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A NEW APPROACH TO POTENTIAL WELL BUNCH DEFORMATION

Johannes Hagel, Bruno Zotter

Abstract

A method is described which permits one to obtain analytic expressions for the bunch shape of an electron bunch in a storage ring. This is achieved by converting the Haissinski integral equation, which is of the Fredholm type, into a Volterra equation by multiplying the integrand with a step-function which is then expanded into a Fourier series. After exchanging integration and summation, the resulting set of nonlinear algebraic equations can be solved numerically by iteration. Approximate analytic solutions can be obtained in the form of rapidly converging sequences for a resonator impedance. For a typical parameter range of high-energy electron storage rings, the first term is often sufficient. The limits of stability of the solution and the generalization to arbitrary impedances are discussed. The method is applied to calculate the evolution of the bunch shape with current in LEP at injection energy.

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1. Haissinski's Integral Equation.

The longitudinal phase-space distribution of an electron bunch in a high-energy storage ring or accelerator is described by a Fokker-Planck equation. In particular, the energy distribution is determined by the equilibrium between radiation damping and quantum excitation of the synchrotron radiation. In the absence of other forces, the energy distribution is therefore Gaussian. The stationary bunch shape or "line density" $\lambda(t)$ is then described by an equation containing a double integral, which was published in 1973 by Haissinski ¹⁾. It can be simplified to an equation with a single integral ²⁾³⁾

$$\lambda(\tau) = K \cdot \exp \left[-U_0(\tau) - \xi \int_{-\infty}^{\tau} dt \cdot S(\tau-t) \cdot \lambda(t) \right] \quad (1)$$

where $U_0(\tau)$ is the potential due to the applied RF Voltage, divided by the square of the zero-current RMS bunch length σ_0^2 . For a (locally) linear RF voltage - usually a very good approximation in electron storage rings where the bunch length is small compared to the RF wave length and the stable phase angle ψ_s is close to π . In order to avoid a negative cosine we put $\phi_s = \pi - \psi_s$. For a linear RF voltage, the potential is parabolic : $U_0(\tau) = \tau^2 / 2\sigma_0^2$, which we shall usually assume for simplicity;

$$\xi = 2\pi I_b / (h_{RF} V_{RF} \cos \phi_s \omega_0^2 \sigma_0^2) \quad (2)$$

is a strength parameter proportional to the bunch current I_b . For a parabolic potential and vanishing current ($\xi = 0$), the Haissinski equation then shows that also the line density is Gaussian ;

$$S(\tau) = \int_0^{\tau} G(t) dt \quad (3)$$

is the "step-function response", defined as the integral over the wake function (= point bunch wake potential or "Green function") which is the Fourier transform of the impedance $Z(\omega)$

$$G(\tau) = \int_{-\infty}^{\infty} d\omega Z(\omega) \exp(\omega\tau) / 2\pi ; \quad (4)$$

1) A. Haissinski, Nuovo Cimento 18B (1973) p.72

2) Chatard-Moulin, A. Jecko, A. Papiernik, Univ. Limoges Reports A1/2 (1973)

3) K. Bane, P. Wilson, Trans. IEEE-NS 24 (1977) p.1485

finally, K is a normalization constant defined by

$$\int_{-\infty}^{\infty} dt \lambda(t) = 1. \quad (5)$$

This integral equation can be solved numerically²⁾⁴⁾⁵⁾, for which purpose it is advantageous to eliminate the normalization constant with the substitution $\lambda(t) = K.p(t)$. Furthermore one may change the integration variable to $u = \tau - t$ to get

$$p(\tau) = \exp \left[- U_0(\tau) - \xi' \int_0^{\infty} du S(u) p(\tau+u) \right] \quad (6)$$

where $\xi' = \xi / \int_{-\infty}^{\infty} p(t) dt$ contains the normalization integral.

One can now start the integration well ahead of the bunch, where the line-density vanishes, and work successively backwards. One thus obtains the bunch shape as function of the parameter ξ' , and the corresponding current from the normalization condition.

