DUALITY-INVARIANT PARTITION FUNCTIONS
AND AUTOMORPHIC SUPERPOTENTIALS
FOR (2, 2) STRING COMPACTIFICATIONS (*)

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ABSTRACT

We define the topological free energy for string compactifications, which is relevant for the discussion of perturbative as well as non-perturbative effects in string theory. This moduli-dependent functional, originating from the integration over massive string modes, is determined by automorphic functions of the target-space duality group. We explicitly construct these automorphic functions for symmetric orbifold and Calabi-Yau compactifications.

(*) Work supported in part by the Department of Energy of USA under the contract DOE-AT03-88ER40 384, TASK E, and by the Grant-in-Aid for Scientific Research (International Scientific Research Program) of the Japanese Ministry of Education, Science and Culture (No. 02045009).

1. Introduction

Four-dimensional string vacua with $N \geq 1$ space-time supersymmetry can be obtained via different approaches (for reviews and references, see e.g. ref. [1]). Infinitely many of them are connected by continuous deformations of the underlying two-dimensional superconformal field theory, parametrized by coupling parameters called *moduli*. At the level of the four-dimensional effective field theory, and neglecting non-perturbative effects, moduli are described by massless scalar fields with flat effective potential, whose vacuum expectation values parametrize the continuous deformations. Keeping track correctly of the moduli-dependence is essential for writing down a sensible low-energy effective field theory, since both gauge couplings and the Yukawa couplings of the matter fields are in general moduli-dependent.

A peculiar feature of the moduli spaces of string vacua is their invariance under some discrete reparametrizations, which generalize the simple $R \leftrightarrow 1/R$ duality of circle compactifications [2-4]. In the four-dimensional effective supergravity theories, these invariances correspond to the so-called generalized duality symmetries. These target-space dualities have been intensely studied in the recent past, not only in toroidal and orbifold compactifications, but also in Calabi-Yau compactifications, and their deep implications are just starting to emerge. In particular, in $N = 1$ models they put rather stringent constraints on the moduli-dependence of the Kähler potential and of the superpotential, both at the classical level and after the inclusion of string loop corrections and possible non-perturbative effects [5-9].

A well-known example is the dependence of the $d = 4$, $N = 1$ supergravity action on the complex modulus T , whose real part describes the breathing mode of the internal six-dimensional torus or orbifold. Neglecting the dependence on all other moduli and matter fields, the Kähler function for T describes [10,11] a $SU(1,1)/U(1) \simeq SL(2, \mathbf{R})/U(1)$ sigma-model. Any non-trivial superpotential would break the continuous $SL(2, \mathbf{R})$ symmetry, but the quantum string symmetry is only the discrete duality group $SL(2, \mathbf{Z})$. Since under $SL(2, \mathbf{Z})$ the Kähler potential transforms as $K \rightarrow K + \Lambda(T) + \bar{\Lambda}(\bar{T})$, any possible non-perturbative superpotential has to transform as $W \rightarrow W e^{-\Lambda(T)}$. This transformation law es-

establishes the connection with the theory of modular functions. The constraint on W is satisfied by the Dedekind eta function, i.e. the choice $W(T) \propto \eta(T)^{-6}$ respects the $SL(2, \mathbf{Z})$ duality invariance.

It is known that any effective field theory with a finite number of fields cannot give a global description of the moduli space. This is made evident by the string Higgs phenomenon: there is in general an infinite number of massive lattice states which may become massless at special points of moduli space. (However only a finite number of states can become massless at some special point.) So, if one wants to formulate a string effective field theory which is valid for large field deformations, not only for small fluctuations around a given solution, one must include at least all the states which may become massless somewhere in moduli space. Another reason to include an infinite number of states in the effective theory is because of the target-space duality symmetries: in general, the action of the full duality group on the whole string spectrum mixes massless modes with massive ones, which may become massless at some point of moduli space. The problem of writing down a fully duality-invariant effective field theory has been recently solved, for $d = 4$ string vacua with unbroken $N = 4$ space-time supersymmetry, by Giveon and Porrati [12], who were also able to obtain some information about the untwisted sectors of $N = 2$ and $N = 1$ vacua.

Nevertheless, if one is not interested in describing the string Higgs effect, but only the full moduli-dependence of the low-energy effective theory around a class of vacua with given gauge group, one can still work in an effective theory with a finite number of fields. However, since such an effective theory is obtained by integrating out all massive degrees of freedom, one expects it to be formulated in terms of automorphic functions of the full duality group acting on moduli. As far as the explicit form of these functions is concerned, in some simple cases one can deduce it from direct string calculations [8].

In this paper, we introduce the topological free energy F of string compactifications. This quantity is given in terms of the determinant of the moduli-dependent chiral mass matrix for the massive compactification modes, and therefore corresponds to the generating functional of the effective action after integrating out all the corresponding massive modes. Since F is a moduli-dependent

functional, it has to be expressed in terms of automorphic functions of the duality group. For symmetric orbifold compactifications, we compute F starting from an effective theory as constructed in ref.[12]. In the case of Calabi-Yau compactifications, we use the powerful constraints of special Kähler geometry to construct explicitly automorphic functions of the Calabi-Yau duality group. Furthermore, we give strong arguments to relate these functions to the topological free energy of Calabi-Yau compactifications, thus gaining important insight into the structure of the Calabi-Yau mass spectrum and moduli-dependent loop amplitudes.

In section 2 we consider symmetric orbifold compactifications of the ten-dimensional heterotic string with $d = 4$, $N = 1$ supersymmetry. Following the lines of ref.[12], we deduce the Kähler function G associated to massless and massive scalar fields in the untwisted sector. This allows us to compute explicitly, at the field-theoretical level but including all Kaluza-Klein and winding modes (vortices), the moduli-dependent chiral mass terms around supersymmetric vacua: the spectrum agrees with the mass formulae of string theory and is invariant under the appropriate duality transformations. In section 3 we use the results of the previous section to write down the moduli-dependent partition function for symmetric orbifold compactifications, and to give it a topological interpretation. Again, in some simple cases the result can be matched with explicit string computations. We observe that from the orbifold free-energy one can get information about the form of the non-perturbative superpotential, which must be an automorphic function of the target-space duality group: the latter can be identified with the determinant of the chiral masses which arise in the compactification, including both Kaluza-Klein and winding states. Finally, in section 4 we extend our considerations to general Calabi-Yau compactifications. Using the fact that the metric of Calabi-Yau moduli spaces is not only Kähler, as implied by $N = 1$ space-time supersymmetry, but “special Kähler”, as implied by the relation with $N = 2$ compactifications of type-II strings, we construct explicitly automorphic functions of the Calabi-Yau duality group. In section 5 we summarize our conclusions.

2. The effective $N = 1$ supergravity action for the untwisted massive orbifold excitations

In this section we consider symmetric orbifold compactifications [13] of the ten-dimensional heterotic string [14] with $d = 4$, $N = 1$ space-time supersymmetry. Our aim is to derive, in the formalism of ref. [15], the Kähler function

$$G(\phi^I, \bar{\phi}^{\bar{I}}) = K(\phi^I, \bar{\phi}^{\bar{I}}) + \log |W(\phi^I)|^2, \quad (2.1)$$

where ϕ^I ($I = 1, \dots, \infty$) denote the massless as well as the infinite number of massive scalar fields in the untwisted sector. The superpotential $W(\phi^I)$ and the Kähler potential $K(\phi^I, \bar{\phi}^{\bar{I}})$ contain the information about the masses of the heavy string modes, which will be used in the next section to compute the free energy of the compactified string. Notice that the masses of the twisted states do not depend on the moduli, and therefore are not relevant for the moduli-dependence of the free energy.

Consider first four-dimensional heterotic string solutions with $N = 4$ space-time supersymmetry, which are obtained via toroidal compactification [16] of the ten-dimensional heterotic string on a Lorentzian lattice $\Gamma(6, 22)$. For this case, it was shown in ref.[12] how to determine the effective action (neglecting higher derivative terms) which contains all lattice string states belonging to massive and massless $N = 4$ vector supermultiplets. The method of ref.[12], which extends to the massive states the method of ref.[11], is based on the fact that in $N = 4$ supergravity coupled to Yang-Mills [17] all interactions (up to two derivatives in the bosonic fields) are uniquely determined, if one knows the three-point couplings among the gauge bosons of the graviton supermultiplet (graviphotons) and of the matter (massless or massive) vector supermultiplets. It is then sufficient to derive, by the simple fusion of any three gauge boson vertex operators, the desired “gauging” of $N = 4$ supergravity, i.e. the structure constants f_{ABC} of the $N = 4$ gauge group. The knowledge of f_{ABC} uniquely fixes the $N = 4$ effective action, which is highly non-linear in terms of the physical fields.

The $N = 4$ “stringy” gauging is a non-compact one; it corresponds to the group $SO(6, \infty)$. In the Cartan-Weyl basis and in the case of unbroken super-

symmetry, the only non-vanishing structure constants are

$$f_{\vec{p}\vec{q}}^i = p^i \delta_{\vec{p}+\vec{q},0} \quad \text{and} \quad f_{\vec{p}\vec{q}}^{\vec{r}} = \epsilon(\vec{p}, \vec{q}) \delta_{\vec{p}+\vec{q}+\vec{r},0}, \quad (2.2)$$

where $i = (i_L, i_R)$ labels the Cartan generators, $i_L = 1, 2, \dots, 6$ for the graviphotons and $i_R = 1, 2, \dots, 22$ for the matter gauge group, and $\vec{p} = (p^{i_L}, p^{i_R})$ is the momentum of a point of the Lorentzian lattice $\Gamma(6, 22)$. Finally, $\epsilon(\vec{p}, \vec{q})$ is a cocycle phase factor associated with the vertex operators of the $\vec{p}$ and $\vec{q}$ momentum states[★].

The obtained “stringy” supergravity action is very similar to the Kaluza-Klein field theory defined by $p^{i_L} = p^{i_R}$ ($i_L = i_R = 1, 2, \dots, 6$), with p^{i_R} ($i_R = 7, \dots, 22$) being an $E_8 \times E_8$ lattice vector. In this theory one would keep all Kaluza-Klein massive vector supermultiplets, i.e the zero-winding sector of the theory. In the T^6 internal space, however, there exist “vortex” configurations (non-trivial semi-classical solutions in T^6). One has to include also all these non-trivial configurations in the field-theory path-integral. Including these extra degrees of freedom is equivalent to including in the effective field theory the non-zero winding sectors, which provide the theory with the target-space duality symmetries. We thus observe that the $N = 4$ “stringy” action can be identified with the Kaluza-Klein field-theory action, once we include all T^6 vortex configurations. This shows how deep the connection between string theory and field theory is, at least in the case of extended supergravity models obtained via toroidal compactifications.

