CERN-TH.6119/91

GAUGE-INDEPENDENT ANALYSIS OF 2D-GRAVITY

E. Abdalla*

University of Sao Paulo
Cep:20516, Sao Paulo, Brazil

M.C.B. Abdalla*

Instituto de Fisica Teorica/UNESP
Cep:01405, Sao Paulo, Brazil

J. Gamboa*

Departamento de Fisica Teorica, Univ. de Zaragoza
Zaragoza 50009, Spain

A. Zadra⁺

CERN - Theory Division
CH-1211, Geneve 23, Switzerland

ABSTRACT

In this note we show that the induced 2D-gravity $SL(2, R)$ currents can be defined in a gauge-independent way although they manifest themselves as generators of residual symmetries only in some special gauges. In the Coulomb gas representation we investigate two approaches, namely one resembling string field theory and another that emphasizes the $SL(2, R)$ structure in the phase space. In the conformal gauge we propose a solution of the Liouville theory in terms of the $SL(2, R)$ currents.

CERN-TH.6119/91
June 1991

---

* Partially supported by CNPq.

* Partially supported by MEC- CICYT.

+ Supported by World Laboratory/CERN.

1. Introduction

Two dimensional induced gravity has received much attention in the last few years¹. Although Einstein gravity in two dimensional space-time is dynamically trivial (at least at the classical level, since the action is a topological invariant²), integration over the matter fields induces a non-trivial effective action for the metric. It is given by³

$$S = \kappa \int \frac{d^2 p}{(2\pi)^2} \left(\sqrt{-\tilde{g}} \tilde{R} \frac{1}{-p^2} \tilde{R} - \mu^2 \sqrt{-\tilde{g}} \right) , \quad (1)$$

where $\tilde{R}$ is the curvature, $-p^2$ is the d'Alembert operator (all in momentum space), $\kappa = c/96\pi$ is a function of the central charge of the original theory, and μ^2 is the cosmological constant.

As discussed at length by several authors⁴, there exists a (hidden) $SL(2, R)$ current algebra in the action (1). We will show that those $SL(2, R)$ currents - first emerging in the light-cone formulation (section 1) - can be defined in a gauge-independent way (section 2), although they appear as residual symmetry generators only in some special gauges.

Using the Coulomb gas representation⁵ of (1) and applying well known Hamiltonian methods for constrained systems⁶, we investigate two approaches for induced 2D-gravity, namely one that resembles string theory (section 3) and another that emphasizes the underlying $SL(2, R)$ currents structure (section 4).

Section 5 is devoted to the conformal gauge and we propose a solution of the Liouville theory in terms of the $SL(2, R)$ currents. Conclusions are drawn in section 6.

1. $SL(2, R)$ symmetry in the light-cone gauge

We define the metric as usual

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} \\ g_{01} & g_{11} \end{pmatrix} \quad ; \quad g = g_{00}g_{11} - g_{01}^2 , \quad (2)$$

taking g_{00} , g_{01} and g_{11} as independent fields. The action (1) may be written as a local functional if we introduce an auxiliary scalar field φ , as in

$$S = \frac{1}{2} \int d^2 x \sqrt{-g} (-\varphi \square \varphi - \alpha R \varphi + \alpha^2 \beta) , \quad (3)$$

where

$$\alpha^2 = 8\kappa - \frac{1}{12\pi} = \frac{c-1}{12\pi} , \quad (4)$$

$$\beta = -\mu^2 \left(\frac{2\kappa}{\alpha^2} \right) , \quad (5)$$

The term $-1/12\pi$ in (4) - a shift in the central charge value - arises from the determinant of φ and compensates the contribution of this auxiliary scalar field to the central charge¹⁾. After some integrations by parts the action (3) can be written as

$$S = \int d^2x \mathcal{L} \quad ,$$

$$\mathcal{L} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{\alpha}{2\sqrt{-g}} (\epsilon^{\mu\alpha} \partial_\alpha \varphi) (\epsilon^{\nu\beta} \partial_\beta g_{\mu\nu}) + \frac{1}{2} \alpha^2 \beta \sqrt{-g} - \frac{\alpha}{8} \frac{\varphi}{\sqrt{-g}} g^{\alpha\beta} \epsilon^{\rho\sigma} \epsilon^{\mu\nu} (\partial_\mu g_{\alpha\rho}) (\partial_\nu g_{\beta\sigma}) \quad , \quad (6)$$

with $\epsilon^{01} = 1$. The Lagrangian above is suitable for the analysis of gauge fixing, residual symmetries and the Hamiltonian description of the model. Let us start from the light-cone gauge (where the theory simplifies dramatically), defined by the choice²⁾

$$g_{\mu\nu} = \begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--} \end{pmatrix} = \begin{pmatrix} -2h_{++} & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad (7)$$

so that we have the (very simple) expressions

$$R = -2\partial_-^2 h_{++} \quad , \quad \sqrt{-g} = 1 \quad , \quad (8)$$

$$\square \varphi = 2\partial_- (\partial_+ + h_{++} \partial_-) \varphi \quad , \quad (9)$$

and the Lagrangian (6) is reduced to

$$\mathcal{L} = \partial_+ \varphi \partial_- \varphi + h_{++} (\partial_- \varphi)^2 - \alpha \partial_- h_{++} \partial_- \varphi + \frac{1}{2} \alpha^2 \beta \quad . \quad (10)$$