2 Analytic Solutions

Exact analytic solutions of the Haissinski equation are known only for two impedances : purely **inductive**¹⁾⁶⁾ or purely resistive⁷⁾. In the first case, the step-function response is a (Dirac) delta function. With $\xi' = \xi L/2$ one obtains an implicit algebraic equation for the line density. We further introduce the dimensionless variable

$$\phi = \tau / (\sigma_0 \sqrt{2}) \quad (7)$$

to get

$$\lambda(\phi) = K. \exp [- \phi^2 - \xi' \lambda(\phi)] . \quad (8)$$

The line density remains always symmetric $\lambda(-\phi) = \lambda(\phi)$. For low current it is Gaussian, but deviates more and more with increasing current by a widened center part (see Fig.1).

4) E. Keil, SLAC-PEP Note 126 (1975)

5) K.Bane, R.Ruth, SLAC Preprint (1989)

6) G.Besnier , Proc. ESRF workshop on impedances (1988)

7) A.Ruggiero, Trans.IEEE-NS 24 (1977) p.1205

For a **resistive** impedance the wake function itself is a delta function. It is therefore advantageous to take the logarithmic derivative of the Haissinski equation. With $\xi'' = \xi R/2$ we get

$$\lambda'(\phi) / \lambda(\phi) = -2\phi - \xi'' \lambda(\phi) \quad (9)$$

This is a nonlinear differential equation of "Bernoulli" type, which can be made linear by a substitution for $1/\lambda(\phi)$. It can then be solved explicitly to yield the normalized line-density

$$\lambda(\phi) = \frac{\exp(-\phi^2)}{\xi' [\operatorname{erf}(\phi) + \operatorname{Coth}(\xi'/\sigma_0 \sqrt{2\pi})]} \quad (10)$$

With increasing current, the distribution becomes more and more asymmetric, resembling a Gaussian which leans forward (see Fig.2).

For a purely **capacitive** impedance, the wake function is a constant. An approximate solution has been published recently⁸⁾, but is incorrectly stated and valid only under very restrictive conditions. The differential form of Haissinski's equation can be converted into a second order equation which contains no explicit parameters:

$$u''(\phi) = [u(\phi) - \phi] u'(\phi) \quad (11)$$

$$\text{where } u(\phi) = (\xi/K\sigma) \int_{-\infty}^{\phi} \lambda(\phi) d\phi \quad (\text{with } u''(0) = 0) \quad (12)$$

The function $u(\phi)$ defined by this equation can be used to express the line density. It remains symmetric at higher currents where it resembles a Gaussian with a narrowed center (see Fig.3).

For a more realistic **resonator** impedance with the 3 parameters: shunt impedance R_s , resonant frequency $\omega_r/2\pi$, and quality factor Q ($>1/2$), the step-function response can be expressed as the imaginary part of an exponential

$$S(\phi) = (2R_s/W) \operatorname{Im}\{ \exp(iv_r\phi) \} \quad (13)$$

$$\text{where } v_r = v_0(i + W), \quad v_0 = \omega_r \sigma_0 / (Q \sqrt{2}), \quad W^2 = 4Q^2 - 1 \quad (14)$$

$$\text{With } \varepsilon = 2R_s \xi / W = R_s I_b / [\pi h V_{RF} \cos \phi_s \cdot f_0^2 \sigma_0^2 W] \quad (15)$$

8) A. Burov , Part. Accel. 28 (1990) p.525

the Haissinski equation then can be written

$$\lambda(\phi) = K.\exp\{ - \phi^2 - \varepsilon \operatorname{Im} \{ \exp (iv_r\phi) \int_{-\infty}^{\phi} d\phi \exp (iv_r\phi) \lambda(\phi) \} \} \quad (16)$$

which constitutes the basic equation for a resonator impedance.