To obtain the effective $N = 1$ supergravity action for the untwisted sector of $(2, 2)$ orbifold theories, one can perform suitable truncations of the massless and massive spectra [12]. At the $N = 4$ level, the four-dimensional dilaton-axion field parametrizes the manifold $\mathcal{M}_0 = \frac{SU(1,1)}{U(1)}$, whereas the 36 massless moduli associated to the internal metric, antisymmetric tensor and breathing mode parametrize (locally) the manifold $\mathcal{M}_{N=4} = \frac{SO(6,6)}{SO(6) \times SO(6)}$. The $N = 1$

★ The method of ref.[11] was further generalized to the case of spontaneously broken $N = 4$ solutions in ref. [18], and the inclusion of massive states was performed in ref. [19]. In this case, non-zero structure constants proportional to the supersymmetry breaking parameter(s) are also present, giving rise to a generalization of the flat-group gaugings of extended supergravity theories [20,21].

orbifolds always reduce this space to a Kählerian submanifold $\mathcal{M} = \mathcal{M}_0 \otimes \mathcal{M}_{N=1}$. There are five generic cases [11], [22], [23], [24] for $\mathcal{M}_{N=1}$.

$$(1) \ 3h_{1,1} + 3h_{1,2} \text{ moduli} \quad \mathcal{M}_1 = \left[\frac{SO(2,2)}{SO(2) \times SO(2)} \right]^3. \quad (2.3)$$

By fixing some of the $h_{1,2}$ moduli U_A ($A = 1, 2, 3$) one has

$$(2) \ 3h_{1,1} + rh_{1,2} \text{ moduli} \quad \mathcal{M}_2 = \left[\frac{SO(2,2)}{SO(2) \times SO(2)} \right]^r \otimes \left[\frac{SU(1,1)}{U(1)} \right]^{3-r}, \quad r=0, 1, 2. \quad (2.4)$$

$$(3) \ 5h_{1,1} + 1h_{1,2} \text{ moduli} \quad \mathcal{M}_3 = \frac{SO(2,2)}{SO(2) \times SO(2)} \otimes \frac{SU(2,2)}{SU(2) \times SU(2) \times U(1)}. \quad (2.5)$$

By fixing the single $h_{1,2}$ modulus U one obtains

$$(4) \ 5h_{1,1} \text{ moduli} \quad \mathcal{M}_4 = \frac{SU(1,1)}{U(1)} \otimes \frac{SU(2,2)}{SU(2) \times SU(2) \times U(1)}. \quad (2.6)$$

Finally

$$(5) \ 9h_{1,1} \text{ moduli} \quad \mathcal{M}_5 = \frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)}. \quad (2.7)$$

The correspondence with the Abelian orbifolds [13] (which were catalogued in ref. [25]) is the following:

$$\begin{aligned} (1) \ & Z_2 \times Z_2 \quad (\text{fermionic construction [26], [22]}) \\ (2) \ & r = 1 \quad Z_6, Z_8, Z_4 \times Z_2, Z_6 \times Z_2, Z_{12} \\ & r = 0 \quad Z_7, Z'_8, Z'_{12}, Z'_6 \times Z_2, Z_3 \times Z_3, \\ & \quad Z_6 \times Z_3, Z_4 \times Z_4, Z_6 \times Z_6 \\ (3) \ & Z_4 \\ (4) \ & Z'_6 \\ (5) \ & Z_3 \end{aligned} \quad (2.8)$$

To include also the massive states, let us discuss first the simplest case

(1). The massless and massive scalars of the $N = 4$ vector supermultiplets

are coordinates of the manifold $\mathcal{M}_{N=4} = \frac{SO(6,n)}{SO(6) \times SO(n)}$ ($n = \infty$). The $Z_2 \times Z_2$ action reduces the $N = 4$ manifold to the Kählerian submanifold $\mathcal{M}_{N=1} = \frac{SO(2,n_1)}{SO(2) \times SO(n_1)} \otimes \frac{SO(2,n_2)}{SO(2) \times SO(n_2)} \otimes \frac{SO(2,n_3)}{SO(2) \times SO(n_3)}$ ($n_A = \infty$, $A = 1, 2, 3$). It is very useful to introduce constrained fields which linearize the above scalar manifold and provide the connection with the $N = 4$ gauging. The $SO(2, n_A)$ constraints are those used in ref.[23] but including now all the massive fields

$$\begin{aligned} |\phi_A^{i_L}|^2 - |\phi_A^{i_R}|^2 - |\phi_A^{\vec{p}}|^2 &= \frac{1}{2}, \\ (\phi_A^{i_L})^2 - (\phi_A^{i_R})^2 - (\phi_A^{\vec{p}})^2 &= 0. \end{aligned} \quad (2.9)$$

The fields $\phi_A^{i_L}, \phi_A^{i_R}$ ($i_L, i_R = 1, 2$) correspond to six unphysical compensator fields and to the six (complex) massless moduli of the manifold $\mathcal{M}_1$, eq.(2.3), whereas the $\phi_A^{\vec{p}}$ are (generically) massive states associated to the $\vec{p}$ momentum vector of $\Gamma(2, 2)_A$. One can also introduce analytic fields $y_A^I = \phi_A^I (Y_A)^{\frac{1}{2}}$ ($I = i_L, i_R, \vec{p}$), which satisfy the corresponding rescaled constraints. Similarly, the $SU(1, 1)$ constraint reads $|\phi_0^0|^2 - |\phi_0^1|^2 = \frac{1}{2}$, and its standard solution is $\phi_0^0 = \frac{S+1}{2Y_0^{1/2}}, \phi_0^1 = \frac{S-1}{2Y_0^{1/2}}$, with $Y_0 = S + \bar{S}$. The $N = 1$ supergravity action is then fully specified in terms of the $N = 4$ gauging. Indeed, the G -function of eq.(2.1) is given by [27],[11]

$$e^G = |W|^2 e^K = |(\phi_0^0 - \phi_0^1) f_{IJK} \phi_1^I \phi_2^J \phi_3^K|^2, \quad (2.10)$$

where

$$K = -\log Y_0 Y_1 Y_2 Y_3 \quad (2.11)$$

and

$$W = f_{IJK} y_1^I y_2^J y_3^K. \quad (2.12)$$

For simplicity, in the rest of this paper we will neglect the S -dependent part of the G -function, Y_0 , and we will also reabsorb, whenever possible, moduli-independent constants into redefinitions of the S field. Using the specific form of the structure constants f_{IJK} , eq.(2.2), one can write

$$W e^{K/2} = [(p_1^{i_L} \phi_1^{i_L} - p_1^{i_R} \phi_1^{i_R}) \phi_2^{\vec{p}_1} \phi_3^{-\vec{p}_1} + \dots] + \phi_1^{\vec{p}} \phi_2^{\vec{q}} \phi_3^{\vec{r}} \epsilon(\vec{p}, \vec{q}) \delta_{\vec{p}+\vec{q}+\vec{r}, 0}, \quad (2.13)$$

where the dots denote permutations of the indices (123). Notice that W in eq.(2.12) is trilinear in terms of the constrained analytic fields y^I . However,

we shall see that in terms of the unconstrained, physical fields it becomes non-homogeneous for the massive fields $\phi^{\vec{p}}$. Another important observation is that the functional form of K and W in terms of the physical fields depends on the particular solution of the constraints in eq.(2.9) and on the particular choice of the $\Gamma(2, 2)$ lattice point. All these apparent dependences are just artifacts. The lattice momenta satisfy constraints conjugate to that of the moduli fields ϕ_A^i ($i = i_L, i_R$):

$$(p_A^{i_L})^2 - (p_A^{i_R})^2 = 2\mathbf{Z}. \quad (2.14)$$

The different points of the lattice $\Gamma(2, 2)$ are reached by $SO(2, 2)$ transformations. Therefore any $SO(2, 2)$ transformation $p_A^{i'} = p_A^j R_j^i$ can be absorbed by a redefinition $(\phi_A^i)' = (R^{-1})_k^i \phi_A^k$, which is also a solution of the $SO(2, 2)$ constraints.

In order to verify the reparametrization invariance of the theory, we shall make two different choices for the solutions y_A^I of the $SO(2, n_A)$ constraints, eq.(2.9):

$$\begin{aligned} (i) \quad y_A^{1_L} &= \frac{1}{2} \left[1 + (\omega_A^I)^2 \right], \\ y_A^{2_L} &= \frac{i}{2} \left[1 - (\omega_A^I)^2 \right], \\ y_A^I &= -\omega_A^I, \\ Y_A &= \left[1 - 2\omega_A^I \bar{\omega}_A^I + (\omega_A^I \omega_A^I)(\bar{\omega}_A^J \bar{\omega}_A^J) \right], \quad I = (i_R, \vec{p}). \end{aligned} \quad (2.15)$$

$$\begin{aligned} (ii) \quad y_A^{1_L} &= \frac{1}{2} \left[1 + z_A^2 - x_A^2 - (\omega_A^{\vec{p}})^2 \right], \\ y_A^{2_L} &= iz_A, \\ y_A^{1_R} &= -\frac{1}{2} \left[1 - z_A^2 + x_A^2 + (\omega_A^{\vec{p}})^2 \right], \\ y_A^{2_R} &= -ix_A, \\ y_A^{\vec{p}} &= -i\omega_A^{\vec{p}}, \\ Y_A &= (z_A + \bar{z}_A)^2 - (x_A + \bar{x}_A)^2 - (\omega_A^{\vec{p}} + \bar{\omega}_A^{\vec{p}})^2. \end{aligned} \quad (2.16)$$

The first solution [28] is $SO(n_A)$ symmetric, thus it is convenient for the study of extended gauge symmetry points [29]. The second one [18] is $SO(1, n_A - 1)$

symmetric and is useful when studying the target space duality symmetries (see later).

Now we would like to determine the masses of the $\omega_A^{\vec{p}}$ charged fields. For this purpose we need the metric $K_{\vec{p}_i \vec{p}_j}^A|_{\omega_A^{\vec{p}}=0}$. For both solutions we have

$$K_{\vec{p}_i \vec{p}_j}^A = \delta_{\vec{p}_i \vec{p}_j} \frac{2}{Y_A}, \quad (2.17)$$

since $K_{ij}^A = \frac{Y_{Ai} Y_{Aj}}{Y_A^2} - \frac{Y_{Aij}}{Y_A} (Y_{Ai} = \frac{\partial Y_A}{\partial y_A^i})$, and it is easy to show that around $y_A^{\vec{p}} = 0$ the first term vanishes, whereas $Y_{ij} = -2\delta_{\vec{p}_i \vec{p}_j}$ (this result would be valid for all solutions of the constraints such that $\omega_A^{\vec{p}} = e^{i\phi} y_A^{\vec{p}}$). Thus the metric is canonical up to the $2Y_A^{-1}$ rescaling.