The equation of motion for h_{++} defines the $(--)$ component of the energy-momentum tensor

$$T_{--} = (\partial_- \varphi)^2 + \alpha \partial_-^2 \varphi \quad . \quad (11)$$

With an appropriate normalization, we find that the central charge is c , which corresponds to the “semi-classical” result.

It is easy to verify³⁾ that the Lagrangian (10) is invariant³⁾ under the (residual) symmetry transformations

$$\delta \varphi = \varepsilon^- \partial_- \varphi - \frac{1}{2} \alpha \partial_- \varepsilon^- \quad , \quad (12a)$$

$$\delta h_{++} = -\partial_+ \varepsilon^- + \varepsilon^- \partial_- h_{++} - h_{++} \partial_- \varepsilon^- \quad , \quad (12b)$$

¹⁾ We could as well consider a classical auxiliary field, but we will rather stay with the above procedure.

²⁾ $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$; $\partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$.

³⁾ Up to divergencies.

with

$$\partial_-^3 \varepsilon^- = 0 \quad . \quad (12c)$$

Following the canonical procedure, we find the momenta conjugate to φ and h_{++} ,

$$\pi_\varphi = (1 + h_{++})(\dot{\varphi} - \varphi') - \varphi' - \frac{1}{2}\alpha(\dot{h}_{++} - h'_{++}) \quad , \quad (13a)$$

$$\pi_{h_{++}} = -\frac{1}{2}\alpha(\dot{\varphi} - \varphi') \quad . \quad (13b)$$

and equal-time Poisson brackets $\{\varphi(x), \pi_\varphi(y)\} = \delta(x - y)$, $\{h_{++}(x), \pi_{h_{++}}(y)\} = \delta(x - y)$. The transformations (12) are generated by the charge $Q_{\varepsilon^-} = \int dx \mathcal{J}^0$, where

$$\begin{aligned} \mathcal{J}^0 = & \varepsilon^- \left[-\frac{1}{\sqrt{2}}[(1 + h_{++})T_{--} - \frac{1}{2}\alpha^2 \partial_-^2 h_{++}] \right] \\ & + \partial_- \varepsilon^- \left[-\frac{1}{2\sqrt{2}}\alpha^2 \partial_- h_{++} \right] + \partial_-^2 \varepsilon^- \left[\frac{1}{2\sqrt{2}}\alpha^2 h_{++} \right] \quad . \end{aligned} \quad (14)$$

If we solve (12c) defining the decomposition

$$\varepsilon^- = \frac{1}{\sqrt{2}} [\zeta^-(x^+) + 2x^- \zeta^0(x^+) + (x^-)^2 \zeta^+(x^+)] \quad , \quad (15)$$

the generator can be correspondingly decomposed as

$$\mathcal{J}^0 = \frac{1}{2}\zeta^-(x^+)J^+ - \zeta^0(x^+)J^0 + \frac{1}{2}\zeta^+(x^+)J^- \quad , \quad (16)$$

where

$$J^+ = - \left[(1 + h_{++})T_{--} - \frac{1}{2}\alpha^2 \partial_-^2 h_{++} \right] \quad , \quad (17a)$$

$$J^0 = \frac{1}{2\sqrt{2}}\alpha^2 \partial_- h_{++} - x^- J^+ \quad , \quad (17b)$$

$$J^- = \alpha^2 h_{++} - 2x^- J^0 - (x^-)^2 J^+ \quad . \quad (17c)$$

In terms of the canonical commutation relations defined by (13), the currents (17) turn out to obey an $SL(2, R)$ Kac-Moody algebra,

$$\{J^a(x), J^b(y)\} = -2\sqrt{2}\epsilon^{abc}\eta_{cd}J^d(x)\delta(x - y) + \alpha^2\eta^{ab}\delta'(x - y) \quad , \quad (18)$$

with

$$\eta^{ab} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \quad , \quad \epsilon^{-0+} = 1 \quad .$$

After building the Hamiltonian corresponding to (10) we also verify that the currents (17) satisfy the following off-shell condition

$$\frac{\partial}{\partial t} J^a + \{J^a, H\} = \frac{\partial}{\partial x} J^a \quad , \quad a = -, 0, + \quad , \quad (19)$$

implying that they are chiral (and therefore conserved) on-shell, $\partial_- J^a = 0$, which together with (17c) gives the well-known solution for h_{++}

$$\frac{\alpha^2}{\sqrt{2}} h_{++} = J^-(x^+) + 2x^- J^0(x^+) + (x^-)^2 J^+(x^+) \quad . \quad (20)$$

Now we relate this theory to that defined in the conformal gauge in order to understand the results obtained here in the framework of other gauge conditions.

