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INTEGRABILITY IN RANDOM MATRIX MODELS^{*}

L. ALVAREZ-GAUMÉ

Theory Division, CERN
CH-1211 Geneva 23, Switzerland

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Some aspects of the recently discovered non-perturbative solutions to non-critical strings [1] can be better understood and clarified directly in terms of the integrability properties of the random matrix model used to define the coupling of (p, q) -conformal matter to $2D$ -gravity. Soon after the appearance of the original papers on the subject, Douglas [2] showed that the continuum limit of these models can be described in terms of special representations of the Heisenberg algebra. He also exhibited some interesting connections with the KdV hierarchy [3] which strongly indicate a deep interplay still to be elucidated between string theory and integrable models. In [4] we have made a modest attempt to clarify these connections. This lecture will be based on this work.[†]

1. *Basic Definition of Integrability.*

The notion of integrability for a classical mechanical system of n degrees of freedom goes back to the nineteenth century and is due to J. Liouville. A system with n degrees of freedom described by the Hamiltonian $H(p, q)$ and the Poisson Bracket $\{, \}$ is said to be integrable if one can find n conserved quantities $I_i(p, q)$, $i = 1 \dots n$ in involution, i.e. with vanishing Poisson brackets $\{I_i, I_j\} = 0$, $i, j = 1, \dots n$. When these conditions are met, the general solution to Hamilton's equations can in principle be reduced to quadratures. An efficient procedure to construct integrable systems is to write them in terms of Lax pairs [7]. If the dynamical equations of motion can be written in terms of two matrices L, B with L symmetric and B anti-symmetric as

$$\frac{dL}{dt} = [B, L] \quad (1)$$

then the quantities

$$I_k = \frac{1}{k} \text{tr} L^k \quad (2)$$

are automatically conserved. This is proved by showing that the eigenvalues of L are time independent. Defining the matrix U according to

$$\frac{dU}{dt} = BU \quad (3)$$

the time evolution of L implied by (1) is $L(t) = UL(0)U^{-1}$ and therefore the eigenvalues of L , $L(t)\phi_n(t) = \lambda_n\phi_n(t)$ are time independent. This way one relates the theory of Integrable Systems with isospectral problems. In general we have to check whether the eigenvalues of L Poisson-commute before we can claim that Liouville integrability is satisfied.

[†] The use of discrete integrable systems to clarify some of the properties of matrix models is also described in [5, 6] where use is made of the properties of the Toda lattice.

One of the basic examples of an integrable system is provided by the Toda lattice [8]. The Hamiltonian is given by

$$H = \frac{1}{2} \sum_n p_n^2 + \sum_n (e^{-(q_{n+1}-q_n)} - 1) \quad (4)$$

with equations of motion

$$\begin{aligned} \frac{dq_n}{dt} &= p_n \\ \frac{dp_n}{dt} &= e^{-(q_n-q_{n-1})} - e^{-(q_{n+1}-q_n)} \end{aligned} \quad (5)$$

The Lax pair form of these equations was found by Flaschka (for references see for example [8]). Defining the variables

$$a_n = \frac{1}{2} e^{(q_n-q_{n+1})/2} \quad b_n = p_n \quad (6)$$

(5) becomes

$$\begin{aligned} \dot{a}_n &= a_n(b_n - b_{n+1}) \\ \dot{b}_n &= 2(a_{n-1}^2 - a_n^2) \end{aligned} \quad (7)$$

the equations (7) can be written in matrix form (1) by introducing the matrices

$$\begin{aligned} L &= \sum_i a_i E_{i,i+1}^+ + \sum_i b_i E_{ii} \\ B &= - \sum_i a_i E_{i,i+1}^- \end{aligned} \quad (8)$$