The fermionic mass terms of $\omega_A^{\vec{p}}$ are then given, around supersymmetric vacua characterized by $W = 0$ (and ignoring constant and S -dependent factors) by

$$M_A^\dagger M_A = e^K Y_B Y_C |W_{BC}|^2 = \frac{|W_{BC}|^2}{Y_A} \quad (A \neq B \neq C \neq A). \quad (2.18)$$

($W_{BC} = \frac{\partial^2 W}{\partial \omega_B^{\vec{p}} \partial \omega_C^{\vec{p}}}$.) Using the expressions for W and Y_A in terms of y_A^I and the structure constants of eq.(2.2), one obtains

$$M_A^\dagger M_A = \frac{|p_A^i \eta_{ij} y_A^j|^2}{\bar{y}_A^i \eta_{ij} y_A^j}, \quad (2.19)$$

where $\eta_{ij} = \text{diag}(1, 1, -1, -1)$ is the Lorentzian metric of $SO(2, 2)$. Notice that this expression is invariant under rescaling of $y_A^i \rightarrow \Lambda y_A^i$. Also, a pair (p_A^i, y_A^i) with $p_A^i \in \Gamma(2, 2)$ and y_A^i solution of the rescaled analytic constraints of $SO(2, 2)$, is equivalent to the pair $(p_A^{i'}, y_A^{i'})$ obtained by $p_A^{i'} = R^i_j p_A^j$ and $y_A^{i'} = (R^{-1})^i_k y_A^k$. This permits us to choose any point of $\Gamma(2, 2)$ and to express $M_A^\dagger M_A$ in terms of the *vev* of y_A^i . Consider for instance the second solution, eq.(2.16), of the $SO(2, n_A)$ constraint. Then $Y_A = (z + \bar{z})_A^2 - (x + \bar{x})_A^2$ and the chiral (holomorphic) masses are

$$p_A^i \eta_{ij} y_A^j = \frac{1}{\sqrt{2}} \left\{ m + n(z^2 - x^2) + i[(z + x)m' + (z - x)n'] \right\}_A, \quad (2.20)$$

where we have chosen $p_{L,R}^1 = \frac{1}{\sqrt{2}}(m \pm n)$, $p_{L,R}^2 = \frac{1}{\sqrt{2}}(m' \pm n')$. The constraint (2.14) now reads $mn + m'n' = \mathbf{Z}$. However, to exclude oscillator contributions to

the spectrum one has to impose the stronger constraint

$$mn + m'n' = 0. \quad (2.21)$$

Introducing the familiar background fields [3] $T = z - x = \sqrt{g} + ib$, $U = z + x = \frac{\sqrt{g}}{g_{22}} + i\frac{g_{12}}{g_{22}}$, where g_{ij} is the internal metric of the two-torus and $b = B_{12}$ the antisymmetric tensor field, we finally obtain

$$M_A^\dagger M_A = \left[\frac{|m + nTU + i(Um' + Tn')|^2}{(T + \bar{T})(U + \bar{U})} \right]_A. \quad (2.22)$$

This is nothing other than the standard $\Gamma(2, 2)$ parametrization of the string mass spectrum. In fact the splitting into the fields T and U is suggested by the fact that the moduli space $\frac{SO(2,2)}{SO(2) \times SO(2)}$ is just a product space, $\left(\frac{SU(1,1)}{U(1)} \right)_T \otimes \left(\frac{SU(1,1)}{U(1)} \right)_U$.

The above expression (2.22) is duality and shifting invariant:

$$\begin{aligned} m' &\leftrightarrow n' & T &\leftrightarrow U, \\ m &\rightarrow m - n', n' \rightarrow n' & T &\rightarrow T + i, \\ m &\rightarrow n', n \rightarrow m', m' \rightarrow -n, n' \rightarrow -m & T &\rightarrow \frac{1}{T}. \end{aligned} \quad (2.23)$$

In addition, one has the “charge conjugation” symmetry

$$m' \leftrightarrow -m', n' \leftrightarrow -n' \quad U \leftrightarrow \bar{U}, T \leftrightarrow \bar{T}. \quad (2.24)$$

These are the four independent generators of $\Gamma = [SO(2, 2, \mathbf{Z})]_A = [PSL(2, \mathbf{Z})_T \times PSL(2, \mathbf{Z})_U \times \mathbf{Z}_2 \times \mathbf{Z}_2]_A$, which define the (T, U) target-space duality transformations.

Comparing the “stringy” mass spectrum, $M_A^\dagger M_A(m_A, n_A, m'_A, n'_A)$ (eq.(2.22)) with the Kaluza-Klein one, $M_A^\dagger M_A(m_A, n_A = 0, m'_A, n'_A = 0) = \left[\frac{|m + im'U|^2}{(T + \bar{T})(U + \bar{U})} \right]_A$, we observe that in the latter the $T_A \leftrightarrow U_A$ symmetry is absent, as well as the $T_A \rightarrow \frac{1}{T_A}$ one. There is however a remaining symmetry, namely $T \rightarrow T + i\alpha$, which is the classical Peccei-Quinn symmetry for the $h_{1,1}$ forms ($\alpha \in \mathbf{R}$). The $h_{1,2}$

forms U_A possess the full duality symmetry $U_A \rightarrow U_A + i$ and $U_A \rightarrow \frac{1}{U_A}$. Therefore the Kaluza-Klein model has the $(PSL(2, \mathbf{Z})_U)^3$ duality symmetry. Including the non-trivial vortex configurations the classical symmetry $T_A \rightarrow T_A + i\alpha$ is broken to $T_A \rightarrow T_A + i$, and the enlarged symmetry $T_A \leftrightarrow U_A$ is generated. This phenomenon is more general, and is valid in more complicated Calabi-Yau compactifications. Specifically, many compactifications appear in mirror pairs [30] [31]: the so-called mirror symmetry is the invariance under the exchange of the $(1, 1)$ and $(1, 2)$ moduli fields. Thus the knowledge about the duality symmetries involving the $(1, 2)$ moduli, which is of geometrical origin, provides important information about the “stringy” duality transformations involving the $(1, 1)$ moduli in the corresponding mirror compactification. In our particular example, which is in fact a “self-mirror” manifold, the modular group $PSL(2, \mathbf{Z})$ for the U -field is just that of two-dimensional gravity, where U plays the role of the Teichmüller parameter which parametrizes the shape of the torus. On the other hand, $(T + \bar{T})$ is just the $\sqrt{\det g}$ of the two-dimensional metric. The Kaluza-Klein spectrum is then extended to the “stringy” mass spectrum, involving also the winding states, by the action of the mirror symmetry $T \leftrightarrow U$.

As a further example, we briefly determine the chiral masses for the cases (3) and (4) with moduli space $\frac{SU(2,2)}{SU(2) \times SU(2) \times U(1)}$. (We leave out the already discussed $\frac{SU(1,1)}{U(1)}$ part of these cases.) Since this space is isomorphic to $\frac{SO(2,4)}{SO(2) \times SO(4)}$ this example is a quite straightforward generalization of the previous example with moduli space $\frac{SO(2,2)}{SO(2) \times SO(2)}$. Specifically, the generalization of eq.(2.20) for this case looks like

$$p^i \eta_{ij} y^j = \frac{1}{\sqrt{2}} \left(M_0 - N_0(z^2 - \sum_{a=1}^3 y^{a2}) + i p_L z - i \sum_{a=1}^3 p_R^a y^a \right). \quad (2.25)$$

To make the $SU(2, 2)$ structure more transparent it is useful to introduce the following four (complex) fields T_{ij} and four winding/momentum numbers M_{ij} ($i, j = 1, 2$):

$$\begin{aligned} T_{ij} &= z \delta_{ij} + \sigma_{ij}^a y^a, \\ M_{ij} &= p_L \delta_{ij} + \sigma_{ij}^a p_R^a. \end{aligned} \quad (2.26)$$

Then the chiral masses take the final form

$$M = M_0 - N_0 \det T + i \text{Tr}(MT). \quad (2.27)$$

This expression provides the generalization of the chiral masses $M = m + inT$ for the case $\mathcal{M} = \frac{SU(1,1)}{U(1)}$. However the six integers in eqs.(2.25),(2.27) are not independent but subject to the level matching constraint

$$-M_0 N_0 + \det M = 0. \quad (2.28)$$

In the next section we will discuss the transformation properties of (2.27) under the generalized target space duality transformations.

Finally case (5), the only one still missing, with $\mathcal{M}_5 = \frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)}$, leads to similar expressions.

3. The topological free energy of $N = 1$ orbifolds

Here we want to define a very useful topological quantity, namely the free energy F of the compactified heterotic string. This always appears when computing amplitudes with light states as external legs, and all massive states entering in loops. Therefore the free energy F must be a target-space duality invariant function of the moduli fields.

In the field theory context, the (bosonic) free energy F_B is obtained by functionally integrating out all massive (bosonic) states,

$$e^{F_B} = \int [\mathcal{D}\phi] e^{\phi M_\phi^2 \phi^\dagger + \dots}, \quad (3.1)$$

where the dots indicate higher orders in ϕ which are neglected around the $\phi = 0$ vacuum. We are also neglecting derivative terms by means of the $\frac{i\partial}{M_\phi}$ expansion of the effective action. We call the free energy a topological quantity because the path integral should be performed only over those massive states which are due to the compactification, like Kaluza-Klein and winding modes, but not over states which are also present in the uncompactified theory, like all massive oscillator

states. Since we only consider backgrounds leading to space-time supersymmetry, the fermionic free energy (F) is opposite to the bosonic one (F_B). However, what we are interested in is the functional dependence of F or F_B on the moduli, which generically plays an important role when calculating loop corrections in string theory, like the one-loop moduli dependence of the gauge coupling constant and of many N -point functions. Integration over the massive fermions leads to

$$F = \log \det M^\dagger M, \quad (3.2)$$

where M is the fermionic mass matrix.

In $N = 1$ supergravity, M takes the following form [15]:^{*}

$$M_{ij} = e^{G/2} G_{i\bar{m}}^{-1/2} [G_{mn} + G_m G_n - G_{mn\bar{k}} G_{\bar{k}l}^{-1} G_l] G_{\bar{n}j}^{-1/2}. \quad (3.3)$$

At supersymmetry preserving minima $e^{G/2} G_i = 0$, thus $(M^\dagger M)_{\bar{i}j} = e^K G_{i\bar{r}}^{-1/2} \bar{W}_{\bar{r}\bar{s}} G_{s\bar{k}}^{-1} W_{kl} G_{\bar{l}j}^{-1/2}$. Observe that, since the massive fields have non-canonical kinetic terms, their physical masses are obtained from the mass terms in the supergravity lagrangian by rescaling the massive fermions by the appropriate power of their Kähler metrics. One has then the following expression for the determinant of the squared fermion mass matrix:

$$\det(M^\dagger M) = \det[e^K (K^{-2})_{i\bar{j}}] |\det W_{ij}|^2. \quad (3.4)$$

We observe that $\det M^2$ nicely splits into a holomorphic square, $|\det W_{ij}|^2$, and a non-holomorphic piece which is due to the e^K rescaling in (3.3) and to the non-canonical kinetic terms of the massive matter fields, with inverse Kähler metrics $(K^{-1})_{i\bar{j}}$. For the untwisted sector of symmetric orbifold models, the Kähler metric of the massive matter fields is just proportional to the metric of the massless moduli. This follows from the $SO(6, \infty)$ structure of the initial $N = 4$ toroidal compactification. Then it is not difficult to show that for the

^{*} We are interested here in the contributions to the free energy of those fermionic mass terms which originate from the effective superpotential, so we do not consider D -term masses.

maximally symmetric untwisted moduli spaces, the determinant of the moduli metric is proportional to $e^K = \frac{1}{Y}$, so that eq.(3.4) can be finally written as

$$\det(M^\dagger M) = \det \frac{|W_{ij}|^2}{Y}. \quad (3.5)$$

We now apply the above formula and compute the free energy of the simplest orbifold compactification with moduli space $\mathcal{M}_1 = \left[\frac{SO(2,2)}{SO(2) \times SO(2)} \right]^3$. The fermionic mass was given in the previous section, eq.(2.22). Since in this case the determinant in (3.5) simply reduces to a triple-product of the masses in the three subsectors, the free energy is just the sum of the free energies of each of the three subtori. Therefore it is sufficient to consider first a single two-dimensional torus with the following free energy:

$$F = \sum_{m,m',n,n'} \log \frac{|m + nTU + i(m'U + n'T)|^2}{(T + \bar{T})(U + \bar{U})}. \quad (3.6)$$

(Here and in the following the sum will only go over the massive states with $(m, m', n, n') \neq (0, 0, 0, 0)$.) This expression has already appeared in different places in the literature. For instance, the authors of ref.[8] obtained exactly this expression when calculating one-loop corrections to gauge coupling constants. Also in the recent work of ref. [32], F is the free energy of compactified $N = 2$ critical strings.