2. Gauge-independent Hamiltonian formulation

We have considered the Lagrangian (6) as a starting point for the Hamiltonian formulation of 2D-gravity before fixing the gauge. After a quite long algebraic exercise, the following results are obtained:

- (a) the momenta conjugate to g_{00} and g_{01} (π^{00} and π^{01} respectively) are constrained (a pair of abelian primary constraints);
- (b) the primary constraints generate two secondary constraints;
- (c) altogether the constraints form a consistent first-class system, which generates local coordinate transformations (gauge symmetry);
- (d) the light-cone and conformal gauges can then be worked out by fixing g_{00} and g_{01} as appropriate functions of g_{11} .

All these results are consistent with previous analysis but the expressions are not transparent enough to lead us to any further conclusions. However it turns out that the Hamiltonian formulation simplifies if we add a surface term⁷ as follows

$$\mathcal{L} \rightarrow \bar{\mathcal{L}} = \mathcal{L} + \partial_\mu \left[\frac{\varphi}{2\sqrt{-g}} \frac{1}{g_{11}} \varepsilon^{\mu\nu} \varepsilon^{\alpha\beta} g_{1\alpha} \partial_\nu g_{1\beta} \right] \quad , \quad (21)$$

or explicitly

$$\begin{aligned} \bar{\mathcal{L}} = \frac{1}{2\sqrt{-g}} \Big[& (-g_{11}\dot{\varphi}^2 + 2g_{01}\dot{\varphi}\varphi' - g_{00}\varphi'^2) + \alpha(\dot{g}_{11}\dot{\varphi} - 2g'_{01}\dot{\varphi} + g'_{00}\varphi') + \\ & + \alpha \frac{g_{01}}{g_{11}} (g'_{11}\dot{\varphi} - \dot{g}_{11}\varphi') - \alpha^2 \beta g \Big] \quad . \quad (22) \end{aligned}$$

The expressions for the momenta conjugate to φ and g_{11} become

$$\pi_\varphi = \frac{1}{\sqrt{-g}}(g_{01}\varphi' - g_{11}\dot{\varphi}) + \frac{\alpha}{2\sqrt{-g}}\left(\dot{g}_{11} - 2g'_{01} + \frac{g_{01}}{g_{11}}g'_{11}\right) , \quad (23a)$$

$$\pi^{11} = \frac{\alpha}{2\sqrt{-g}}\left(\dot{\varphi} - \frac{g_{01}}{g_{11}}\varphi'\right) , \quad (23b)$$

from which $\dot{\varphi}$ and $\dot{g}_{11}$ can be inverted. On the other hand the “velocities” $\dot{g}_{00}$ and $\dot{g}_{01}$ disappear after a total divergence is added as in (21), implying that the momenta conjugate to g_{00} and g_{01} are constrained to vanish,

$$\Gamma_1 = \pi^{00} \approx 0 , \quad (24a)$$

$$\Gamma_2 = \pi^{01} \approx 0 . \quad (24b)$$

The canonical Hamiltonian now reads

$$H_c = \int dx \left[-\frac{\sqrt{-g}}{g_{11}}\phi_1 + \frac{g_{01}}{g_{11}}\phi_2 \right] , \quad (25)$$

where

$$\phi_1 = \frac{1}{2} \left[\varphi'^2 - \frac{4}{\alpha^2}(g_{11}\pi^{11})^2 - \frac{4}{\alpha}(g_{11}\pi^{11})\pi - \alpha \frac{g'_{11}}{g_{11}}\varphi' + 2\alpha\varphi'' + \alpha^2\beta g_{11} \right] , \quad (26a)$$

$$\phi_2 = \pi\varphi' - 2g_{11}\pi^{11'} - \pi^{11}g'_{11} . \quad (26b)$$

Notice that g_{00} and g_{01} are absent in (26). These metric components show up in the Hamiltonian (25) as sort of Lagrange multipliers. Therefore consistency conditions for the primary constraints (24) lead to the following secondary constraints,

$$\phi_1 \approx 0 , \quad (27a)$$

$$\phi_2 \approx 0 , \quad (27a)$$

which obey the diffeomorphism algebra,

$$\begin{aligned} \{\phi_1(x), \phi_1(y)\} &= [\phi_2(x) + \phi_2(y)]\delta'(x-y) , \\ \{\phi_1(x), \phi_2(y)\} &= -[\phi_1(x) + \phi_1(y)]\delta'(x-y) , \\ \{\phi_2(x), \phi_2(y)\} &= [\phi_2(x) + \phi_2(y)]\delta'(x-y) . \end{aligned} \quad (28)$$

