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General Renormalization of the Gauge-Invariant Perturbation
Expansion near the Z^0 Resonance

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ABSTRACT

It was recently shown how to construct, to arbitrary accuracy, a perturbative expansion near the Z^0 resonance that is exactly gauge-invariant at each order. Here, a scheme-independent renormalization of the Standard Model is performed and the counterterm structure arising in the expansion for calculations to $\mathcal{O}(\alpha)$ is given. The results can be easily specialized to the on-shell and $\overline{MS}$ renormalization schemes and are useful for studying the numerical differences between the two.

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1. Introduction

In a recent article [1] it was pointed out that most calculations of the annihilation process, $e^+e^- \rightarrow f\bar{f}$ near the Z^0 resonance, contain spurious higher-order gauge-dependent terms arising from the incorrect use of a truncated perturbation expansion in the Dyson summation. It was then shown how to construct a perturbation expansion near a gauge resonance to arbitrary accuracy that at each order is exactly gauge-invariant by means of a Laurent expansion of the full scattering amplitude followed by expansion in the coupling constant.

In that analysis the renormalization and counterterm structures, generated in the expansion, were not considered in detail. In the present letter a general renormalization of the Standard Model is carried out and the one-loop counterterms appearing in the gauge-invariant perturbation expansion of Ref. 1 are given. The form of the expressions obtained is renormalization scheme-independent. In particular they are valid in both in the popular on-shell and $\overline{MS}$ schemes and can be used for analyses in either one just by changing the finite parts of the counterterms.

The layout of the paper is as follows; in section 2 the gauge-invariant perturbation expansion of Ref. 1 is reviewed, its properties and features discussed and a comparison made with certain other calculations. In section 3, a general renormalization framework for the bosonic sector of the Standard Model is set up. Scheme-independent equations for a set of physical input data are given from which values for the renormalized parameters can be extracted once the renormalization scheme has been specified. The general counterterms that arise in the gauge-invariant perturbation expansion are given in section 4.

2. Gauge-invariant perturbation expansion near the Z^0 resonance

The full scattering amplitude for $e^+e^- \rightarrow f\bar{f}$ is known to be an analytic function of s and t . For simplicity in the following it will normally be assumed that the external fermions are massless with $f \neq e^-$ and that the t dependence has been integrated out. The $\mathcal{O}(\alpha)$ QED corrections will not be considered here. Starting from the known structure of the scattering amplitude, $A(s)$, and extracting the leading simple pole in its Laurent expansion near the Z^0 resonance, one obtains,

$$A(s) = \frac{R}{s - s_p} + B(s) \quad (2.1)$$

which is just a Breit-Wigner with a constant width plus a nonresonant background. It is easy to see that the position of the pole s_p , the residue, R , and the background, $B(s)$, are all separately gauge-invariant. There can be no gauge cancellation between the resonant term and the background because of their differing pole structure. The position of the pole on the complex plane must gauge-invariant because, in principle, it can be determined from physical cross sections using dispersion relations and analytic continuation. Finally since the full scattering amplitude is gauge-invariant, the residue must be as well. As s_p , R and $B(s)$ are separately gauge-invariant they can be independently expanded to any required order in the coupling constant without inducing gauge-dependence in the scattering amplitude.

In Ref. 1 the expansion just described was applied to the Standard Model to obtain expressions for s_p , R and $B(s)$ in terms of the bare parameters; in particular the bare Z^0 mass, M_{Z0} . The bare mass and pole position were then eliminated in favour of the renormalized mass, M_Z . The counterterms generated by this procedure were not given explicitly there but are derived in section 4 of the present work.

The extraction of the leading pole in the Laurent expansion of the scattering amplitude in this way, provides a means of isolating the dominant physics of a resonant process in a single term. The remaining corrections should then be small. This is much in the spirit of perturbation theory except that the separation is not done at the Lagrangian level.

The extreme simplicity, and hence utility, of (2.1) should be emphasized. To $\mathcal{O}(\alpha)$ accuracy the background can be taken as a constant leading to,

$$A(s) = \frac{R}{s - s_p} + B_0 \quad (2.2)$$

This form is completely general in the resonance region, being independent of the model. The general scattering amplitude is then fully defined by the three complex constants, s_p , R and B_0 , and accordingly the cross section or lineshape is determined by five real numbers as the overall phase is not observed. R may then be taken to be real. The s -dependence of $A(s)$ at this order is trivial and lies just in the leading resonant term. Both the simplicity and exact gauge-invariance are direct consequences of having *fully* expanded the complete scattering amplitude about the pole.