With free boundary conditions the L -eigenvalues Poisson-commute and we have an integrable system. In the usual Toda lattice the index n runs from $-\infty$ to $+\infty$, but in the applications to the matrix models we are interested in, n runs from 0 to ∞ . The Lax pair [(1)] can also be written as the compatibility condition for an auxiliary linear problem

$$L\phi = \lambda\phi \quad \frac{d\phi}{dt} = B\phi \quad (9)$$

These two equations are compatible as long as the Lax equation is satisfied. Notice that if we think of the ϕ variables as the amplitudes of oscillators at sites n of the lattice, the

eigenvalue equation (9) takes the form of a three-step recursion relation

$$\lambda\phi_n = (L\phi)_n = a_{n+1}\phi_{n+1} + b_n\phi_n + a_{n-1}\phi_{n-1} \quad (10)$$

reminiscent of the recursion relation for orthogonal polynomials [9]. We will see that this analogy is not a mere coincidence. Although we have written the simplest time evolution equation generated by the Hamiltonian $H = H_2 = \text{tr} L^2/2$, we could instead define the time evolution equation in terms of the higher conservation laws H_k obtained by taking higher order traces of the operator L . These evolution equations are known as the higher Toda flows

$$\frac{\partial a_n}{\partial t_k} = \{H_k, a_n\} \quad \frac{\partial b_n}{\partial t_k} = \{H_k, b_n\} \quad (11)$$

where the Poisson brackets are computed with the standard Poisson bracket between p and q . In the original Toda equation we cannot consistently set $b = 0$; however, if we look at the higher flows, it is possible to construct another integrable hierarchy known as the Volterra hierarchy defined in terms of a Lax pair

$$L = \sum_{k \geq 1} \sqrt{R_k} E_{k,k+1}^+$$

satisfying $\text{tr} L^{2n+1} = 0$ and

$$B = \sum_k \sqrt{R_k R_{k+1}} E_{k,k+1}^- \quad (12)$$

yielding the equations of motion

$$\dot{R}_n = R_n(R_{n+1} - R_{n-1}) \quad (13)$$

In the continuum limit this equation reproduces the Korteweg-DeVries equation. Some of the conserved charges in the Volterra hierarchy are

$$\begin{aligned} H_1 &= \frac{1}{2} \text{tr} L^2 = \sum_n R_n \\ H_2 &= \frac{1}{4} \sum_n \frac{1}{2} R_n^2 + R_n R_{n+1} \\ H_3 &= \sum_n \left(\frac{1}{3} R_n^3 + R_n R_{n+1}^2 + R_n R_{n+1} R_{n+2} \right) \end{aligned} \quad (14)$$

The Hamiltonian structure which produces (13) out of H_1 is

$$\{R_n, R_m\} = R_n R_m (\delta_{n,m+1} - \delta_{n,m-1}) \quad (15)$$

It is a relatively tedious exercise to verify the Jacobi identity for this Poisson bracket. The explicit form of the Hamiltonians (14) can be obtained using the staircase procedure explained in [10]. The basic properties of the Volterra hierarchy can be summarized in the following results:

Theorem 1. The k -th flow of the Volterra hierarchy can be written as a Lax pair

$$\frac{\partial L}{\partial t_k} = [(L_+^{2k}, L] \quad (16)$$

where L_+^{2k} is the antisymmetric matrix whose upper triangular part coincides with L^{2k} .

The second important result is

Theorem 2. There is a second Hamiltonian structure for the Volterra hierarchy compatible with the first (15) one [11]

$$\{R_n, R_m\}_2 = R_n R_m (R_n + R_m) (\delta_{n,m+1} - \delta_{n,m-1}) + R_n R_m (\delta_{n,m+2} - \delta_{n,m-2}) \quad (17)$$

It is an unpleasant exercise to verify the Jacobi identity for [(17)]. The compatibility between the two Poisson structures is equivalent to saying that any linear combination of the two is again a Poisson structure. It satisfies the Jacobi identity. When this happens, this is a very strong signal about the Liouville integrability of the theory. Finally

Theorem 3. The following identity (analogous to the Gel'fand-Dickii [12] relation in KdV) is satisfied,

$$\{H_n, R_m\}_2 = \{H_{n+1}, R_m\}_1 \quad (18)$$

These three theorems guarantee the complete integrability of the Volterra hierarchy. Notice that if we assume some homogeneity properties of the Hamiltonians H_n with respect to their R_n dependence, we can use (18) to compute all Hamiltonians starting from H_1 . These properties play an important role in the analysis of the double scaling limit for this hierarchy. We point out that the second Poisson structure produces in the continuum limit the Virasoro algebra [13], and, as stressed by the authors of reference [13] it can be taken as the starting point in the lattice quantization of Liouville theory.