Since F is infinite due to the infinite lattice sum, one needs a regularization procedure which respects the target space duality symmetry $\Gamma = PSL(2, \mathbf{Z})_T \times PSL(2, \mathbf{Z})_U \times \mathbf{Z}_2 \times \mathbf{Z}_2$. Since we have defined F to be a topological quantity, the sum in the above formula does not run over oscillator excitations and is therefore subject to the constraint $\vec{p}_L^2 - \vec{p}_R^2 = mn + m'n' = 0$. Following [32], it is important to recognize that the constraint on the summation over n, m, n', m' can be solved in two inequivalent orbits, namely $n = m' = 0$ and $n = n' = 0$. Therefore, the free energy is just the sum of these two orbits:

$$F = \sum_{m,n'} \log \frac{|m + in'T|^2}{(T + \bar{T})} + \sum_{m,m'} \log \frac{|m + im'U|^2}{(U + \bar{U})}. \quad (3.7)$$

These two sums can be evaluated using ζ -function regularization. As described

in the appendix, one obtains[†]

$$F \Big|_{\text{reg.}} = \log[|\eta(T)|^4 |\eta(U)|^4 (T + \bar{T})(U + \bar{U})], \quad (3.8)$$

where $\eta(T)$ is the Dedekind eta function. F diverges exactly at the cusp points of the moduli space, $T, U = 0, \infty$, which correspond to the degeneration points of the spectrum. For example, for $T = \infty$ ($T = 0$) the Kaluza-Klein (winding) modes become massless and thus give an infinite contribution to the free energy. Using the well-known transformation properties of the Dedekind function under the modular transformations $PSL(2, \mathbf{Z})$, it is easy to show that the free energy is invariant under all target space duality transformations. In fact, this result could have been already anticipated from the general requirement of modular invariance of F . Specifically, the appearance of the factor $Y = (T + \bar{T})(U + \bar{U})$ in eq.(3.6) implies that $\det W_{ij}$, i.e the determinant of all holomorphic string masses, must be given by an automorphic function of the modular group $SL(2, \mathbf{Z})_{T,U}$ with modular weight 1. Moreover, the additional requirement that $\det W_{ij}$ can only diverge at $T = (\infty, 0)$ (and analogously for the U -field), where the Kaluza-Klein or the winding modes become massless and thus give an infinite contribution to the sum (3.7), and has no zeroes inside the fundamental region, uniquely implies that $\det W_{ij} = [\eta(T)\eta(U)]^2$.

It is important to emphasize that if one freezes all $h_{1,2}$ moduli, i.e. $U_A = 1$, thus truncating the moduli space to $\mathcal{M}_2 = \left[\frac{SU(1,1)}{U(1)} \right]^3$, the topological constraint $mn + m'n' = 0$ implies that the free energy can be written as a sum over six completely unconstrained integers m_A, n_A (two for each subsector):

$$F \Big|_{\text{reg.}} = \left[\sum_{i=1}^3 \sum_{m_i, n_i} \log \frac{|(m_i + in_i T_i)|^2}{(T_i + \bar{T}_i)} \right]_{\text{reg.}} = \sum_{i=1}^3 \log |\eta(T_i)|^4 (T_i + \bar{T}_i). \quad (3.9)$$

Since all orbifold compactifications contain at least these three $(1, 1)$ moduli fields T_i , this expression provides the universal part of the free energy common to all

[†] Note that the bosonic free energy, $F_B = -F$, agrees with the world-sheet string calculation in [8].

orbifold compactifications. Finally, keeping only a single modulus $T = T_1 = T_2 = T_3$ which corresponds to the overall volume of the six-space, one obtains

$$F \Big|_{\text{reg.}} = \left[\sum_{n,m} \log \frac{|m + inT|^6}{(T + \bar{T})^3} \right]_{\text{reg.}} = \log[|\eta(T)|^{12}(T + \bar{T})^3]. \quad (3.10)$$

Let us now briefly derive the free energy for a model with more than three (1,1) moduli fields, for example the case with moduli space $\mathcal{M}_4 = \frac{SU(1,1)}{U(1)} \otimes \frac{SU(2,2)}{SU(2) \times SU(2) \times U(1)}$. Since we know already the free energy of the $\frac{SU(1,1)}{U(1)}$ part, we concentrate on the second factor. Again the free energy has the form $F = \log \det \frac{|W_{ij}|^2}{Y}$ with $Y = \det(T_{ij} + \bar{T}_{ij})$ ($i, j = 1, 2$). Thus $\det W_{ij}$ has to be an automorphic function of the relevant target space duality group Γ , which is the maximal discrete subgroup of $SU(2, 2)$, namely $\Gamma = SU(2, 2, \mathbf{Z})$. Then $\det W_{ij}$ provides just the generalization of the Dedekind function to this discrete group.

To obtain the explicit form of the free energy we use the result eq.(2.27) of the previous section about the holomorphic masses:

$$F = \sum_{M_0, N_0, M_{ij}} \log \frac{|M_0 - N_0 \det T + i \text{Tr}(MT)|^2}{\det(T + \bar{T})}. \quad (3.11)$$

As before, the sum does not go over six independent integers, but the integers have to satisfy the topological level matching constraint $\vec{p}_L^2 - \vec{p}_R^2 = 0$. This feature is in fact true for all the various lattices, i.e. orbifold models, and defines the orbits which contribute to the sum in the corresponding automorphic functions. For this particular example this condition reads

$$-M_0 N_0 + \det M_{ij} = 0. \quad (3.12)$$

Before discussing the solutions for this constraint, we first determine the duality transformation properties of M_0 , N_0 and M_{ij} . First, the two $\mathbf{Z}_2$ symmetries $T_{11} \leftrightarrow T_{22}$ and $T_{12} \leftrightarrow T_{21}$ clearly have to be accompanied by $M_{11} \leftrightarrow M_{22}$, $M_{12} \leftrightarrow M_{21}$ whereas M_0 and N_0 do not transform. Second, the transformation $T_{ij} \leftrightarrow (T^{-1})_{ij}$ induces $M_0 \leftrightarrow N_0$, $M_{11} \leftrightarrow M_{22}$ and $M_{21} \leftrightarrow -M_{12}$. Third, the shift $T_{11} \rightarrow T_{11} + i$ must be absorbed by $M_0 \rightarrow M_0 + M_{11}$ and $M_{22} \rightarrow M_{22} + N_0$ and analogously for the shifts of T_{22} , T_{12} and T_{21} .

Unlike the case of a single complex modulus T , the constraint (3.12) cannot be uniquely solved. This means that there exist more than one inequivalent orbit in which N_0 , M_0 and M_{ij} are expressed by unconstrained momentum and winding numbers. For example, one can solve (3.12) by replacing N_0 , M_0 and M_{ij} by two unconstrained winding and momentum numbers m_i , n_i in the following way:

$$\begin{aligned} M_0 &= m_1 m_2, & N_0 &= n_1 n_2, \\ M_{11} &= n_1 m_2, & M_{22} &= n_2 m_1, \\ M_{12} &= M_{21} = 0. \end{aligned} \tag{3.13}$$

Making this choice one recognizes that the free energy eq.(3.11) splits in the limit $T_{12} = T_{21} = 0$ into the following sum:

$$F = \sum_{m_1, n_1} \log \frac{|m_1 + i n_1 T_{11}|^2}{T_{11} + \bar{T}_{11}} + \sum_{m_2, n_2} \log \frac{|m_2 + i n_2 T_{22}|^2}{T_{22} + \bar{T}_{22}} \tag{3.14}$$

This is just the free energy corresponding to the case $\mathcal{M} = \left(\frac{SU(1,1)}{U(1)} \right)^2$. However the constraint (3.12) is equally well solved in the inequivalent orbit

$$\begin{aligned} M_0 &= m_1 m_2, & N_0 &= n_1 n_2, \\ M_{12} &= n_2 m_1, & M_{21} &= -n_1 m_2, \\ M_{11} &= M_{22} = 0. \end{aligned} \tag{3.15}$$

Obviously, this choice does not lead in the limit $T_{12} = T_{21} = 0$ to eq.(3.14). This strange-sounding result has the following explanation. Setting $T_{12} = T_{21} = 0$, i.e. truncating the moduli space from $\frac{SU(2,2)}{SU(2) \times SU(2) \times U(1)}$ to $\left(\frac{SU(1,1)}{U(1)} \right)^2$, does not imply an analogous truncation on the massive spectrum. This means the massive spectrum obeying the constraint (3.12) is not identical to the massive spectrum of the $\left(\frac{SU(1,1)}{U(1)} \right)^2$ model after the truncation $T_{12} = T_{21} = 0$. Only the sector with the choice eq.(3.13) provides the proper truncation on the massive spectrum and leads to the factorization (3.14) of e^F .

So far we have defined and computed the free energy of string compactifications starting from the effective field theory for the massive modes. However,

identical results are obtained via string computations. Working directly in string theory with D compactified coordinates, the free energy is a ratio of world-sheet determinants of the $\partial_z \partial_{\bar{z}}$ operator, the one in the flat D -dimensional space $\mathbf{R}^D$ over the one in the D -dimensional internal space $\mathbf{M}^D$:

$$F = \int_{\mathcal{F}} \frac{d^2\tau}{(\text{Im}\tau)^2} \left(\frac{(\det \partial_z \partial_{\bar{z}})_{\mathbf{R}^D}}{V(\mathbf{R}^D)(\det \partial_z \partial_{\bar{z}})_{\mathbf{M}^D}} - 1 \right). \quad (3.16)$$

Notice that in the case of $\mathbf{R}^D$ the volume of the target space is infinite and must be divided out. Using $\frac{(\det \partial_z \partial_{\bar{z}})_{\mathbf{R}^D}}{V(\mathbf{R}^D)} = (\text{Im}\tau)^{D/2} |\eta(\tau)|^{2D}$, we see that the oscillator contribution $|\eta|^{2D}$ is dropped out and only the classical, zero-mode part remains. Thus, the free energy is purely topological and is given in terms of the Ray-Singer torsion T [33]:

$$F = \int_{\mathcal{F}} \frac{d^2\tau}{(\text{Im}\tau)^2} (T - 1), \quad T = \frac{(\det \partial_z \partial_{\bar{z}})_{\mathbf{R}^D}}{V(\mathbf{R}^D)(\det \partial_z \partial_{\bar{z}})_{\mathbf{M}^D}}. \quad (3.17)$$

For example, consider the two-dimensional toroidal compactification with moduli space $\mathcal{M} = \frac{SO(2,2)}{SO(2) \times SO(2)}$. Here we obtain that

$$(\det \partial_z \partial_{\bar{z}})_{T_2}^{-1} = \frac{\Gamma(2,2)}{(\eta\bar{\eta})^2} \quad (3.18)$$

where $\Gamma(2,2)$ is the lattice partition function of the two-torus,

$$\Gamma(2,2) = \sum_{\vec{p}_L, \vec{p}_R} e^{i\pi\tau \vec{p}_L^2} e^{-i\pi\bar{\tau} \vec{p}_R^2}. \quad (3.19)$$

This world-sheet integral was computed in ref.[8] for another purpose, and the result exactly coincides with eq.(3.8).

4. Automorphic functions of general Calabi-Yau compactifications

In the last section we have shown that, for $N = 1$ symmetric orbifold compactifications, the target-space partition function, i.e. the exponential of the topological free energy, always takes the form

$$e^{-F} \Big|_{\text{reg.}} = \frac{|\Delta(T_i)|^2}{Y}, \quad (4.1)$$

where $\Delta(T_i)$, defined as the determinant of all chiral masses in the compactification spectrum, is a holomorphic function of the moduli fields T_i . $\Delta(T_i)$ is an automorphic function of the discrete target-space duality group Γ . More precisely, it is inversely proportional to some power of the cusp form (the generalization of $\eta(T)^{24}$) of the duality group, since only for infinite (zero) radius of the internal space does the contribution of the Kaluza-Klein (winding) modes create a pole for Δ .