The complete set of constraints is of first class type and generates gauge transformations. We can choose a gauge by fixing the components g_{00} and g_{01} . The light-cone gauge, for instance, is obtained from

$$g_{00} = g_{11} + 2 , \quad (29a)$$

$$g_{01} = g_{11} + 1 , \quad (29b)$$

followed by the canonical transformations

$$\begin{aligned}
\varphi &\rightarrow \varphi \quad , \\
\pi_\varphi &\rightarrow \pi = \pi_\varphi - \frac{\alpha}{2} \frac{g'_{11}}{g_{11}} \quad , \\
g_{11} &\rightarrow h_{++} = -(g_{11} + 1) \quad , \\
\pi^{11} &\rightarrow \pi_{h_{++}} = - \left(\pi^{11} + \frac{\alpha}{2} \frac{\varphi'}{g_{11}} \right) \quad ,
\end{aligned} \tag{30}$$

while the conformal gauge is given by

$$g_{00} = -g_{11} \quad , \tag{31a}$$

$$g_{01} = 0 \quad , \tag{31b}$$

plus the following canonical change of variables,

$$\begin{aligned}
\varphi &\rightarrow \chi = \varphi - \frac{\alpha}{2} \ln |g_{11}| \quad , \\
\pi_\varphi &\rightarrow \pi_\chi = \pi_\varphi \quad , \\
g_{11} &\rightarrow \phi = \frac{\alpha}{2} \ln |g_{11}| \quad , \\
\pi^{11} &\rightarrow \pi_\phi = \frac{2}{\alpha} g_{11} \pi^{11} + \pi_\varphi \quad .
\end{aligned} \tag{32}$$

Before imposing either (29) or (31), let us explore the gauge-independent properties of the secondary constraints.

3. String-like approach

In this section we wish to point out that the secondary constraints ϕ_1 and ϕ_2 may be rewritten in such a way that their functional forms resemble string theory. Indeed, in terms of the canonical variables defined in (32), we have

$$\phi_1 = \frac{1}{2} \chi'^2 + \frac{1}{2} \pi_\chi^2 + \alpha \chi'' - \frac{1}{2} \phi'^2 - \frac{1}{2} \pi_\phi^2 + \alpha \phi'' - \frac{\alpha^2 \beta}{2} e^{\frac{2}{\alpha} \phi} \approx 0 \quad , \tag{33a}$$

$$\phi_2 = \pi_\chi \chi' + \alpha \pi'_\chi + \pi_\phi \phi' - \alpha \pi'_\phi \approx 0 \quad , \tag{33b}$$

which is equivalent to imposing the constraints

$$\begin{aligned}
T_\pm &= \frac{\phi_1 \pm \phi_2}{2} \\
&= \left(\frac{1}{4} h_\pm^2 \pm \frac{\alpha}{2} h'_\pm \right) - \left(\frac{1}{4} j_\pm^2 \pm \frac{\alpha}{2} j'_\pm \right) - \frac{\alpha^2 \beta}{4} e^{\frac{2}{\alpha} \phi} \approx 0 \quad ,
\end{aligned} \tag{34}$$

where

$$h_{\pm} = (\pi_{\chi} \pm \chi') \quad , \quad (35a)$$

$$j_{\pm} = (\pi_{\phi} \mp \phi') \quad . \quad (35b)$$

Given the boundary conditions, the ϕ and χ fields (and their conjugate momenta) can be expressed in terms of $h_{\pm}$ and $j_{\pm}$. We notice that the constraints (34) have the form of a “Sugawara-type” term plus total derivatives and a cosmological term. From the Poisson algebra of the new set of variables,

$$\begin{aligned} \{h_a(x), h_b(y)\} &= 2a\delta_{ab}\delta'(x-y) \quad , \\ \{j_a(x), j_b(y)\} &= -2a\delta_{ab}\delta'(x-y) \quad , \quad a, b = \pm \quad , \end{aligned} \quad (36)$$

we verify that the constraints obey a pair of (mutually commuting) Virasoro algebras,

$$\begin{aligned} \{T_{\pm}(x), T_{\pm}(y)\} &= \mp(T_{\pm}(x) + T_{\pm}(y))\delta'(x-y) \quad , \\ \{T_+(x), T_-(y)\} &= 0 \quad . \end{aligned} \quad (37)$$