2. Relation with One-Matrix Models

Given some even potential $V(\lambda) = \sum g_p \lambda^{2p}$ and a basis of orthonormal polynomials $P_n(\lambda)$ with respect to the measure $e^{-V(\lambda)}$, the operation of multiplication by λ is represented by a Jacobi matrix L of the same form as in the previous section. As we change the couplings g_i , the polynomials $P_n(\lambda)$ also change:

$$\frac{\partial P_n}{\partial g_i} = (M_i)_{n\ell} P_\ell \quad ()$$

Since λ is independent of the couplings, differentiating $\lambda P_n = L_{nk} P_k$ we obtain

$$\frac{\partial L}{\partial g_i} = [M_i, L] \quad (19)$$

The couplings g_i are all independent and therefore we can vary them independently, implying that the g_i -flows commute. In terms of the matrices M_i this implies

$$\frac{\partial}{\partial g_i} M_j - \frac{\partial}{\partial g_j} M_i - [M_i, M_j] = 0 \quad (20)$$

In the space of couplings $A = M_i dg^i$ represents a flat $O(n)$ gauge field and (19) means that L changes by parallel transport in the presence of the gauge connection $A(g)$. The matrices M_i are antisymmetric and they should be local to have a continuum limit. In this context, locality means that $(M_i)_{mn} = 0$ for $|m - n| > p$ finite. We could modify M_i by adding a symmetric matrix X . However, the symmetry of $\partial L / \partial g_i$ requires $[X, L] = 0$. If L is generic (all its eigenvalues are distinct) then X has to be a polynomial in L and therefore it is irrelevant in the construction of the flows. The matrices M_i are determined by the conditions (1) M_i is antisymmetric, (2) $[M_i, L]$ is a Jacobi matrix of locality one, and (3) M_i is homogeneous of degree $2i$ in the matrix elements of L . This is so because $V = \sum g_p \lambda^{2p}$ is unchanged under $\lambda \mapsto \alpha \lambda$ accompanied by $g_p \mapsto g_p \alpha^{-2p}$. These three conditions fix M_i to be given by L_+^{2i} , the matrices defined in the previous section:

$$\frac{\partial L}{\partial g_i} = [L_+^{2i}, L] \quad (21)$$

The flows are generated only by the even powers L_+^{2i} . We know that the only non-vanishing matrix elements of $[L_+^{2i}, L]$ are $(m, m \pm 1)$. If instead we considered $[L_+^{2i+1}, L]$, only the diagonal matrix elements would be different from zero. Consequently, L_+^{2i} generates flows and L_+^{2i+1} provides initial conditions. This cannot be seen so clearly in the continuum limit, where the operators generated by $[L_+^{2n}, L]$ and $[L_+^{2n+1}, L]$ differ only by a total derivative.

Summarizing, $L(g_i)$ is an orbit of the Volterra flows. What is left to do is to determine the initial conditions for the flows. If $V(\lambda)$ is a polynomial, then the operator $d/d\lambda$ is also local (in the Jacobi sense), and it can be chosen antisymmetric. Write

$$\frac{d}{d\lambda}P_k = \sum N_{k,r}P_r \quad (22)$$

Then

$$\begin{aligned} 0 &= \int d\lambda \frac{d}{d\lambda} \left(e^{-V} P_k P_l \right) \\ &= - \int d\lambda e^{-V(\lambda)} V'(\lambda) P_k P_l + N_{kl} + N_{lk} \end{aligned} \quad (23)$$

and hence $N - V'(L)/2$ is antisymmetric, and if $n = \text{degree}(V)$, then $N_{kr} = 0$ unless $|k - r| \leq n - 1$. We can choose N to be antisymmetric. Therefore, we can expand N as a linear combination of L_+^p . Since $[N, L] = \mathbf{1}$, this means that only odd powers of p appear, and therefore $[N, L] = \mathbf{1}$ becomes a set of initial conditions for the Volterra hierarchy

$$\sum_{j=1}^n 2j g_j^{(0)} [L_+^{2j-1}, L] = \mathbf{1} \quad (24)$$

In conclusion, the generic one-matrix model is equivalent to a Volterra hierarchy with a particular initial condition. It is possible to show that this initial condition does not intersect any of the multi-soliton sectors of the Volterra equation. To study the mechanical properties of this system, we may define a Volterra equation with a finite number of points, instead of considering the equation on the semi-infinite line. For a finite number of particles the initial condition becomes

$$[N, L] = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & -n+1 \end{pmatrix} \quad (25)$$

