2)$ Bhabha event generator is needed. However, we think that it is fair to say that something can be done along the preceding lines. As an example, we have taken the ALEPH analysis ([11]) and computed the efficiency corrections. The correction in this case turns out to be rather small (one per mil) with an estimated error of plus or minus two per mil. It is very difficult to reach a better precision with the existing Monte Carlo programs.

To end this section, let's point out that these difficulties when computing the efficiency corrections have nothing to do with our treatment of the Bhabha lineshape fitting and that they appear in all the other approaches, for instance, the widely used "t-channel subtraction" method.

7 Checks

A number of checks have been done to test the program. First of all a simulated lineshape has been generated using [3] (assuming the current experimental errors [11] and refitted using our program. The difference between the generated values for the Z parameters and the fitted values is shown in table 2. Since all the experiments are basically obtaining M_Z and Γ_Z from the hadron lineshape, it is more meaningful fitting only Γ_e . The difference between the electron partial width of the Z (the main parameter measured with the electron lineshape) and the fitted value is then 13 KeV. This implies a bias smaller than 0.2 per mil and, then, completely negligible. The χ^2 of the fit per degree of freedom is about 10^{-5} (there are no statistical fluctuations) and shows also that the two lineshapes agree with high precision.

Other checks have been done using data from the ALEPH collaboration at LEP [11]. Those checks include looking for strange behaviour of the fitted values when taking only the peak points,

only the wings points, when modifying the angular range, etc. All the results are consistent with what can be expected and, then, the program can be used for all the energies scanned around the Z up to a forward angle of 25° or even lower.

Finally, we have explicitly checked that freezing the interference term does not change the result of the fit in a sizable way. This relies in the fact that the luminosity taken at LEP is more or less symmetric in the center of mass energy around the peak. Since this freezing is effectively done in the commonly used t-channel subtraction method to fit the Bhabha lineshape, this means that the results of both methods should agree, as we have explicitly seen using again ALEPH data [10].

8 Conclusions

The implications of different treatments of the $e^+e^- \rightarrow e^+e^-$ cross sections in the context of Z parameter fitting have been presented. The use of the sensitivity function language has enabled us to compare quantitatively the fitting to a complete description of the process with the more commonly used t-channel subtraction method. We have shown that using the first approach might become important for an accurate determination of the Z parameters.

The function MIBA ("Model Independent Bhabha Approximation") describing the $e^+e^- \rightarrow e^+e^-$ lineshape in terms of the Z parameters has been presented. This formula has been built by properly dressing a simple Born-like structure. The implementation of the QED corrections has been done following basically the approach of ref. [3]. This approximate formula agrees with the most accurate calculation in the Standard Model to within 1 per mil.

Few important technical details of the implementation of MIBA as a fitting formula have been discussed as well as the bias in the fitted parameters as compared to the most complete Standard Model calculation. They remain well under the experimental errors for the future LEP running.

Finally, we have discussed the way in which the small corrections needed to obtain a proper matching between our fitting formula and the data corrected with an $O(\alpha)$ M.C. event generator can be estimated.

The computer program MIBA is available as a FORTRAN function from any of the authors.

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Figure 1: Feynman diagrams for Bhabha scattering at lowest order.

Figure 2: Contribution of the different amplitudes squared and interferences to the Bhabha cross section for $|\cos \theta| < 0.9$. Upper full line: total cross section; dashed line: s-channel Z amplitude squared; dotted line: t-channel photon amplitude squared; dash-dotted line: interference between s-channel Z and t-channel photon amplitudes; lower full line: all the other contributions.

$$\sigma/\sigma_{\text{TOT}}(*100)$$

Figure 3: Relative contribution of the different amplitudes squared and interferences to the Bhabha cross section for $|\cos\theta| < 0.9$. Full line: s-channel Z amplitude squared; dashed line: t-channel photon amplitude squared; dotted line: interference between s-channel Z and t-channel photon amplitudes; dash-dotted line: all the other contributions.

$S_e(\Gamma_e)$ ($\text{GeV}^{-1}\text{nb}^{1/2}$)

Figure 4: Sensitivity as a function of the minimum polar angle at 1 GeV below the Z peak. Full line: taking into account the complete matrix element squared; dotted line: taking into account only the s-channel diagrams (like $\mu^+\mu^-$ production); dashed line: taking into account the complete matrix element squared but freezing the photon-Z interference (like in the "t-channel subtraction" scheme)

$S_e(\Gamma_e) \text{ (GeV}^{-1}\text{nb}^{1/2}\text{)}$

Figure 5: Sensitivity as a function of the minimum polar angle at the Z peak. Full line: taking into account the complete matrix element squared; dotted line: taking into account only the s-channel diagrams (like $\mu^+\mu^-$ production); dashed line: taking into account the complete matrix element squared but freezing the photon-Z interference (like in the “t-channel subtraction” scheme). Since, in a tree level calculation, this interference is zero, this line is completely superimposed to the full line.

$S_e(\Gamma_e)$ ($\text{GeV}^{-1}\text{nb}^{1/2}$)

Figure 6: Sensitivity as a function of the minimum polar angle at 1 GeV above the Z peak. Full line: taking into account the complete matrix element squared; dotted line: taking into account only the s-channel diagrams (like $\mu^+\mu^-$ production); dashed line: taking into account the complete matrix element squared but freezing the photon-Z interference (like in the “t-channel subtraction” scheme)

$S_e(\Gamma_e) \text{ (GeV}^{-1}\text{nb}^{1/2}\text{)}$

Figure 7: Sensitivity as a function of the center of mass energy for $|\cos \theta| < 0.9$. Full line: taking into account the complete matrix element squared; dotted line: taking into account only the s-channel diagrams (like $\mu^+ \mu^-$ production); dashed line: taking into account the complete matrix element squared but freezing the photon-Z interference (like in the “t-channel subtraction” scheme)

Figure 8: Relative difference between the cross sections predicted by ALIBABA and MIBA (in per mil) without QED corrections in the Z peak region for $|\cos \theta| < 0.9$.

Figure 9: Relative difference between the cross sections predicted by ALIBABA and MIBA (in per mil) without QED corrections in a wide range of center of mass energies for $|\cos \theta| < 0.9$.

Figure 10: Relative difference between the complete cross sections predicted by ALIBABA and MIBA (in per mil) in the Z peak region for $-0.9 < \cos \theta^* < 0.7$ and $\xi < 20^\circ$.