More generally, $\Delta(T_i)$ can be regarded as a holomorphic section of a line bundle over the moduli space $\mathcal{M}$. In particular, since the target space duality transformations $\Gamma : T_i \rightarrow \tilde{T}_i(T_j)$ act as Kähler transformations on the Kähler potential $K = -\log Y$,

$$Y \rightarrow Y |e^{-\Lambda(T_i)}|^2 \quad \text{for} \quad T_i \rightarrow \tilde{T}_i, \quad (4.2)$$

the holomorphic section Δ has to transform as

$$\Delta(T_i) \rightarrow \Delta(T_i) e^{-\Lambda(T_i)}. \quad (4.3)$$

The duality invariant norm of the holomorphic section is defined by the Kähler potential:

$$||\Delta(T_i)||^2 = \Delta(T_i) e^K \bar{\Delta}(\tilde{T}_i) = e^{-F} \Big|_{\text{reg.}}. \quad (4.4)$$

Note that it is impossible to write the modular invariant target-space partition function $e^{-F}|_{\text{reg.}} = ||\Delta||^2$ as a perfect holomorphic square. The non-holomorphic

part, namely e^K , describes a holomorphic anomaly and is related to a Kähler anomaly of the moduli space $\mathcal{M}$. More precisely, the non-holomorphicity of $e^{-F}|_{\text{reg.}}$ is due to a $U(1)$ anomaly (see also the discussion at the end of this section). This universal $U(1)$ is just related to the $U(1)$ symmetry of the internal $(2,2)$ superconformal algebra. For the case of $\mathcal{M} = \frac{SU(1,1)}{U(1)}$ with modular group $SL(2, \mathbf{Z})$, this anomaly is given by Quillen’s holomorphic anomaly, which means that the Eisenstein function $G_2 = \sum_{m,n} (m + inT)^{-2}$ can either be regularized in a holomorphic way, which however destroys the modular covariance (this regularization leads to $G_2(T)|_{\text{reg.}} = -4\pi \frac{\eta'(T)}{\eta(T)}$), or in a non-holomorphic way, which leads to an automorphic function with modular weight two (namely to $\hat{G}_2(\bar{T}, T)|_{\text{reg.}} = -4\pi \frac{\eta'(T)}{\eta(T)} - \frac{2\pi}{T+\bar{T}}$). Thus, in general, the regularization procedure which is necessary to define $\Delta(M_i)$ will either destroy holomorphicity or modular invariance. However we have to insist on the latter.

The form of the orbifold free energy is completely analogous to the form of the G -function itself: $e^G = \frac{|W|^2}{Y}$. Specifically, as discussed in ref.[5], the purely moduli-dependent superpotential, which may arise due to non-perturbative (∞ -genus) string effects, must be an automorphic function of the target-space duality group. For example, for all $N = 1$, $d = 4$ orbifold models, keeping only the overall $(1,1)$ modulus $T = R^2 + iB$, we have that $\Gamma = SL(2, \mathbf{Z})$ and $Y = (T + \bar{T})^3$. Then the only choice for $W(T)$ which is finite and non-zero inside the fundamental region is fixed to be

$$W(T) = \frac{1}{\eta(T)^6} = \left[\prod_{m,n} (m + inT)^{-3} \right]_{\text{reg.}}. \quad (4.5)$$

This expression provides a modular-invariant completion of the “field-theoretical” T -independent superpotential, which is obtained neglecting all string winding modes by setting $n = 0$ in the above equation. Thus, for all symmetric orbifold compactifications, the non-perturbative superpotential is determined by exactly the same holomorphic section as the free energy, i.e. the non-perturbative superpotential is nothing other than the determinant of all fermionic chiral masses which are due to the compactification, including momentum as well as winding states: $W = (\det M)^{-1}$. In the following we will give strong arguments for the conjecture that this fact is also true for general Calabi-Yau compactifications.

The identification of the non-perturbative superpotential with the determinant of the chiral masses can be understood in the context of gaugino condensation [34],[9]. The free energy we have computed can be viewed as the stringy version of Weinberg’s formula [35] for the moduli-dependent threshold corrections to the gauge coupling constants at one loop [8]. Specifically, $\log \Delta(T_i) = -\log \det M_i$ is proportional to the holomorphic part of the threshold correction to the inverse gauge coupling constant, $g^{-2}(T_i) \propto \log \Delta(T_i)$. It follows that the non-perturbative superpotential, which is proportional to the “holomorphic” gaugino condensate $\langle \lambda\lambda \rangle \propto e^{1/g^2(T_i)}$, is given by $W(T_i) = \Delta(T_i) \propto \lambda\lambda$.

We now construct the holomorphic section $\Delta(T_i)$ for all (2,2) symmetric Calabi-Yau compactifications [36]. This then provides an expression for the non-perturbative superpotential, $W = \Delta$, and, if our above conjecture is true, an expression for the determinant of all chiral masses of the Calabi-Yau compactification, $(\det M)^{-1} = \Delta$, i.e. the free energy of the compactified system. For this purpose we will use the important observation [37] that the metric of the moduli space of Calabi-Yau manifolds should not only be Kähler, as implied by $N = 1$ space-time supersymmetry, but also “special Kähler”, as implied by $N = 2$ space-time supersymmetry when one regards the (2,2) theories as possible vacua for type II superstrings [38]. Specifically, in type IIA theories the (1,1) moduli belong to vector multiplets, whereas the (2,1) moduli belong to hypermultiplets. In type IIB theories the situation is reversed. Since the particular form of the moduli space does not depend on whether we deal with a heterotic, type IIA or type IIB theory, the couplings of the moduli fields are strongly constrained by the possible couplings of $N = 2$ vector multiplets.

In the $N = 2$ superconformal tensor calculus [39], one describes the couplings of n physical vector multiplets by first introducing $n + 1$ vector multiplets, whose scalar components we denote by L^I , $I = 0, 1, \dots, n$. The vector component of the extra (compensating) multiplet is the spin-one field of the $N = 2$ supergravity multiplet. The L^I are complex, covariantly holomorphic functions of the complex physical moduli fields T^i , $i = 1, \dots, n$: $L^I = L^I(\bar{T}^i, T^i)$ (see ref. [40] for details). The special Kähler potential can then be written as

$$K = -\log Y = -\log \frac{L^I \bar{L}^J N_{IJ}}{L^0 \bar{L}^0} \quad (4.6)$$

where

$$N_{IJ} = \mathcal{F}_{IJ}(L) + \bar{\mathcal{F}}_{IJ}(\bar{L}) \quad (4.7)$$

$$\mathcal{F}_{IJ} = \frac{\partial}{\partial L^I} \frac{\partial}{\partial L^J} \mathcal{F}(L) \quad (4.8)$$

and $\mathcal{F}(L)$ is a homogeneous holomorphic function of degree 2 in the $n+1$ variables L . One can also introduce completely holomorphic fields (sections) $X^I = X^I(T^i)$,

$$X^I = e^{-K/2} L^I, \quad (4.9)$$

and the Kähler potential looks like (using the homogeneity properties of $\mathcal{F}$)

$$K = -\log(X^I \bar{\mathcal{F}}_I + \bar{X}^I \mathcal{F}_I). \quad (4.10)$$

These formulas are independent of a particular coordinate system. However, to compare with the previous formulas it is very convenient to choose a “special” coordinate system, the “special” gauge, in the following way:

$$T^i = -iX^i/X^0. \quad (4.11)$$

In this gauge the Kähler potential takes the form

$$K = -\log \left[\sum_i (T^i + \bar{T}^i)(f_i + \bar{f}_i) - 2(f + \bar{f}) \right] \quad (4.12)$$

where $f(T) = (X^0)^{-2} \mathcal{F}(X)$.

After this brief introduction to the special Kähler geometry, we are now ready to construct the automorphic function $\Delta(M^i)$ of the Calabi-Yau modular group Γ . Since the function $\frac{|\Delta|^2}{Y}$ must be duality-invariant, $\Delta(T^i)$ must be a holomorphic section of holomorphic degree one (Y has degree $(1,1)$). Holomorphic sections of exactly this degree are X^I and $\mathcal{F}_I$. Therefore the most natural ansatz for Δ

is to take a product of all possible linear combinations of these two holomorphic functions:

$$\log \frac{|\Delta|^2}{Y} = \left[- \sum_{M_I, N^I} \log \frac{|M_I X^I + i N^I \mathcal{F}_I|^2}{X^I \bar{\mathcal{F}}_I + \bar{X}^I \mathcal{F}_I} \right]_{\text{reg.}} \quad (4.13)$$

Using the covariantly holomorphic functions L^I and $P_I = e^{K/2} \mathcal{F}_I$, this expression can be written in a slightly simpler form:

$$\log \frac{|\Delta|^2}{Y} = \left[- \sum_{M_I, N^I} \log |M_I L^I + i N^I P_I|^2 \right]_{\text{reg.}} \quad (4.14)$$

Finally, in the special gauge eq.(4.11) we obtain the following expression:

$$\log \frac{|\Delta|^2}{Y} = \left[- \sum_{M_I, N^I} \log \frac{|M_0 + M_i T^i + i N^0 f + i N^i f_i|^2}{(T^i + \bar{T}^i)(f_i + \bar{f}_i) - 2(f + \bar{f})} \right]_{\text{reg.}} \quad (4.15)$$

As already discussed, the holomorphic section Δ is an automorphic function, inversely proportional to the cusp form, of the Calabi-Yau duality group Γ . Δ has to transform under target space duality transformations in such a way that $\frac{|\Delta|^2}{Y}$ is a completely invariant function. This requirement implies that the product over M_I and N^I in eqs.(4.13), (4.14), (4.15) does not go over all but only over a restricted set of integers. This restriction is equivalent to the topological level matching condition $\vec{p}_L^2 - \vec{p}_R^2 = 0$ for orbifold compactifications. To understand this, recall (see for example [40]) that the target space duality transformations act (up to a holomorphic function) as particular $Sp(2n+2)$ transformations on the vector $(X^I, i\mathcal{F}_I)$:

$$\begin{pmatrix} X^I \\ i\mathcal{F}_I \end{pmatrix} \rightarrow \begin{pmatrix} B & D \\ C & A \end{pmatrix} \begin{pmatrix} X^I \\ i\mathcal{F}_I \end{pmatrix}. \quad (4.16)$$

A, B, C, D are real constant matrices and provide a linear realization of the target space modular group, i.e. $\Gamma \subset Sp(2n+2)$. The symplectic condition implies that

A, B, C, D have to satisfy

$$\begin{aligned} B^T C - C^T B &= 0, \\ D^T A - A^T D &= 0, \\ B^T A - C^T D &= \alpha 1, \end{aligned} \tag{4.17}$$

where α is a real constant. These symplectic transformations on X^I , $i\mathcal{F}_I$ leave the Kähler potential invariant up to a Kähler transformation. Note that not any symplectic transformation corresponds to a duality transformation; the specific form of the matrices A, B, C, D depends on the particular holomorphic function $\mathcal{F}$ and on the relevant duality group. It is now evident that eqs.(4.13), (4.14), (4.15) provide a duality invariant expression only if the symplectic transformations on X^I and $i\mathcal{F}_I$ are accompanied by the following transformations on the integers M_I and N^I :

$$\begin{pmatrix} M_I \\ -N^I \end{pmatrix} \rightarrow \begin{pmatrix} C & A \\ B & D \end{pmatrix} \begin{pmatrix} M_I \\ -N^I \end{pmatrix}. \tag{4.18}$$

This means that (M_I, N^I) must build proper representations of the symplectic duality transformations, i.e. they form a restricted set of integers. In other words, take a given pair of (M_I, N^I) ; then the action of the symplectic matrices A, B, C, D , which is in general of infinite order, generates all possible integers N_I and M^I which may contribute to Δ . This means that the product in (4.13), (4.14), (4.15) has to go over the specific orbits $\mathcal{O} (M^I, N_I \in \mathcal{O})$ which are generated by the particular symplectic transformations.