The converse of this result can also be proven. Starting with the Volterra hierarchy and taking into account that $R_n > 0$, we can use Favard's theorem [9] which guarantees that, given a set $\phi_n(\lambda)$ ($n \geq 0$) satisfying $\lambda\phi_n = \phi_{n+1} + R_n\phi_{n-1}$ with $R_n > 0$, we can always find a measure $d\lambda e^{-V(\lambda)}$ making the ϕ_n orthogonal. This measure is unique up to an overall

constant depending on the normalization of the ϕ_n 's. The initial condition (24) implies that $V(\lambda) = \sum g_p^{(0)} \lambda^{2p}$. The string equations can be written in Hamiltonian form:

$$\sum_j 2j g_j R_m \frac{\partial H_j}{\partial R_m} = m\beta^{-1} \quad (26)$$

where we have taken the potential to be $V(\lambda) = \beta \sum g_p \lambda^{2p}$ with $\beta = e^{\Lambda_B}$, and Λ_B the bare cosmological constant. In terms of the partition function $Z = \prod_n R_n^{N-n}$, equation (26) becomes

$$\begin{aligned} \sum_j j g_j \frac{\partial \log Z}{\partial u_m} &= m\beta^{-1} \\ u_m &= \log R_m \end{aligned} \quad (27)$$

Next, we want to show that in the double scaling limit the Volterra hierarchy turns into the KdV hierarchy. In this way, we will reproduce directly the results in the original papers on the double scaling limit. The strategy is to use the fact that in the KdV hierarchy we also have a Gel'fand–Dikii [12] relation giving recursion relations among different members of the hierarchy. We recapitulate these results here. The original KdV equation is defined in terms of a Schrödinger operator

$$L = \frac{d^2}{dx^2} - u(x) \quad (28)$$

and a Lax pair representation

$$\frac{\partial L}{\partial t} = [B, L]$$

with B a third-order differential operator

$$\begin{aligned} B &= D^3 - \frac{3}{2}uD - \frac{3}{4}u' \\ D &\equiv \frac{\partial}{\partial x} \quad U' \equiv \frac{\partial u}{\partial x} \end{aligned} \quad (29)$$

For the higher flows it can be shown in general that

$$\frac{\partial L}{\partial t_n} = [L_+^{n+1/2}, L] \quad n = 1, 2, \dots \quad (30)$$

where $L^{1/2}$ is defined as the pseudo-differential operator with expansion

$$L^{1/2} = D + q_1 D^{-1} + q_2 D^{-2} + \dots \quad (31)$$

and for any pseudo-differential operator Q , Q_+ means that we keep only its differential part. Notice that in the scaling limit the operator of multiplication by λ becomes a

second-order differential operator. The two-step recursion relation is a discrete version of the Schrödinger operator (28). The equations (30) can also be derived from a collection of commuting Hamiltonians H_n . There are again two Poisson structures. The first one is

$$\{u(x), u(y)\}_1 = \frac{1}{2}(\partial_x - \partial_y)\delta(x - y) \quad (32)$$

and the second one is the classical version of the Virasoro algebra:

$$\{u(x), u(y)\}_2 = \left(\frac{1}{2}D^3 + 2uD + u'\right)\delta(x - y) \quad (33)$$

These two Poisson structures are compatible and we can construct the commuting Hamiltonians using the analogue of the Gel'fand–Dik'ii relation. Using the first Hamiltonian structure,

$$\frac{\partial u}{\partial t_n} = D \frac{\delta H_n}{\delta u(x)} \equiv DR_{n+1}[u] \quad (34)$$

and the Gel'fand–Dik'ii relation now implies

$$DR_{k+1} = \left(\frac{1}{2}D^3 - 2uD - u'\right)R_k(u) \quad (35)$$

Starting with the simplest possible $R_0 = 1$, we obtain all others $R_1 = -u$, $R_2 = \frac{3}{2}u^2 - \frac{1}{2}u''$, etc. The variables R_m are not good scaling variables. For the k -th multi-critical point we have

$$\begin{aligned} x = \frac{n}{\beta} &= 1 - \beta^{-\frac{2k}{2k+1}}t \\ R &= 1 - \beta^{-\frac{2}{2k+1}}f(t) \end{aligned} \quad (36)$$