3 Volterra Integral Equation

Since the step-function response vanishes for $\tau < 0$, the integral in Eq.(1) has to be written with the variable upper limit τ and the integral equation is of the "Fredholm" type. If the upper limit of the integral could be extended to infinity, the integral would be a constant (with respect to the independent variable τ). The integral equation would become a set of algebraic equations - this technique to solve a Volterra integral equation is sometimes called the "degenerate kernel method".

However, if we multiply the integrand in the Haissinski equation with a unit (Heavyside) step function H of argument $(\tau-t)$, we may extend the upper limit to infinity.

$$\lambda(\tau) = K.\exp\{ - \tau^2/(2\sigma_0^2) - \xi \int_{-\infty}^{\infty} d\tau \lambda(\tau) S(\tau-t) H(\tau-t) \} \quad (17)$$

In terms of the dimensionless variable $\phi = \tau / (\sigma_0 \sqrt{2})$ introduced before, this integral equation becomes

$$\lambda(\phi) = K.\exp\{ - \phi^2 - \xi \int_{-\infty}^{\infty} d\phi \lambda(\phi) S(\phi-\psi) H(\phi-\psi) \} \quad (18)$$

and for a resonator impedance we get thus

$$\lambda(\phi) = K.\exp\{ - \phi^2 - \varepsilon \operatorname{Im} \{ \exp (iv_r\phi) \int_{-\infty}^{\infty} d\phi \exp (iv_r\phi) \lambda(\phi) \} \} \quad (19)$$

We shall further express the step function by an exponential Fourier series with a period $T = v \cdot \sigma_0$ which must be large compared to the bunch length - in particular, one may take the machine period T_0 :

$$H(\phi) = 1/2 + (i/\pi) \sum_{k=-\infty}^{\infty} \exp [(2k-1)iv\phi] / (2k-1) \quad (20)$$

One thus obtains for the resonator impedance the set of algebraic equations

$$\lambda(\phi) = K \cdot \exp\{-\phi^2 - \varepsilon \operatorname{Im}\{\exp(iv_r\phi) G(\phi)\}\} \quad (21)$$

$$\text{where } G(\phi) = C_0/2 + (i/\pi) \sum_{k=-\infty}^{\infty} C_{2k-1} \exp \{-(2k-1)iv\phi\} / (2k-1) \quad (22)$$

$$\text{and } C_k = \int_{-\infty}^{\infty} d\phi \lambda(\phi) \exp \{-i(v_r - kv)\phi\} \quad (23)$$

The constants C_k can be determined numerically by iteration. This method has been programmed on a PC and was used to compare the solutions with the straightforward integration. It is much faster, and the two methods show excellent agreement for not too large values of the parameters v_0 and ε (we assume $Q=1$). In particular, for LEP at injection energy and with currents below the turbulent threshold, approximately $v_0 \approx 1/4$ and $\varepsilon \approx 10$, in which region the new method yields excellent results (see Fig.4).

4 Shift of Stable Phase Angle

If the two parameters v_0 and ε are not too large, we can use a perturbation method to approximate the solution analytically. In first order we assume that the Gaussian shape remains unchanged, but is shifted backwards. We can find the value of the shift $\Delta\phi$ by looking for the value where the derivative of the line-density vanishes:

$$-2\Delta\phi - \varepsilon \operatorname{Im}\{\exp(iv_r\Delta\phi) [ivG(\Delta\phi) + G'(\Delta\phi)]\} = 0 \quad (24)$$

For a small phase shift, we may in lowest order approximation replace $\Delta\phi$ in the arguments of G and G' by zero to get an explicit expression for $\Delta\phi$

$$\Delta\phi = -(\varepsilon/2) \operatorname{Im} \{ iv_r G(0) + G'(0) \} =$$

$$-(\varepsilon/2) \operatorname{Im} \{ iv_r [C_0/2 + (i/\pi) \sum C_{2k-1}/(2k-1)] + (v/\pi) \sum C_{2k-1} \} \quad (25)$$