The most general Z^0 lineshape to $\mathcal{O}(\alpha)$ then takes the form,

$$\sigma(s) = \frac{4\pi\alpha^2}{3}s \left\{ \left| \frac{r}{s - s_p} \right|^2 + 2 \operatorname{Re} \left(\frac{b_0}{s - s_p} \right) \right\}. \quad (2.3)$$

The constant b_0 is complex and r can be taken as real. The quantities s_p , r and b_0 are ideal model-independent resonance parameters that could be extracted from experimental data. Conversely their values could be calculated in a particular model under study and used as input for comparison with data.

Borrelli *et al.* [2] have undertaken a model-independent analysis of the Z^0 lineshape using field-theoretical arguments. They also find that to $\mathcal{O}(\alpha)$ the lineshape is specified by 5 real constants.

Other authors have also made use of an expansion about the pole but only for the propagator [3] and not for the full scattering amplitude. Starting from a Breit-Wigner with an s -dependent width as an approximation to the Z^0 resonance, Bardin *et al.* [4] showed that it could be written as a Breit-Wigner with a constant width by means of a transformation of the Z^0 parameters. They emphasized that the simplicity of the latter form readily lends itself to use in convolution integrals for the inclusion of initial state radiation effects and to other calculations where a constant width has been used. Their Z^0 parameter transformation amounts to an expansion of the propagator about the pole.

In Ref. 1 it was shown that the mass of the Z^0 boson, as traditionally defined for an unstable particle from the position of the pole, differs by 35 MeV from the on-shell definition extracted by LEP experiments. This has also been pointed out by Willenbrock and Valencia [5].

Berends *et al.* [6] observed that the use of an s -dependent width in the Z^0

propagator lowers the position of the peak cross section by 35 MeV with respect to M_Z for which they implicitly took the on-shell scheme definition.

The authors of Ref. 4 note that the ‘formal’ Z^0 mass introduced in their Z^0 boson parameter transformation is shifted by 34 MeV from M_Z , again implicitly defined in the on-shell scheme. This formal Z^0 mass does, however, have a physical interpretation as it is just the mass as defined from the position of the pole.

Recently, Sirlin [7] has shown that the definition of the Z^0 mass, normally adopted in the on-shell scheme, is gauge-dependent.

3. General Renormalization framework for the Standard Model

In the renormalization of the bosonic sector of the Standard Model it is convenient to write the bare $U(1)$ and $SU(2)_L$ coupling constants, g'_0 and g_0 , in terms of the renormalized coupling constants and counterterms as,

$$g_0 = g + \delta g, \quad (3.1a)$$

$$g'_0 = g' + \delta g'. \quad (3.1b)$$

Similarly, the bare boson masses in terms of the renormalized masses and counterterms are,

$$M_{W0}^2 = M_W^2 + \delta M_W^2, \quad (3.2a)$$

$$M_{Z0}^2 = M_Z^2 + \delta M_Z^2. \quad (3.2b)$$

Of course, δM_W^2 and δM_Z^2 are related to the counterterms of the Higgs sector [8] but this fact can be ignored for the present purposes. Note that the renormalized quantities, g , g' , M_W^2 and M_Z^2 are not the same as the ‘physical’ quantities. Setting the physical and renormalized quantities to be equal constitutes a choice of renormalization scheme. This specialization will not be made yet, so that the expressions obtained will be quite general and scheme-independent.

It is sometimes convenient to perform wavefunction renormalization on the unrotated W and B fields that appear in the Standard Model. Thus the bare wavefunctions are written,

$$W_0 = W + \delta Z_W, \quad (3.3a)$$

$$B_0 = Z + \delta Z_B. \quad (3.3b)$$

The chief advantage is that calculations are finite at the Green's function level rather than only in the full S -matrix. However, in the calculation of physical processes the wavefunction renormalization counterterms, δZ_W and δZ_B , will cancel. This can provide a useful calculational check if they are kept in intermediate steps. Note that calculations that suffer from the presence of spurious higher-order gauge-dependent terms, as described in Ref. 1, normally also contain spurious higher-order dependence on δZ_W and δZ_B . In the gauge-independent perturbation expansion discussed here the wavefunction renormalization counterterms cancel exactly at each order.