In terms of the variable $u_m = \log R_m$ and setting $\lambda = \beta^{-1/2k+1}$, the first Poisson structure

$$\{u_n, u_m\} = \delta_{n,m+1} - \delta_{n,m-1}$$

becomes in the continuum limit $u_m \mapsto -\lambda^2 f_m \mapsto -\lambda^2 f(t)$:

$$\{f(t), f(t')\}_1 = \lim_{\lambda \rightarrow 0} \lambda^2 \{f_n, f_m\} \quad (37)$$

Next we construct the scaling limit of the second Poisson structure. To obtain a second Poisson structure with good scaling properties, we have to subtract the first Poisson

structure with a coefficient depending appropriately on the scaling parameter λ . This is legitimate because the two Hamiltonian structures are compatible for any value of λ . Thus the second Hamiltonian structure becomes in the continuum limit:

$$\{f(t), f(t')\}_2 = \left(\frac{1}{2} D^3 - 2fD - \dot{f} \right) \delta(t - t') \quad (38)$$

and now we can immediately construct the combination of Hamiltonians H_n with the good scaling properties (with respect to (36)). These Hamiltonians are the ones which will give the string equations at each of the multi-critical points:

$$\mathcal{H}_k = \sum_{j=1}^k 2j \bar{g}_j^{(k)} H_j \quad (39)$$

where $\bar{g}_j^{(k)}$ are the couplings chosen to give good scaling properties. We actually do not need to do any work to find the explicit form of $\mathcal{H}_k$. It is given automatically by the Gel'fand–Dik'ii relation in the continuum limit. For the k -th multi-critical point the string equation is written as

$$\sum_j 2j \bar{g}_j^{(k)} \frac{\partial H_j}{\partial u_m} = m\beta^{-1} = \frac{\partial \mathcal{H}_k}{\partial u_m}$$

which in the scaling limit becomes, up to some constants,

$$\delta \mathcal{H}_k / \delta f(t)$$

and $\mathcal{H}_k$ is the KdV Hamiltonian containing a term proportional to f^k . Hence the string equation for the k -th Kazakov multi-critical point becomes

$$R_k[f] = t \quad (40)$$

Defining now $D \equiv d/dt$, we can also differentiate (40) to obtain

$$\frac{\partial f}{\partial t_k} = D R_k[f] = 1 \quad (41)$$

and using the Lax pair representation of the KdV hierarchy this implies

$$\frac{\partial L}{\partial t_k} = [L_+^{k+1/2}, L] = 1 \quad (42)$$

Hence, in the continuum limit the operator $d/d\lambda$ becomes $L_+^{k+1/2}$. If we write

$$Q \equiv L \quad P \equiv L_+^{k+1/2}$$

then (42) takes the form of a Heisenberg algebra

$$[P, Q] = 1 \tag{43}$$

The string equations were written in this form for all multi-matrix models in a beautiful paper by M. Douglas [2] . Once we know the scaling operators $\mathcal{H}_k$, we can perturb the critical point by adding them with arbitrary couplings

$$\sum_{l,k} \mu_l \alpha_l^{(k)}(\beta) \frac{\partial \mathcal{H}_k}{\partial u_m} = m\beta^{-1} \tag{44}$$

where the constants $\alpha_l^{(k)}(\beta)$ have a well-defined scaling limit. The answer one obtains is then

$$\sum_{n=1}^{\infty} (2n+1)t_n R_n[u] = -t$$

where $t_k = -1/(2k+1)$, $t_n = 0$ for $n \neq k$, defines the k -th critical point.

3. Discussion

We have shown in the previous paragraphs that the integrability structure present in the double scaling limit of the one matrix model is a consequence of the fact that this model for an arbitrary even potential is equivalent to an integrable Volterra hierarchy, even before we take the continuum limit. From this point of view it is natural to expect that the continuum partition function satisfies a set of Virasoro constraints as found originally by [14, 15] because they are induced by the integrability structure already included in the Volterra hierarchy and its second Hamiltonian structure. When one studies the multimatrix models, the corresponding constraints can be written in terms of W -algebras. Similar structures however do not seem to appear in the case of the $c = 1$ theory. It may be true that the integrable structures presented in this lecture are only possible for strings propagating in discrete space-times. It is nevertheless quite intriguing that the second Hamiltonian structure for the Volterra hierarchy is a lattice representation of the Virasoro algebra, and it can be taken as the starting point for the quantization of Liouville theory. This approach may provide new insights into the understanding of the connection between matrix models and Liouville theory.

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