The Hamiltonian can be written as

$$H_c = \int dx (u_- T_+ + u_+ T_-) \quad , \quad (38)$$

where

$$u_{\pm} = -\frac{(\sqrt{-g} \pm g_{01})}{g_{11}} \quad (39)$$

play the role of Lagrange multipliers, so that fixing $u_{\pm}$ corresponds to choosing a gauge in this formulation⁴⁾. If the Hamiltonian (38) clearly depends on such a choice, we remark that the constraints (34) as well as their Virasoro-type algebra are actually gauge-independent, and in analogy with string theory they can be taken as a starting point for quantization. If we take the Fourier transformation of the constraint T_+ ,

$$L_n = \int_{-\pi}^{\pi} dx e^{inx} T_+(x) \quad , \quad (40)$$

we have

$$\{L_n, L_m\} = i(n-m)L_{n+m} \quad , \quad (41)$$

and if now we carry on the analogy with strings, defining the oscillator variables

$$\alpha_n = \int_{-\pi}^{\pi} dx e^{inx} h_+(x) \quad , \quad (42a)$$

$$\beta_n = \int_{-\pi}^{\pi} dx e^{inx} j_+(x) \quad , \quad (42b)$$

---

⁴⁾ The conformal gauge corresponds to $u_+ = u_- = 1$, while the light-cone gauge is given by $u_+ = \frac{(1-h_{++})}{(1+h_{++})}$ and $u_- = 1$ (recall (29)).

which obey

$$\{\alpha_n, \alpha_m\} = -2in\delta_{n,-m} \quad , \quad (43a)$$

$$\{\beta_n, \beta_m\} = 2in\delta_{n,-m} \quad , \quad (43b)$$

and considering the special case where the cosmological term is absent ($\beta = 0$), then the Virasoro operators assume a familiar form

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \alpha_m - \frac{1}{2} \sum_m \beta_{n-m} \beta_m - in(\alpha_n - \beta_n) \quad . \quad (44)$$

At the quantum level the operators (44) must be ordered. If we consider a 2D quantum gravity theory described by the two sets of operators $\alpha_n^\pm$ and $\beta_n^\pm$ (creating and annihilating quanta associated with the fields $h_\pm$ and $j_\pm$) with commutation relations taken from (43), then we can define the quantum Virasoro generators using normal ordering,

$$L_n = \frac{1}{2} \sum_m : \alpha_{n-m} \alpha_m : - \frac{1}{2} \sum_m : \beta_{n-m} \beta_m : - in(\alpha_n - \beta_n) \quad , \quad (45)$$

and the vacuum state $|0\rangle$ obeying

$$L_n|0\rangle = 0 \quad , \quad n > 0 \quad (46)$$

and

$$L_0|0\rangle = \alpha_0|0\rangle \quad . \quad (47)$$

If we follow such a prescription, we find a non-zero central charge in the Virasoro algebra and we conclude that this quantum theory of gravity is anomalous.

The next section will be devoted to an alternative gauge-independent formulation of the model, which replaces the string-like variables by a set of current-like fields, in such a way that we are led to a consistent quantum theory (even if the cosmological constant is non-vanishing).

4. $SL(2, R)$ description

Inspired by earlier analysis in the light-cone gauge, we define the following set of variables

$$\begin{aligned} J^+ &= \frac{1}{g_{11}}(\phi_2 - \phi_1) + \frac{\alpha^2 \beta}{2} \\ &= -2 \left(\pi^{11} + \frac{\alpha}{2} \frac{\varphi'}{g_{11}} \right)' + \frac{2}{\alpha} \left(\pi^{11} + \frac{\alpha}{2} \frac{\varphi'}{g_{11}} \right) \left[\left(\pi - \frac{\alpha}{2} \frac{g'_{11}}{g_{11}} - \varphi' \right) + \frac{1}{\alpha} g_{11} \left(\pi^{11} + \frac{\alpha}{2} \frac{\varphi'}{g_{11}} \right) \right] \end{aligned} \quad (48)$$

$$J^0 = j^0 - x^- J^+ \\ j^0 = \sqrt{2} \left[g_{11} \left(\pi^{11} + \frac{\alpha}{2} \frac{\varphi'}{g_{11}} \right) + \frac{\alpha}{2} \left(\pi - \frac{\alpha}{2} \frac{g'_{11}}{g_{11}} - \varphi' \right) \right] \quad ; \quad (49)$$

$$J^- = j^- - 2x^- J^0 - (x^-)^2 J^+ \quad , \\ j^- = \alpha^2 (g_{11} + 1) \quad ; \quad (50)$$

$$b = \pi - \frac{\alpha}{2} \frac{g'_{11}}{g_{11}} + \varphi' \quad . \quad (51)$$

We can easily invert eq. (50) and express the graviton field in terms of the J -variables:

$$g_{11} = -1 + \frac{1}{\alpha^2} [J^- + 2x^- J^0 + (x^-)^2 J^+] \quad , \quad (52)$$

while π^{11} , φ and π can be solved using (48), (49) and (51) with appropriate boundary conditions. It is straightforward to verify that the J -variables coincide with the currents (16) found in the light-cone gauge if we make the canonical transformation (30), but we emphasize that the set of variables (J^+, J^0, J^-, b) is now defined in a gauge-independent way, and we describe the whole theory in terms of them.