If q_μ is the vector boson momentum, then making the substitutions eqs. (3.1)–(3.3), the boson self-energy counterterms obtained are,

$$\begin{array}{c} W_\mu \quad W_\nu \\ \text{---}\times\text{---} \end{array} = \delta Z_W (q^2 g_{\mu\nu} - q_\mu q_\nu) - \delta M_W^2 g_{\mu\nu} - \delta Z_W M_W^2 g_{\mu\nu}, \quad (3.4a)$$

$$\begin{array}{c} Z_\mu \quad Z_\nu \\ \text{---}\times\text{---} \end{array} = \frac{1}{2} \delta Z_Z (q^2 g_{\mu\nu} - q_\mu q_\nu) - \frac{1}{2} \delta M_Z^2 g_{\mu\nu} - \frac{1}{2} \delta Z_Z M_Z^2 g_{\mu\nu}, \quad (3.4b)$$

$$\begin{array}{c} A_\mu \quad A_\nu \\ \text{---}\times\text{---} \end{array} = \frac{1}{2} \delta Z_\gamma (q^2 g_{\mu\nu} - q_\mu q_\nu), \quad (3.4c)$$

$$\begin{array}{c} Z_\mu \quad A_\nu \\ \text{---}\times\text{---} \end{array} = \delta Z_{Z\gamma} (q^2 g_{\mu\nu} - q_\mu q_\nu) - \frac{1}{2} M_Z^2 \delta Z_{Z\gamma} g_{\mu\nu} \\ - M_Z^2 s_\theta c_\theta \left(\frac{\delta g}{g} - \frac{\delta g'}{g'} \right) g_{\mu\nu}, \quad (3.4d)$$

where $s_\theta \equiv \sin \theta_W$, $c_\theta \equiv \cos \theta_W$ and,

$$\delta Z_Z = \delta Z_W c_\theta^2 + \delta Z_B s_\theta^2, \quad (3.5a)$$

$$\delta Z_\gamma = \delta Z_W s_\theta^2 + \delta Z_B c_\theta^2, \quad (3.5b)$$

$$\delta Z_{Z\gamma} = s_\theta c_\theta (\delta Z_W - \delta Z_B). \quad (3.5c)$$

Similarly the fermion vertex counterterms are,

$$Z_\mu = ig\gamma_\mu\gamma_L t_3 c_\theta \left(\frac{\delta g}{g} + \frac{1}{2}\delta Z_W \right) + ig\gamma_\mu\gamma_L t_3 \frac{s_\theta^2}{c_\theta} \left(\frac{\delta g'}{g'} + \frac{1}{2}\delta Z_B \right) - ig\gamma_\mu Q \frac{s_\theta^2}{c_\theta} \left(\frac{\delta g'}{g'} + \frac{1}{2}\delta Z_B \right), \quad (3.6a)$$

$$A_\mu = ig\gamma_\mu\gamma_L t_3 s_\theta \left(\frac{\delta g}{g} + \frac{1}{2}\delta Z_W \right) - ig\gamma_\mu\gamma_L t_3 s_\theta \left(\frac{\delta g'}{g'} + \frac{1}{2}\delta Z_B \right) + ig\gamma_\mu Q s_\theta \left(\frac{\delta g'}{g'} + \frac{1}{2}\delta Z_B \right), \quad (3.6b)$$

where t_3 and Q are the weak isospin and electric charge of the fermion and fermion wavefunction renormalization counterterms have been ignored.

As physical input parameters in to the Standard Model one normally chooses the three well-measured quantities, α , G_μ and M_Z . As discussed elsewhere [1, 5] the definition of the mass of the Z^0 is not unambiguous. It will be assumed that the value of the renormalized mass, M_Z , is known in some specified renormalization scheme. In the case of LEP experiments, the value quoted is the renormalized mass in the on-shell scheme and differs significantly both from the $\overline{MS}$ definition and the traditional one based on the position of the pole of a scattering amplitude.

A straightforward calculation of the matrix element to $\mathcal{O}(\alpha)$ for Thomson scattering leads to the result,

$$\sqrt{4\pi\alpha} = e \left(1 + \frac{1}{2}\Pi_{\gamma\gamma}^{(1)'}(0) + \frac{s_\theta}{c_\theta} \frac{\Pi_{Z\gamma}^{(1)}(0)}{M_Z^2} + s_\theta^2 \frac{\delta g}{g} + c_\theta^2 \frac{\delta g'}{g'} \right) \quad (3.7)$$

and of the one-loop corrections to muon decay yields,

$$\frac{\pi\alpha}{\sqrt{2}G_\mu} = \frac{M_W^2 s_\theta^2}{(1 + \Delta r)} \quad (3.8)$$

where,

$$\Delta r = \frac{\Pi_{WW}^{(1)}(0)}{M_W^2} + \frac{g^2}{16\pi^2} \left\{ 4 \left(\Delta - \ln M_Z^2 \right) + 6 + \left(4 + c_\theta^2 + \frac{6c_\theta^2}{s_\theta^2} \right) \ln \frac{1}{c_\theta^2} \right\} + \frac{g^2}{16\pi^2} \frac{5c_\theta^4 - 3s_\theta^2}{2s_\theta^2} \ln \frac{1}{c_\theta^2} + \frac{\delta M_W^2}{M_W^2} + 2 \frac{\delta g}{g} \quad (3.8a)$$