As already said, it is very natural to regard $\Delta(T_i)^{-1} = [\prod_{M_I, N^I} (M_I X^I + i N^I \mathcal{F}_I)]_{\text{reg.}}$ as the determinant of all chiral masses of the compactified string which provides at the same time an expression for the non-perturbative, moduli-dependent superpotential. We shall now see that this identification becomes very plausible by discussing the form of the possible mass terms as they arise in the corresponding type II compactifications with enlarged $N = 2$ space-time supersymmetry. (Of course, the chiral masses and therefore also the free energies of the corresponding heterotic and type II theories are identical.)

Consider first the four-dimensional type II A theories with the (1,1) moduli fields belonging to $N = 2$ vector multiplets. Then the chiral masses are described by the couplings between one $N = 2$ vector multiplet and two $N = 2$

hypermultiplets. This means that the relevant massive string states are hypermultiplets with masses given by the *vev*'s of the moduli scalars inside the $(1, 1)$ vector multiplets. The general field theoretical couplings among the vector- and hypermultiplets in “gauged” $N = 2$ supergravity theories were recently discussed in ref. [41] and also already some time ago in ref.[39]. These are of the form $X^I H H'$ where H, H' denote the massive hypermultiplet fields. Thus we see that the chiral $N = 2$ masses are just given by the holomorphic degree-one functions $X^I(T_i)$. However if one only considers these “field theoretical” masses X^I it is clear that the spectrum is not invariant under the target space duality transformations which mix X^I with the other degree-one functions X^J and $i\mathcal{F}_I$ in a non-trivial way. In fact, the duality transformations act on an infinite number of massive $N = 2$ hypermultiplets. This means only an infinite number of massive hypermultiplets provide a duality invariant $N = 2$ spectrum with masses given by the X^I and also by the “dual” fields $\mathcal{F}_I$. Therefore the most general ansatz for the hypermultiplet masses M is to take $M = M_I X^I + iN^I \mathcal{F}_I$, where the M_I, N^I are a restricted set of integers satisfying the constraints inferred by the symplectic modular transformations. It is important to note that this expression for the chiral masses exactly provides the duality invariant generalization of the $N = 4$ gauging procedure of ref.[12] for the case of the special Kähler geometry of $N = 2$ supergravity.

To obtain a clearer understanding about the generalized momentum (Kaluza-Klein) and winding spectrum of a Calabi-Yau compactification let us consider the large radius, i.e field theory, limit (the weak coupling limit of the corresponding σ -model) of the Calabi-Yau compactification. In this limit $Y_{(1,1)}$ just becomes the volume of the six-dimensional Calabi-Yau space,

$$Y_{(1,1)} = \int J \wedge J \wedge J = d_{ijk}(T_i + \bar{T}_i)(T_j + \bar{T}_j)(T_k + \bar{T}_k), \quad (4.19)$$

where $J = \sum_{i=1}^n (T^i + \bar{T}^i) V_i$ and V_i being the harmonic $(1,1)$ forms in $H^{(1,1)}$. The corresponding holomorphic function $f(T_i)$ which leads to (4.19) has the form $f(T_i) = d_{ijk} T_i T_j T_k$. In the field theory limit d_{ijk} coincide with the intersection matrices of V_i : $d_{ijk} = \int V_i \wedge V_j \wedge V_k$. For the case of a single modulus the Kähler potential becomes $K = -3 \log(T + \bar{T})$ in the large radius limit. Now choose the

special gauge eq.(4.11) and consider those “masses” in eq.(4.15) which have only non-vanishing M_0 :

$$M_{M_0}^2 = \frac{M_0^2}{Y} = \frac{M_0^2}{\int J \wedge J \wedge J}. \quad (4.20)$$

We recognize that in the field theory limit the masses $M_{M_0}^2$ are inversely proportional to the volume of the Calabi-Yau space. This fact strongly suggests that $M_{M_0}^2$ are just the ordinary field-theoretical Kaluza-Klein masses, i.e. that the momentum spectrum of the Calabi-Yau compactification is given by the states with only non-vanishing M_0 .

On the other hand, for small radii, non-perturbative effects, i.e. instanton configurations [42] which give a non-trivial map of the world-sheet onto the Calabi-Yau space modify eqs.(4.19),(4.20). (They also break the continuous Peccei-Quinn symmetry $T^i \rightarrow T^i + iC^i$, $C^i \in \mathbf{R}$, to a discrete subgroup.) At the same time the generalized winding states become light. The symplectic duality invariance enforces to also include the winding states with non-vanishing M_i and N^I into the target space free energy. Thus, the masses of the winding states seem to be determined by $M_i T^i$ and in particular by $N^I \mathcal{F}_I$. In summary, just like for toroidal/orbifold compactifications we can regard the states with non-vanishing M_0 as momentum states, and the states with non-vanishing M_i and N^I as generalized winding states (vortex configurations). (In the context of the following orbifold example M_0 is a pure momentum number whereas N^0 is a pure winding number. The M_i , N^i ($i = 1, \dots, n$) are products of winding and momentum numbers.) Thus, the M_I and N^I define, in some sense, a generalized Calabi-Yau lattice.

A very similar interpretation is valid when regarding $\Delta(T_i)$ as the non-perturbative superpotential. As discussed in [43], the most general, “field theoretical” superpotential compatible with the special Kähler geometry has the form $\log W \sim -\log M_I X^I$. (In the special gauge, the constant superpotential is proportional to X^0 .) Thus, Δ again provides the duality invariant completion of this already known expression. This also shows that the orbifold superpotential $\log W = -\sum_{m,n} \log(m + inT)^3 = -6 \log \eta(T)$ is compatible with the special Kähler structure of $N = 2$ supersymmetry (see also the following example).

So far we have been discussing the couplings of the $(1, 1)$ moduli fields. However, analogous arguments are also true for the $(2, 1)$ moduli fields. This comes from the fact that we can use the same Calabi-Yau background (the same $(2, 2)$ superconformal field theory) for type IIB compactifications, where the $(2, 1)$ moduli fields belong to $N = 2$ vector multiplets. Remember that the $(2, 1)$ moduli describe the deformations of the complex structure of the Calabi-Yau space. The moduli space for these fields reflects the homology of the underlying Calabi-Yau manifold. More precisely, the complex structure can be described by the periods of the holomorphic three-form Ω over a canonical homology basis. Choosing a symplectic basis (A^I, B_I) such that $A^I \cap B_J = \delta_B^A$, $A^I \cap A^J = 0$ and $B_I \cap B_J = 0$, then the $2n + 2$ functions $(X^I, \mathcal{F}_J)$ are nothing other than the periods of Ω :

$$X^I = \int_{A^I} \Omega, \quad i\mathcal{F}_J = \int_{B_J} \Omega. \quad (4.21)$$

The symplectic modular transformations on the $X^I, i\mathcal{F}_I$ therefore just describe the action of the target space duality group on the homology cycles. (This is in complete analogy to the action of the so-called Dehn twists around the non-trivial cycles of a higher genus Riemann surface.) Thus the holomorphic section Δ is just the product of all possible linear combinations of the periods of Ω , and the periods are just the chiral masses of the massive hypermultiplets in the corresponding $N = 2$, type IIB theory.

From eqs.(4.10) and (4.21) it follows that the Kähler potential of the $(2, 1)$ moduli is given as

$$K_{(2,1)} = -\log(-i \int \Omega \wedge \bar{\Omega}). \quad (4.22)$$

Then the “masses” M_{M_I, N^I}^2 look like

$$M_{M_I, N^I}^2 = \frac{|M_I \int_{A^I} \Omega + N^I \int_{B_I} \Omega|^2}{-i \int \Omega \wedge \bar{\Omega}}. \quad (4.23)$$

In contrast to the $(1, 1)$ moduli, the Kähler potential eq.(4.22) as well as the periods $X^I, i\mathcal{F}_I$ do not receive any quantum corrections from world-sheet instantons. They are entirely determined by their field-theoretical expressions. This is also

a reflection of the fact that the duality transformations involving the $(2,1)$ fields have a geometrical origin and do not involve the exchange of momentum and topological winding modes. Thus we expect that mass spectrum and the free energy in the $(2,1)$ sector is also of pure field theoretical nature and can be computed by algebraic methods as demonstrated for a specific example in [31]. Then, for the compactifications which appear in mirror pairs the knowledge about the $(2,1)$ mass spectrum provides important information about the winding spectrum in the $(1,1)$ sector of the corresponding mirror image Calabi-Yau space.

Now we show that eq.(4.13) reproduces the results for the free energy as computed in the previous section for the case of the orbifold compactifications thus giving support for its general validity. First, consider the orbifolds with moduli space $\mathcal{M} = \left(\frac{SU(1,1)}{U(1)} \right)^3$. The corresponding holomorphic function $\mathcal{F}(X)$ takes the form [43]:

$$\mathcal{F}(X) = i \frac{X^1 X^2 X^3}{X^0}. \quad (4.24)$$

In the special gauge the standard moduli fields are given by

$$T_i = -i \frac{X^i}{X^0}, \quad i = 1, 2, 3. \quad (4.25)$$

Then, using eq.(4.12), it is not difficult to show that $Y = \prod_{i=1}^3 (T_i + \bar{T}_i)$. Eq.(4.13) now leads to the following expression for $\frac{|\Delta|^2}{Y}$:

$$\log \frac{|\Delta|^2}{Y} = \left[- \sum_{M_I, N^I} \log \frac{1}{Y} |M_0 + iM_1 T_1 + iM_2 T_2 + iM_3 T_3 - iN^0 T_1 T_2 T_3 + N^1 T_2 T_3 + N^2 T_1 T_3 + N^3 T_1 T_2|^2 \right]_{\text{reg.}}. \quad (4.26)$$

On the other hand, the corresponding orbifold calculation of the previous section led to the expression (3.9) for the free energy. The two formulae (3.9) and (4.26) coincide, provided the following relations among the M_I , N^I and the unrestricted

integers m_i, n_i hold:

$$\begin{aligned}
M_0 &= m_1 m_2 m_3, & N^0 &= n_1 n_2 n_3, \\
M_1 &= n_1 m_2 m_3, & N^1 &= -m_1 n_2 n_3, \\
M_2 &= n_2 m_1 m_3, & N^2 &= -m_2 n_1 n_3, \\
M_3 &= n_3 m_1 m_2, & N^3 &= -m_3 n_1 n_2.
\end{aligned} \tag{4.27}$$

We recognize that M_0 is a product of three momentum numbers and N^0 is a product of three winding numbers, whereas the others contain both momentum and winding numbers. Thus the pure Kaluza-Klein spectrum is entirely described by M_0 .