We first work out their Poisson brackets, using the canonical algebra of $(\varphi, \pi, g_{11}, \pi^{11})$; we obtain

$$\{J^a(x), J^b(y)\} = -2\sqrt{2}\epsilon^{abc}\eta_{cd}J^d(x)\delta(x-y) + \alpha^2\eta^{ab}\delta'(x-y) \quad , \quad (53)$$

an $SL(2, R)$ algebra identical to (18), and

$$\{b(x), b(y)\} = 2\delta'(x-y) \quad , \quad (54)$$

a typical chiral field bracket, and finally

$$\{b(x), J^a(y)\} = 0 \quad . \quad (55)$$

Next we write the secondary constraints in terms of the J/b -variables; we find

$$\phi_2 = T_m + T_S - \frac{x^-}{\sqrt{2}} J^{+'} \approx 0 \quad , \quad (56a)$$

$$\phi_1 = \phi_2 - g_{11} \left(J^+ - \frac{\alpha^2 \beta}{2} \right) \approx 0 \quad , \quad (56b)$$

where

$$T_m = \frac{1}{4} b^2 + \frac{\alpha}{2} b' \quad , \quad (57)$$

$$T_S = \frac{1}{2\alpha^2} \eta_{ab} J^a J^b - \frac{1}{\sqrt{2}} J^{0'} \quad . \quad (58)$$

We notice that the old constraints (27) are equivalent to imposing

$$\mathcal{J} = J^+ - \frac{\alpha^2 \beta}{2} \approx 0 \quad , \quad (59a)$$

$$\tau = T_m + T_S \approx 0 \quad , \quad (59b)$$

which turn out to obey a *Current* $\times$ *Virasoro* algebra:

$$\{\mathcal{J}(x), \mathcal{J}(y)\} = 0 \quad , \quad (60a)$$

$$\{\tau(x), \mathcal{J}(y)\} = -\mathcal{J}'(x)\delta(x-y) \quad , \quad (60b)$$

$$\{\tau(x), \tau(y)\} = (\tau(x) + \tau(y))\delta'(x-y) \quad . \quad (60c)$$

Therefore the Hamiltonian can be obtained from (25) and (56),

$$H = \int dx (v_1 \tau + v_2 \mathcal{J}) \quad (61)$$

where

$$v_1 = \frac{g_{01} - \sqrt{-g}}{g_{11}} \quad , \quad (62a)$$

$$v_2 = \sqrt{-g} - \frac{1}{2}v_1 + \frac{x^-}{\sqrt{2}}v_1' \quad (62b)$$

play the role of the Lagrange multipliers (gauge freedom⁵⁾) in this description. Once again we remark that the J/b -variables, the constraints and their algebras are gauge-independent, although the dynamics (Hamiltonian) is not. Indeed, the classical equations of motion (using (61) and the constraints (59)) are:

$$\frac{dJ^+}{dt} = v_1 J^{+'} \quad , (= 0 \text{ according to (59a)}) \quad , \quad (63a)$$

$$\frac{dJ^0}{dt} = v_1 J^{0'} + v_1' J^0 - \frac{\alpha^2}{\sqrt{2}}v_1'' + \sqrt{2}(v_2 - \frac{1}{2})J^+ \quad , \quad (63b)$$

$$\frac{dJ^-}{dt} = v_1 J^{-'} + 2v_1' J^- + 2\alpha^2 v_2' + 2\sqrt{2}(v_2 - \frac{1}{2})J^0 \quad , \quad (63c)$$

$$\frac{db}{dt} = v_1 b' + v_1' b - \alpha v_1'' \quad , \quad (64)$$

from which we see that the $SL(2, R)$ currents are conserved if we choose the gauge $v_2 = \frac{1}{2}$ and $v_1 = \text{constant}$ (if $v_1 = 1$, i.e. light-cone gauge, they satisfy $\partial_- J^a = 0$ as already stated in section 1). But this is not true for other gauges, in which those currents *do not* generate symmetry transformations.

---

⁵⁾ Light-cone gauge: $v_1 = 1$, $v_2 = \frac{1}{2}$. Conformal gauge: $v_1 = 1$, $v_2 = e^{\frac{2}{\alpha}\phi} - \frac{1}{2}$.

The decomposition (52) indicates that the $SL(2, R)$ currents can be taken as the actual gravitational variables: they are the “components” of the graviton field (g_{11}); the “constant curvature condition”⁸ (constraint (59a)) is naturally described in terms of the J^+ component ($J^+ = \alpha^2 \beta / 2$). Therefore we consider the term T_S in the constraint (59a) as the gravitational contribution¹ to the energy-momentum component (τ).