Here $\Pi_{ZZ}(s)$, $\Pi_{\gamma\gamma}(s)$ are the transverse parts of the 1PI Z^0 and photon self-energies and $\Pi_{Z\gamma}(s)$ is the 1PI Z - γ mixing. The prime denotes differentiation with respect to s and the bracketed superscripts indicate the number of loops in a given term.

In eqs. (3.7) and (3.8), the left hand side is a well-determined experimental number and the right hand side is an expression in terms of the renormalized parameters and counterterms in some, as yet, unspecified renormalization scheme. These equations, along with the value of the renormalized Z^0 mass, M_Z , provide sufficient physical input to completely determine the values of the renormalized parameters once the renormalization scheme has been fixed.

In the on-shell scheme one requires at one-loop,

$$\delta M_W^2 = -\text{Re } \Pi_{WW}^{(1)}(M_W^2), \quad (3.9a)$$

$$\delta M_Z^2 = -\text{Re } \Pi_{ZZ}^{(1)}(M_Z^2), \quad (3.9b)$$

$$s_\theta^2 \frac{\delta g}{g} + c_\theta^2 \frac{\delta g'}{g'} = -\frac{1}{2} \Pi_{\gamma\gamma}^{(1)'}(0) - \frac{s_\theta}{c_\theta} \frac{\Pi_{Z\gamma}^{(1)}(0)}{M_Z^2}, \quad (3.9c)$$

and hence,

$$\frac{\delta g}{g} = -\frac{1}{2} \Pi_{\gamma\gamma}^{(1)'}(0) - \frac{s_\theta}{c_\theta} \frac{\Pi_{Z\gamma}^{(1)}(0)}{M_Z^2} - \frac{c_\theta^2}{2s_\theta^2} \text{Re} \left(\frac{\Pi_{WW}^{(1)}(M_W^2)}{M_W^2} - \frac{\Pi_{ZZ}^{(1)}(M_Z^2)}{M_Z^2} \right) \quad (3.10a)$$

$$\frac{\delta g'}{g'} = -\frac{1}{2} \Pi_{\gamma\gamma}^{(1)'}(0) - \frac{s_\theta}{c_\theta} \frac{\Pi_{Z\gamma}^{(1)}(0)}{M_Z^2} + \frac{1}{2} \text{Re} \left(\frac{\Pi_{WW}^{(1)}(M_W^2)}{M_W^2} - \frac{\Pi_{ZZ}^{(1)}(M_Z^2)}{M_Z^2} \right). \quad (3.10b)$$

In the $\overline{MS}$ scheme the counterterms are chosen to be just the divergent parts of eqs. (3.9a, b) and (3.10a, b) proportional to,

$$\Delta = \pi^{n/2-2} \Gamma \left(2 - \frac{n}{2} \right) \quad (3.11)$$

where n is the dimensionality of space-time that arises from the use of dimensional regularization. The difference between any two renormalization schemes lies just in the finite parts of the counterterms.

4. Counterterms for the Gauge-invariant Perturbation Expansion

After elimination of the bare quantities in favour of renormalized ones via eqs. (3.1)–(3.3) the scattering amplitude can be written as,

$$A(s) = \frac{R + \delta R}{s - s_p} + B(s) + \delta B. \quad (4.1)$$

Expressions for R and $B(s)$ in the Standard Model have already been given in Ref. 1, eq. (2.13), and will not be repeated here. By means eqs. (3.4)–(3.6) the counterterms present in (4.1) may be easily derived. For the residue and background,

$$\begin{aligned} \delta R = & V_{iZ}^{(0)} 2 \left(c_\theta^2 \frac{\delta g}{g} + s_\theta^2 \frac{\delta g'}{g'} \right) V_{Zf}^{(0)} + V_{iZ}^{(0)} 2 s_\theta c_\theta \left(\frac{\delta g}{g} - \frac{\delta g'}{g'} \right) V_{\gamma f}^{(0)} \\ & + V_{i\gamma}^{(0)} 2 s_\theta c_\theta \left(\frac{\delta g}{g} - \frac{\delta g'}{g'} \right) V_{Zf}^{(0)}, \end{aligned} \quad (4.2)$$

and

$$\delta B = V_{i\gamma}^{(0)} \frac{2}{s} \left(s_\theta^2 \frac{\delta g}{g} + c_\theta^2 \frac{\delta g'}{g'} \right) V_{\gamma f}^{(0)}. \quad (4.3)$$