Now it is crucial to observe that this constraint on M_I and N^I is just the orbit $\mathcal{O}$ which is inferred by the symplectic modular transformations on M_I and N^I . Specifically, consider the action of the target space duality modular group $\Gamma = [SL(2, \mathbf{Z})]^3 \subset Sp(8, \mathbf{Z})$. First, the transformation $T_1 \rightarrow T_1 - i$ acts as the following $Sp(8, \mathbf{Z})$ transformation on M_I and N^I (see also [44]):

$$\begin{aligned}
M_0 &\rightarrow M_0 - M_1, & N^0 &\rightarrow N^0, \\
M_1 &\rightarrow M_1, & N^1 &\rightarrow N^0 + N^1, \\
M_2 &\rightarrow M_2 + N^3, & N^2 &\rightarrow N^2, \\
M_3 &\rightarrow M_3 + N^2, & N^3 &\rightarrow N^3
\end{aligned} \tag{4.28}$$

and similarly for the shifts of T_2 and T_3 . Second, the inversion $T_1 \rightarrow \frac{1}{T_1}$ acts as the symplectic transformation

$$\begin{aligned}
M_0 &\rightarrow -M_1, & N^0 &\rightarrow -N^1, \\
M_1 &\rightarrow M_0, & N^1 &\rightarrow N^0, \\
M_2 &\rightarrow N^3, & N^2 &\rightarrow -M_3, \\
M_3 &\rightarrow N^2, & N^3 &\rightarrow -M_2
\end{aligned} \tag{4.29}$$

plus analogous transformations for the inversions of T_2 and T_3 . On the other hand, we know already that the modular transformation $T_1 \rightarrow T_1 - i$ acts on the unconstrained integers as $m_1 \rightarrow m_1 - n_1$ where the remaining m_i, n_i are

not transformed. Analogously the transformation $T_1 \rightarrow \frac{1}{T_1}$ induces $m_1 \rightarrow -n_1$, $n_1 \rightarrow m_1$. However this transformation on the unconstrained integers exactly induces the symplectic transformations (4.28) and (4.29) on M_I, N^I when using the constraint eq.(4.27). This shows that this identification exactly generates the set of integers M_I and N^I which correctly transform under the symplectic modular transformations. Therefore we conclude that the formula (4.26) with restricted M^I, N_I and the orbifold free energy (3.9) agree.

It is also illustrative to consider briefly the simplest case with moduli space $\mathcal{M} = \frac{SU(1,1)}{U(1)}$ of the overall modulus which is obtained by identifying all three moduli fields: $T = T_1 = T_2 = T_3$. The corresponding holomorphic function $\mathcal{F}(X)$ looks like $\mathcal{F} = i \frac{(X^1)^3}{X^0}$. Then the holomorphic section Δ takes the form:

$$\Delta(T) = \left[\prod_{M_I, N^I} (M_0 - iN^0 T^3 + iM_1 T + 3N^1 T^2)^{-1} \right]_{\text{reg.}}. \quad (4.30)$$

In order to reproduce the free energy (3.10) with $\Delta(T) = [\prod_{m,n} (m + inT)^{-3}]_{\text{reg.}} = \frac{1}{\eta(T)^6}$ the product has to go over the following set of restricted integers:

$$\begin{aligned} M_0 &= m^3, & N^0 &= n^3, \\ M_1 &= 3m^2 n, & N^1 &= -mn^2. \end{aligned} \quad (4.31)$$

Again this restriction is exactly the one which defines the $SL(2, \mathbf{Z})$ orbit when demanding the proper transformation properties of M_I, N^I under the symplectic modular transformations. These are for $T \rightarrow T - i$

$$\begin{aligned} M_0 &\rightarrow M_0 - M_1 - N^0 - 3N^1, \\ M_1 &\rightarrow M_1 + 3N^0 + 6N^1, \\ N^0 &\rightarrow N^0, \\ N^1 &\rightarrow N^0 + N^1. \end{aligned} \quad (4.32)$$

and for $T \rightarrow \frac{1}{T}$

$$\begin{aligned} M_0 &\rightarrow -N^0, \\ M_1 &\rightarrow -3N^1, \\ N^0 &\rightarrow M_0, \\ N^1 &\rightarrow \frac{1}{3}M_1. \end{aligned} \quad (4.33)$$

These are exactly the symplectic transformations derived in ref. [45].

Finally it is also not difficult to show that eq.(4.13) also reproduces the orbifold result eq.(3.11) which corresponds to the orbifold compactifications with moduli space $\mathcal{M} = \frac{SU(2,2)}{SU(2) \times SU(2) \times U(1)} \otimes \frac{SU(1,1)}{U(1)}$ using the holomorphic function $\mathcal{F}(X) = i \frac{\det X^{ij}}{X^0} X^3$ ($i, j = 1, 2$).

Let us emphasize that $\Delta(T_i)$ is in general not the unique automorphic function of the Calabi-Yau target space duality group Γ . Remember that $\Delta(T_i)$ is the generalization of the (inverse) Dedekind function such that $|\Delta|^2/Y$ is duality invariant. However, apart from $Y = e^{-K}$, which determines the Kähler metric $G_{ij} = -\partial_i \partial_{\bar{j}} \log Y$ for the moduli and also the massless matter fields, there exists another globally defined real function, now of degree zero, namely $\det N_{IJ} = \det[\mathcal{F}_{IJ}(X) + \bar{\mathcal{F}}_{IJ}(\bar{X})]$. This function provides important information about the moduli-dependent tree-level Yukawa couplings among the charged matter fields:

$$||C||_{ij}^2 = C_{imp} \bar{C}_{\bar{j}\bar{n}\bar{q}} G^{m\bar{n}} G^{p\bar{q}} = -\partial_i \partial_{\bar{j}} \log \det -N_{IJ}. \quad (4.34)$$

Here C_{ijk} are covariantly holomorphic 3-index tensors which are related to the holomorphic Yukawa couplings W_{ijk} of the massless matter fields in the following way (see [40] for details): $W_{ijk}(T_i) = 2e^{-K} C_{ijk}(T_i, \bar{T}_i)$. Eq.(4.34) follows from the fact that the Riemann tensor of a special Kähler space has to satisfy the constraint

$$R_{ijk\bar{l}} = G_{i\bar{j}} G_{k\bar{l}} + G_{i\bar{l}} G_{k\bar{j}} - C_{ikm} \bar{C}_{j\bar{l}\bar{n}} G^{m\bar{n}}. \quad (4.35)$$

Now, using eq.(4.16), the target space duality transformations act on $\mathcal{F}_{IJ}$ in the following way (in matrix notation):

$$\mathcal{F} \rightarrow (A\mathcal{F} - iC)(iD\mathcal{F} + B)^{-1}. \quad (4.36)$$

Remembering that for the case of the $(2, 1)$ moduli the X^I , $i\mathcal{F}_I$ are nothing other than the periods of the holomorphic three form Ω it follows that $\mathcal{F}_{IJ}$ is just the symmetric period matrix of the Calabi-Yau space. (Again, all these features are in complete analogy to the symplectic modular transformations in the theory of higher genus Riemann surfaces.)

Using eq.(4.17) it is not difficult to show that $\det N_{IJ}$ transforms as

$$\det N_{IJ} \rightarrow |\det(iD\mathcal{F} + B)|^{-2} \det N_{IJ}. \quad (4.37)$$

Now, in analogy to our previous discussion, we are looking for automorphic functions $\Pi(T_i)$ of degree zero such that

$$\frac{|\Pi(T_i)|^2}{(\det N_{IJ}(T_i, \bar{T}_i))^r} \quad (4.38)$$

is duality invariant. Thus $\Pi(T_i)$ has to transform (up to a phase) as

$$\Pi(T_i) \rightarrow \det(iD\mathcal{F} + B)^{-r} \Pi(T_i). \quad (4.39)$$

Thus r defines the weight of Π , its exact value depends on the particular model. ($\Pi(T_i)$ transforms analogously to the Riemann theta functions of more than one variable.) Since Π is of holomorphic degree zero it must be a function of $\mathcal{F}_{IJ}$. To explicitly construct Π one can make the following formal ansatz:

$$\Pi(T_i) = [\det \prod_{M_{IJ}, N_{IK}} (M_{IJ} + iN_{IK}\mathcal{F}_{KJ})]_{\text{reg.}}^r. \quad (4.40)$$

It follows that the target space duality transformation (4.36) induces the following transformation on M_{IJ} and N_{IJ} :

$$M \rightarrow MB + NC, \quad N \rightarrow MD + NA. \quad (4.41)$$

Thus, in order to obtain a duality invariant expression, the product again has to go over an infinite, but restricted set of integers M_{IJ} , N_{IJ} .

In general, the automorphic functions $\Pi(T_i)$ are not identical to the previously discussed automorphic functions $\Delta(T_i)$. However, considering maximally symmetric moduli spaces as they arise for example for all orbifold compactifications one can show that $(\det N_{IJ})^r$ is exactly proportional to Y . Thus in these cases $\Delta(T_i)$ and $\Pi(T_i)$ are expected to agree after a proper restriction on the integers M_{IJ} and N_{IJ} .

The non-holomorphic structures K and $\det N_{IJ}$ previously defined are relevant to discuss, on general grounds, moduli-dependent corrections to the gauge couplings or to the dilaton Kähler potential due to one-loop (genus one) string effects. These corrections can be discussed in full generality for smooth Calabi-Yau manifolds, whilst they are somewhat model-dependent for orbifold compactifications. According to refs. [46], [47], [48], these moduli-dependent corrections can be interpreted as coming from supersymmetrization of anomalies for target-space duality transformations. They are due to a mixed $U(1)$ gauge anomaly for the gauge groups E_8 and E_6 (for general Calabi-Yau compactifications), where the $U(1)$ connection is a composite Kähler-holonomy connection. The Kähler connection is the vector component of the superfield, whose first component is the Kähler potential, whilst the $U(1)$ holonomy connection is the vector component of a superfield, whose first component is $\log \det G_{i\bar{j}}$, where $G_{i\bar{j}}$ is the Kähler metric. For general Calabi-Yau manifolds, the $U(1)$ composite connection has two contributions, since the moduli space is a product space for Kähler class $((1,1)$ moduli) and complex structure deformations $((2,1)$ moduli), corresponding to E_6 27 and $\bar{27}$ families, respectively.

Let us discuss first the Kähler anomaly coming from the $U(1)$ Kähler connection, coupled, via a triangular graph, to the $(E_8$ and $E_6)$ gauge fields. For the E_8 gauge group, since only gauginos circulate in the loop, one gets for the $F_{\mu\nu}^2$ term (the superpartner of the anomaly term), up to a normalization factor [46],[48]

$$\text{Tr } F_{\mu\nu}^2|_{E_8} \times C_{(G=E_8)}(K_{(1,1)} + K_{(2,1)}), \quad (4.42)$$

where $K_{(1,1)} + K_{(2,1)}$ is the Kähler potential for the full moduli space and $C_{(G=E_8)}$ is the quadratic Casimir of the adjoint representation of E_8 .