Evaluating the coefficients C_k to lowest order, i.e. replacing $\lambda(\phi)$ by a Gaussian in the integrals of Eq.23, yields

$$C_k^{(0)} = \exp [- (v_r - kv)^2 / 4] . \quad (26)$$

We thus find for the shift $\Delta\phi = -(\varepsilon/4) \cdot \operatorname{Im} \{ iv_r \exp(-v_r^2/4) \}$

$$- (v_r/\pi) \sum \exp(z_k^2)/(2k-1) + (v/\pi) \sum \exp(z_k^2) \} \quad (27)$$

$$\text{where} \quad z_k = kv_r - (v+v_r)/2. \quad (28)$$

These infinite series can be summed with the expressions derived in ref.⁹⁾. With $z_k = \alpha k + \beta$, $z_q = \alpha q + \beta$, the pertinent expressions are

$$\sum \exp(z_k^2) = \alpha \sqrt{\pi} \quad (29)$$

$$\text{and} \quad \sum \exp(z_k^2)/(k-q) = i\pi [w(z_q) - (1 - \cot(\pi q)) \exp(-z_q^2)] \quad (30)$$

where $w(z)$ is the complex error function. For $q=1/2$, the cotangens is zero and further $z_q = -v_r/2$. In order to avoid negative arguments of the complex error function, we will use the relation¹⁰⁾ $w(-z) = 2 \exp(-z^2) - w(z)$ to get for the first sum

$$\sum \exp(z_k^2)/(k-1/2) = i\pi [\exp(-v_r^2/4) - w(v_r/2)] \quad (31)$$

Thus the two exponential terms in the expression for $\Delta\phi$ cancel. The second sum yields $v\sqrt{\pi}$, i.e. it is real and the imaginary part of the last term vanishes. The total result is then independent of v and hence of the period T chosen for the Fourier expansion of the Heavyside function

$$\Delta\phi = -(\varepsilon/4) \operatorname{Re} \{ v_r w(v_r/2) \} \quad (32)$$

9) B. Zotter, CERN Report ISR/TH 80-03 (1980)

10) Gradshteyn-Ryzhik, Tables of Integrals

For small values of v_0 , this first approximation is sufficient and yields a shifted Gaussian for the bunch shape. For larger values of v_0 , we must include the next order term by using this shifted Gaussian on the rhs of Eq.(19). We evaluate $G(\phi)$ with Eq.(18) and obtain the solution

$$\lambda_2(\phi) = K \cdot \exp\{ -\phi^2 - \varepsilon \exp[-(\phi - \Delta\phi)^2] \operatorname{Im} [w(v_r/2 - i(\phi - \Delta\phi))] \} \quad (33)$$

containing an exponential and a complex error function in the argument of the exponential. To simplify this expression, we expand the additional argument into a Taylor series in the shifted phase $(\phi - \Delta\phi)$

$$\lambda(\phi) = K \cdot \exp\{ -\phi^2 - a_0 - a_1(\phi - \Delta\phi) - a_2(\phi - \Delta\phi)^2 - \dots \} \quad (34)$$

where $a_0 = (\varepsilon/2) \operatorname{Im} \{ w(v_r/2) \}$

$$a_1 = (\varepsilon/2) \operatorname{Im} \{ i v_r w(v_r/2) \} = 2 \Delta\phi$$

$$a_2 = (\varepsilon/4) \operatorname{Im} \{ 2i v_r / \sqrt{\pi} - v_r^2 w(v_r/2) \} = \rho - 1 \quad (35)$$

were obtained using a computer algebra program. After reordering the terms in the exponent one gets

$$\lambda(\phi) = K \cdot \exp\{ (a_0 - \Delta\phi^2) - \rho(\phi - \Delta\phi)^2 + \dots \} \quad (36)$$

For the expanded exponential, the normalization condition can be evaluated explicitly

$$K = \sqrt{(\rho/\pi)} \exp\{ - (a_0 - \Delta\phi^2) \} \quad (37)$$

and thus we get up to terms in second order in the exponent

$$\lambda_2(\phi) = \sqrt{(\rho/\pi)} \exp\{ -\rho(\phi - \Delta\phi)^2 + \dots \} \quad (38)$$