Concerning the b variable, we notice that its dynamics depends only on the choice of v_1 . Now if we consider the algebra of the T_m piece (where the b field enters), we get

$$\{T_m(x), T_m(y)\} = (T_m(x) + T_m(y))\delta'(x - y) - \frac{\alpha^2}{2}\delta'''(x - y) \quad , \quad (65a)$$

$$\{T_m(x), T_S(y)\} = 0 \quad , \quad (65b)$$

which indicates that T_m retains some memory of the matter fields through the semi-classical central charge α^2 in (65a). Therefore we can consider the b field to be the one that carries the matter central charge in this representation of the theory.

At this point the quantization procedure should be obvious, since we have a complete analogy with the pioneering light-cone gauge prescription¹. We assume that the J -operators satisfy equal-time commutation relations as in (53),

$$[J^a(x), J^b(y)] = -2i\sqrt{2}\epsilon^{abc}\eta_{cd}J^d(x)\delta(x - y) + i\bar{\alpha}^2\eta^{ab}\delta'(x - y) \quad , \quad (66)$$

allowing a possible renormalization of the constant α ,

$$\alpha^2 = 8\kappa = \frac{c - 1}{12\pi} \rightarrow \bar{\alpha}^2 = 8\bar{\kappa} = \frac{k}{96\pi} \quad , \quad (67)$$

where k is to be determined by consistency conditions. We define the quantum operator constraints,

$$\mathcal{J} = J^+ - \frac{\alpha^2 \beta}{2} \approx 0 \quad (68)$$

$$\tau = :T_m: + :T_G: + :T_S: \approx 0 \quad , \quad (69)$$

where

$$:T_S: = (48\pi) \frac{1}{k + 2} : \eta_{ab} J^a J^b : - \frac{1}{\sqrt{2}} J^{0'} \approx 0 \quad , \quad (70)$$

is the suitably renormalized gravitational energy-momentum contribution and T_G is the ghost piece coming from some gauge fixing. Imposing the usual physical state conditions,

$$\mathcal{J}|\text{physical}\rangle = 0 \quad (71a)$$

$$\tau|\text{physical}\rangle = 0 \quad (71b)$$

demands that the algebra of the constraints has no anomalous terms. The only non-trivial consistency condition comes from imposing the vanishing of the total central charge⁶⁾ of $[\tau, \tau]$,

$$c_{total} = c_m + c_G + c_S(k) = 0 \quad (72)$$

⁶⁾ $c_S(k) = \frac{3k}{k+2} - 6k$, which comes from the $SL(2, R)$ algebraic properties.

which establishes a relation between k and the matter central charge c_m . We emphasize that the above results, previously obtained in some special gauges^{1,9}, are obtained here in a gauge-independent way.

5. Conformal gauge: Liouville theory

It is interesting to speculate about the consequences of the previous discussion for the Liouville theory, defined by⁷⁾

$$S_L = - \int d^2x \left[\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} \alpha^2 \beta e^{\frac{2}{\alpha} \phi} \right] , \quad (73)$$

leading to the equation of motion

$$2\partial_+ \partial_- \phi + \alpha \beta e^{\frac{2}{\alpha} \phi} = 0 . \quad (74)$$

The relation between the conformal and light-cone gauges can be understood¹⁰ from the following coordinates/fields transformation

$$\phi = \frac{\alpha}{2} \ln(\partial_- f) , \quad (75a)$$

$$h_{++} = \frac{\partial_+ f}{\partial_- f} , \quad (75b)$$

$$x^+ \rightarrow x^+ , \quad x^- \rightarrow f(x^+, x^-) . \quad (75c)$$

In the quantum theory we need also to consider the Jacobian $\frac{\mathcal{D}h_{++}}{\mathcal{D}\phi} \sim \det(\partial_+ + h_{++}\partial_-)$ which accounts for the difference between the ghost central charge of the Liouville theory (-26), and that of the $SL(2, R)$ counterpart (-28).

Notice that the above analysis takes place in configuration space. If we go over to the Hamiltonian formalism, the connection between the phase-space variables of the light-cone and conformal gauges (implicitly given by (30) and (32)) is nothing but a canonical transformation,

$$\varphi = \chi + \phi , \quad (76a)$$

$$h_{++} = e^{\frac{2}{\alpha} \phi} - 1 , \quad (76b)$$

$$\pi_\varphi = \pi_\chi - \phi' , \quad (76c)$$

$$\pi_{h_{++}} = \frac{\alpha}{2} e^{-\frac{2}{\alpha} \phi} (\pi_\phi + \phi' - \pi_\chi + \chi') . \quad (76d)$$

---

⁷⁾ The Liouville model comes out after we fix the gauge (31) and choose the variables as in (32): the resulting action is given by (73) plus a free scalar piece, namely $S_\chi = \int d^2x \partial_+ \chi \partial_- \chi$.