For the E_6 gauge group, three types of fermions can circulate in the loop: gauginos, 27 and $\bar{27}$. Let us first consider the contribution to the $U(1)$ anomaly coming from the 27 families. This can be separated into a curvature contribution and a Kähler contribution. The Kähler contribution is easily computed from the fact that, under a Kähler transformation of the moduli space, the 27 and $\bar{27}$

family multiplets transform as [49]

$$\phi_{27} \rightarrow \phi_{27} e^{\frac{-\Lambda_{(1,1)} + \Lambda_{(2,1)}}{3}}, \quad \phi_{\bar{27}} \rightarrow \phi_{\bar{27}} e^{\frac{\Lambda_{(1,1)} - \Lambda_{(2,1)}}{3}}. \quad (4.43)$$

One then gets a contribution to the $E_6 F_{\mu\nu}^2$ term of the form [46], [50]

$$-T(R)\left(\frac{5}{3}K_{(1,1)} + \frac{1}{3}K_{(2,1)}\right) - T(\bar{R})\left(\frac{5}{3}K_{(2,1)} + \frac{1}{3}K_{(1,1)}\right). \quad (4.44)$$

($T(R)$ is the quadratic Casimir of the representation R .) The gaugino contribution gives a term

$$C_{(G=E_6)}(K_{(1,1)} + K_{(2,1)}), \quad (4.45)$$

while the $U(1)$ curvature contribution gives a term [48]

$$2T(R) \log \det G_{i\bar{j}}^{(1,1)} + 2T(\bar{R}) \log \det G_{i\bar{j}}^{(2,1)}, \quad (4.46)$$

where $G_{i\bar{j}}^{(1,1)}$, $G_{i\bar{j}}^{(2,1)}$ are the moduli metrics and we used the fact that, for smooth Calabi-Yau, the 27, $\bar{27}$ family metrics are parallel to the moduli metrics [49]

$$g_{i\bar{j}}^{27} = e^{\frac{-K_{(1,1)} + K_{(2,1)}}{3}} G_{i\bar{j}}^{(1,1)}, \quad g_{i\bar{j}}^{\bar{27}} = e^{\frac{K_{(1,1)} - K_{(2,1)}}{3}} G_{i\bar{j}}^{(2,1)}. \quad (4.47)$$

The above terms are not target space duality invariant. Duality invariance is restored by adding to the superfields K and $\log \det G_{i\bar{j}}$ appropriate holomorphic terms $\log \Delta$ and $\log \Pi$ as previously discussed, in such a way that the coefficients of the $F_{\mu\nu}^2$ terms become modular invariant. In the $F_{\mu\nu} \tilde{F}^{\mu\nu}$ sector this fact can be interpreted as a cancellation of target-space duality anomalies [46],[47],[48], [51].

For example, in the case of the E_8 anomalies, $K_{(1,1)}$ and $K_{(2,1)}$ have to be replaced by

$$K + \log \Delta + \log \bar{\Delta}, \quad (4.48)$$

where $\Delta = W$ is the Calabi-Yau automorphic form defined in eq.(4.13). For the E_6 gauge group, one must also consider the second automorphic function Π defined by eqs.(4.38),(4.39),(4.40), since for special geometry one has [43]

$$\log \det G_{i\bar{j}}^{(1,1)} = (n_{(1,1)} + 1)K_{(1,1)} + \log \det(-N_{IJ}^{(1,1)}), \quad (4.49)$$

and a similar formula for $G_{i\bar{j}}^{(2,1)}$.

5. Conclusions

In this paper we investigated automorphic functions of the discrete target-space duality groups, relevant for closed string compactifications on six-dimensional Calabi-Yau spaces or symmetric orbifolds, corresponding to $(2,2)$ superconformal field theories with central charge $\hat{c} = 6$. These automorphic functions play a very important role when constructing the effective four-dimensional action, since they arise when making loop calculations such as the threshold contributions to the gauge coupling constants. Moreover they provide an expression for the purely moduli-dependent non-perturbative superpotential. Therefore the knowledge about target-space automorphic functions is crucial for the understanding of perturbative as well as non-perturbative effects in string theory, like spontaneous supersymmetry breaking and the vacuum structure of string compactifications.

First we considered $N = 1$ orbifold compactifications, and we showed that the target-space automorphic functions $\Delta(T_i)$ are intimately related to the free energy of the string compactification, i.e. to the target-space partition function. Specifically, we showed that Δ is nothing other than the determinant of the (fermionic) mass matrix of all untwisted string states which arise due to the compactification, i.e. momentum and winding states. The form of the massive string spectrum was derived starting from toroidal compactifications with $N = 4$ space-time supersymmetry, using the field-theoretical method of ref.[12] and successively truncating the theory to the untwisted orbifold spectrum. For orbifold compactifications, analogous computations of threshold corrections to gauge coupling constants directly in the string theory [8] lead to results of the same form.

For arbitrary (six-dimensional) Calabi-Yau compactifications (which include also the case of blown-up orbifolds), the field-theoretical constraints from $N = 4$ supersymmetry can no longer be used to obtain information about the string spectrum. However, we showed that the special Kähler geometry of the Calabi-Yau moduli spaces, which originates from an underlying $N = 2$ supersymmetry, allows us to explicitly construct holomorphic functions (sections) $\Delta(T_i)$ of the moduli fields. They are automorphic functions of the Calabi-Yau target-space duality group, since they lead to a duality invariant G -function, where $\Delta(T_i)$ plays

the role of the superpotential. In addition, we gave very strong arguments for the fact that $\Delta(T_i)$ is again nothing other than the determinant of all chiral masses. Indeed, $\Delta(T_i)$ provides just the duality-invariant generalization of the already known expressions for the masses of the $N = 2$ hypermultiplets in terms of the $N = 2$ moduli vector multiplets. Thus we have found the moduli-dependence of all chiral masses for general Calabi-Yau compactifications, for example the dependence of the string masses on the blown-up moduli, provided one knows the complete moduli-dependence of the holomorphic prepotential $\mathcal{F}[X^I(T_i)]$. In this way we have discovered the duality-invariant $N = 2$ “gauging” procedure which is analogous to the $N = 4$ “gauging” of ref.[12].

Besides the necessity of knowing $\mathcal{F}(X^I)$, the explicit construction of the automorphic functions $\Delta(T_i)$ involves a second key ingredient, namely the fact that the product over the “masses” $M_I X^I(T_i) + iN^I \mathcal{F}_I(T_i)$ must not go over all integers M_I, N^I , but only over those orbits which are generated by the symplectic duality transformations on X^I and $i\mathcal{F}_I$. This restriction on M_I, N^I can be explicitly solved for the orbifold compactifications, and the result agrees with the previous computations for these flat backgrounds. For example, one rederives that $\Delta(T) = 1/\eta(T)^6$ considering only the overall modulus T . However, the constraints are rather difficult to solve for non-flat backgrounds with duality groups different from $PSL(2, \mathbf{Z})$, like the model considered by [52],[31].

In addition to the holomorphic function $\Delta(T_i)$ there exist another type of automorphic function, $\Pi(T_i)$, which is needed to provide a duality invariant completion of the determinant of the Kähler metric. We discussed the relation of both $\Delta(T_i)$ and $\Pi(T_i)$ with the appearance of $U(1)$ Kähler and holonomy anomalies which arise when computing moduli dependent loop corrections to the gauge couplings or to the dilaton Kähler potential.

The free energy F we have computed is a topological index, in the sense that it takes into account only those massive states which are due to the compact target space, and not those states which are also present in a flat background, like all oscillator excitations. Moreover, the general form of the automorphic functions $\Delta(T_i), \Pi(T_i)$ indicates a very close relationship between Calabi-Yau compactifications and topological superconformal field theories [53], [54]. This observation relies on the fact that $\Delta(T_i), \Pi(T_i)$ depends only on the moduli

fields and on the holomorphic function $\mathcal{F}[X^I(T_i)]$. One knows [54] that $\mathcal{F}(X^I)$ is entirely determined by the couplings of the chiral primary fields [55] of the underlying $(2,2)$ $\hat{c} = 6$ superconformal field theory. However, in the topological counterparts of the superconformal models one is essentially left with the chiral ring, and the $\mathcal{F}$ -function plays the role of the generating functional [54] of the topological field theory. Then all S -matrix elements in the topological theory, like the three-point functions of the chiral primary fields (Yukawa couplings), are determined by $\mathcal{F}$. So we recognize that all the chiral masses $M_I X^I + i N^I \mathcal{F}_I$, and thus also the free energy of Calabi-Yau compactifications, are topological, since they are entirely determined by the $\mathcal{F}$ -function, i.e. by the topological sector of the superconformal field theory. It is a very interesting problem to further clarify this relation between Calabi-Yau compactifications and topological field theories.

ACKNOWLEDGEMENTS

We would like to thank L. Alvarez-Gaume, L. Dixon, L. Ibanez, V. Kaplunovsky, J. Louis, M. Porrati, A. Strominger, P. Windey and E. Witten for useful discussions. D.L. would like to thank the Ecole Normale Supérieure in Paris and F.Z. would like to thank the Department of Physics, University of Tokyo, for the kind hospitalities during part of this work. The work of D.L. was performed as an Heisenberg fellow.

APPENDIX

We give here some details about the evaluation of the sum

$$F(T, \bar{T}) = \sum_{(m,n) \neq (0,0)} \log \frac{|m + inT|^2}{T + \bar{T}}. \quad (\text{A1})$$

Using ζ -function regularization and recalling that $\zeta(s) = \sum_{m>0} m^{-s}$, $\zeta(-0) =$

$-\frac{1}{2}$, we can write

$$\sum_{(m,n) \neq (0,0)} 1 = \left[4 \sum_{m>0} m^{-s} \sum_{n>0} n^{-s} + 2 \sum_{m>0} m^{-s} + 2 \sum_{n>0} n^{-s} \right]_{s=0} = -1, \quad (\text{A2})$$

and then

$$\sum_{(m,n) \neq (0,0)} \log \frac{1}{T + \bar{T}} = \log(T + \bar{T}). \quad (\text{A3})$$

Incidentally, notice that including the massless mode ($m = n = 0$) in the above sum would give a vanishing result.

With similar manipulations, we can write

$$\begin{aligned} \mathcal{A}(T) &\equiv \sum_{(m,n) \neq (0,0)} \log(m + inT) = \sum_{m>0} \log m \\ &+ \sum_{m>0, n>0} \log[(m-1+inT)(m-1-inT)(m+inT)(m-inT)]. \end{aligned} \quad (\text{A4})$$

Subtracting the T -independent quantity $\mathcal{A}(\epsilon)$, we find:

$$\begin{aligned} \mathcal{A}(T) - \mathcal{A}(\epsilon) &= \sum_{m>0, n>0} \log \frac{(m-1+inT)(m-1-inT)(m+inT)(m-inT)}{(m-1+i\epsilon)(m-1-i\epsilon)(m+i\epsilon)(m-i\epsilon)} \\ &= \sum_{n>0} \log \frac{\Gamma(in\epsilon)\Gamma(1-in\epsilon)\Gamma(-in\epsilon)\Gamma(1+i\epsilon)}{\Gamma(inT)\Gamma(1-inT)\Gamma(-inT)\Gamma(1+iT)} \\ &= 2 \log \frac{e^{\sum_{n>0} \pi n T} \prod_{n>0} (1 - e^{-2\pi n T})}{e^{\sum_{n>0} \pi n \epsilon} \prod_{n>0} (1 - e^{-2\pi n \epsilon})}, \end{aligned} \quad (\text{A5})$$

where we have used the identities

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)} = \prod_{m>0} \left(\frac{m-1+\alpha+\gamma}{m-1+\alpha} \right) \left(\frac{m-1+\beta-\gamma}{m-1+\beta} \right) \quad (\text{A6})$$

and

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \quad (\text{A7})$$

Remembering now that $\zeta(-1) = -\frac{1}{12}$ and the definition of the Dedekind eta-

function, $\eta(T) \equiv e^{\frac{-\pi T}{12}} \prod_{n>0} (1 - e^{-2\pi n T})$, we can write

$$\mathcal{A}(T) = 2 \log \eta^2(T) + \text{T-independent terms}, \quad (\text{A8})$$

and finally

$$F(T, \bar{T}) \Big|_{\text{reg.}} = \log[(T + \bar{T})|\eta(T)|^4] + \text{T-independent terms}. \quad (\text{A9})$$

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