$V_{iZ}^{(0)}$ and $V_{Zf}^{(0)}$ are the tree-level Ze^+e^- and $Zf\bar{f}$ vertices, respectively, and $V_{i\gamma}^{(0)}$ and $V_{\gamma f}^{(0)}$ are the corresponding photon vertices.

Note that only coupling constant counterterms appear. In δB the counterterms are just those that occur in electric charge renormalization (3.7) and in δR they are the Z coupling constant renormalization and Z - γ mixing counterterms.

For the position of the pole, s_p , matters are slightly more complicated. In terms of the renormalized mass and counterterms,

$$\begin{aligned} s_p = & M_Z^2 + \delta M_Z^2 + \Pi_{ZZ}^{(1)}(M_Z^2) + \Pi_{ZZ}^{(2)}(M_Z^2) \\ & + [\Pi_{ZZ}^{(1)}(M_Z^2) + \delta M_Z^2] \Pi_{ZZ}^{(1)'}(M_Z^2) + \frac{\Pi_{Z\gamma}^{(1)2}(M_Z^2)}{M_Z^2}. \end{aligned} \quad (4.4)$$

For $\text{Re } s_p$, the genuine two-loop contributions contained in $\text{Re } \Pi_{ZZ}^{(2)}(s)$ can be removed by a suitable choice of δM_Z^2 as is done in the on-shell scheme. Note also that it is a simple matter to define a renormalization scheme in which the renormalized mass, M_Z , is exactly equal to the mass defined from position of the pole, i. e. $\text{Re } s_p = M_Z^2$.

To obtain $\mathcal{O}(\alpha)$ accuracy in the scattering amplitude near resonance, $\text{Im } s_p$ must be calculated to two loops and coupling constant counterterms then implicitly arise. One obtains,

$$\begin{aligned} \text{Im } s_p = & \text{Im } \Pi_{ZZ}^{(1)}(M_Z^2) + \text{Im } \Pi_{ZZ}^{(2)}(M_Z^2) + [\text{Re } \Pi_{ZZ}^{(1)}(M_Z^2) + \delta M_Z^2] \text{Im } \Pi_{ZZ}^{(1)'}(M_Z^2) \\ & + \text{Re } \Pi_{ZZ}^{(1)'}(M_Z^2) \text{Im } \Pi_{ZZ}^{(1)}(M_Z^2) + \frac{2 \text{Re } \Pi_{Z\gamma}^{(1)}(M_Z^2) \text{Im } \Pi_{Z\gamma}^{(1)}(M_Z^2)}{M_Z^2}. \end{aligned} \quad (4.5)$$

The $\mathcal{O}(\alpha^2)$ quantity,

$$\text{Im } \Pi_{ZZ}^{(2)}(M_Z^2) + \text{Re } \Pi_{ZZ}^{(1)'}(M_Z^2) \text{Im } \Pi_{ZZ}^{(1)}(M_Z^2) + \frac{2 \text{Re } \Pi_{Z\gamma}^{(1)}(M_Z^2) \text{Im } \Pi_{Z\gamma}^{(1)}(M_Z^2)}{M_Z^2}$$

contains the same combination of coupling constant counterterms as appear in eq. (4.2) since the cut diagrams that contribute to (4.5) are identical to those contributing to δR . This is a consequence of the unitarity constraint, that the total decay width of the Z^0 is the sum of its partial decay widths. Note that the second term on the right hand side of (4.5) vanishes in the on-shell scheme but must be included in the $\overline{MS}$ scheme.

It should again be emphasized that the form of the expressions for the counterterms (4.2)–(4.5) is scheme independent. They are equally valid in the on-shell, $\overline{MS}$ or any other renormalization scheme. Only the definitions of δg , $\delta g'$, δM_W^2 , δM_Z^2 and accordingly the values of the renormalized quantities will change. The latter can be solved for from the physical input relations, (3.7) and (3.8), once the desired renormalization conditions have been prescribed.

The form of in which the counterterms appear is notable for its compactness and simplicity.

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