It is natural to discuss the role of the $SL(2, R)$ currents in the conformal gauge. The general solution for the Liouville field is usually given as

$$\phi = \frac{\alpha}{2} \ln \left[\frac{\partial_+ A \partial_- B}{(1 - AB)^2} \right] , \quad (77a)$$

where A and B satisfy simple linear first order equations,

$$\partial_- A = 0 , \quad (77b)$$

$$\partial_+ B = 0 . \quad (77c)$$

Now we can use (32) and (52) to express the Liouville field in terms of the currents,

$$\phi = \frac{\alpha}{2} \ln \left[1 - \frac{1}{\alpha^2} (J^- + 2x^- J^0 + (x^-)^2 J^+) \right] , \quad (78a)$$

where the J -components obey the following non-linear equations (see (63)),

$$\partial_- J^+ = 0 , \quad (78b)$$

$$\partial_- J^0 = -\frac{1}{\alpha^2} [J^- + 2x^- J^0 + (x^-)^2 J^+] J^+ , \quad (78c)$$

$$\partial_- J^- = -\sqrt{2} [J^- + 2x^- J^0 + (x^-)^2 J^+] - \frac{2}{\alpha^2} [J^- + 2x^- J^0 + (x^-)^2 J^+] J^0 . \quad (78d)$$

In fact the above equations imply

$$2e^{-\frac{2}{\alpha}\phi} \partial_+ \partial_- \phi = -\frac{2}{\alpha} J^+ , \quad (79)$$

i.e. J^+ is the curvature on-shell; if we finally use the constraint (59a) the latter equation becomes a constant curvature condition, which is the Liouville equation (74). The last condition (59b) binds the matter and Liouville pieces of energy-momentum so that covariance is guaranteed. We conclude that, although the $SL(2, R)$ currents are not conserved in this gauge ⁸⁾, they can be used to describe the classical Liouville theory (and they might play an important role in the quantization of that model, as in the light-cone version of 2d-gravity).

6. Conclusions

---

⁸⁾ In configuration space we have $j^0 \sim \partial_- \phi$ and $j^- \sim \text{const.} + e^{\frac{2}{\alpha}\phi} \sim \text{const.} + \partial_+ \partial_- \phi$ on-shell, so that in a description where x^+ is the *time* and x^- is the *space* component, the space-integrals of j^0 and j^- would be constants on-shell, and the currents would be actual symmetry generators.

We have obtained the essential features of two dimensional gravity in a gauge independent way. Although the $SL(2, R)$ currents can be defined in any gauge, they are residual symmetry generators only in some gauges (typically the light-cone one).

We defined the theory using the Coulomb gas representation, in order to deal with a local action. One can obtain a description very similar to string theory. In particular, the constraints algebra is a Virasoro algebra in that case. Alternatively, one can switch to a description that emphasizes the $SL(2, R)$ structure, obtaining the corresponding Kac-Moody algebra.

In the conformal gauge, we can write a general solution of the Liouville equation in terms of $SL(2, R)$ currents which satisfy a certain set of non-linear first-order equations of motion. This might give us a hint for quantizing Liouville theory.

References

- [1] A. M. Polyakov, Mod. Phys. Lett. **A2**(1987)893.
V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. **A3**(1988)819.
- [2] M. Green, J. H. Schwarz and E. Witten, Superstring Theory, Cambridge Univ. Press, 1987.
- [3] E. Abdalla, M. C. B. Abdalla and K. D. Rothe, Non-perturbative Methods in Two Dimensional Quantum Field Theory, World Scientific, 1991.
E. Abdalla, M. C. B. Abdalla and A. Zadra, ICTP Preprint 89/56.
K. Isler and J. M. Lina, U de M -LPN-TH-12 Preprint 1991.
- [4] A. M. Polyakov, Lectures given at Les Houches Summer School, Session XLIX, 1988.
A. Alekseev, S. Shatashvili, Nucl. Phys. **B323**(1989)719.
M. Bershadsky, H. Ooguri, Commun. Math. Phys. 126(1989)49.
- [5] R. Marnelius, Nucl. Phys. **B211**(1983)14; Nucl. Phys. **B211**(1983)409.
- [6] P. A. M. Dirac, Lectures in Quantum Mechanics and Lectures in Quantum Field Theory, Yeshiva Univ., N.Y., 1964.
- [7] Ed. Sh. Egorian, R. P. Manvelian, Mod. Phys. Lett. **A5**(1990)2371.
- [8] R. Jackiw, in Quantum Theory of Gravity, S. Christensen ed., Hilger, 1984.
C. Teitelboim, in Quantum Theory of Gravity, S. Christensen ed., Hilger, 1984; Phys. Lett. **126B**(1983)41.
- [9] F. David, Mod. Phys. Lett. **A3**(1988)1651.
J. Distler, H. Kawai, Nucl. Phys. **B321**(1989)509.
- [10] Al. B. Zamolodchikov, ITEP 89-84 preprint.
K. Harada and D. Dalmazi, unpublished.