Substituting this result again into the rhs. of Eq.(19) one obtains the next higher order result

$$\lambda_3(\phi) = K \cdot \exp\{ -\phi^2 - (\varepsilon/2) \times \operatorname{Im} \{ w[(v_r - 2i\rho(\phi - \Delta\phi))/2\sqrt{\rho}] \exp[-\rho(\phi - \Delta\phi)^2] \} \} \quad (39)$$

This expression yields excellent agreement (error < 10 %) with the direct numerical solution up to values of v_0 up to 1/2 and ε larger than 20. For increasing currents, the bunch becomes a little bit shorter (due

to the capacitive impedance seen by a bunch originally shorter than the wavelength of the resonator impedance), and slightly asymmetric. The strongest effect is the shift of the stable phase angle which we shall discuss again with respect to the stability of the solution.

5 Application of the method to LEP

The two parameters required for the evaluation of the bunch shape are ν_0 and ε . For LEP at injection energy ($E=20$ GeV) are defined in Eqs.(13) and (14). We rewrite them here as

$$\nu_0 = \sqrt{2} \pi f_r \sigma_0 / Q \quad (40)$$

$$\text{and} \quad \varepsilon = h_r I_b (Z/n) / (\pi h_{RF} V_{RF} \cos \phi_s \cdot f_0^2 \sigma_0^2 W') \quad (41)$$

$$\text{where} \quad h_r = f_r / f_0, \quad h_{RF} = f_{RF} / f_0 \quad \text{and} \quad W'^2 = 1 - 1/4Q^2 \quad (42)$$

Below the turbulent threshold ($I_b = 100 \mu A$) we take the following values for LEP at injection energy (20 GeV) , operated without wigglers

Revolution frequency	$f_0 = 11.245$ KHz
RF frequency	$f_{RF} = 352.2$ MHz
RF voltage	$V_{RF} = 80$ MV
Stable phase angle	$\cos \phi_s = 0.996$
Natural bunch length	$\sigma_0 = 18$ psec
Longitudinal impedance	$Z/n = 0.25 \Omega$
Resonant frequency	$f_r = 2.0$ GHz
Quality factor	$Q = 1$

With these numbers we get the auxiliary quantities

RF harmonic number	$h_{RF} = 31324$
Resonator harmonic	$h_r = 1.78 \cdot 10^5$
Reduced quality factor	$W' = 0.866$

and hence the parameters for LEP

$$\nu_0 = 0.16 \quad \text{and} \quad \varepsilon = 16.0 \quad (43)$$

The variation of the bunch shape with increasing current as well as the change of the phase shift are shown in Fig.4 as function of the bunch current.

6 Stability of the Solution

The Haissinski equation yields only the stationary solution ("potential well solution") of the Fokker-Planck equation which describes the phase-space distribution of particles in a bunch. However, it has been observed experimentally¹¹⁾ that the distribution becomes unstable above a threshold value given approximately by using local values for the current and energy spread in the criterion for coasting beams. Together with the appearance of high-frequency signals ("microwave instability"), an increased rate of bunch-lengthening (anomalous or "turbulent") and an increase of the energy spread ("bunch widening") appear above this "turbulent threshold".

Many attempts have been made to understand and calculate the limits of stability of the stationary solution⁷⁾¹²⁾¹³⁾¹⁴⁾¹⁵⁾. However, none of these theories is completely satisfactory, and we hope to extend the present approach to obtain a better estimate of the so-called "turbulent threshold" by using the analytic expression for the bunch shape derived here for the potential well region.

Based on a thermo-dynamic analysis, the following model has been proposed¹⁶⁾ to explain the turbulent threshold by : due to the shift of the bunch to a higher RF voltage, the potential energy of the stationary bunch increases sufficiently to become larger than the total energy of an oscillating bunch, equal to the sum of its kinetic energy and the reduced potential energy of a longer bunch, roughly when the shift is of the order of the RMS bunch length. As can be seen from Fig.4, the shift of the bunch center in LEP is indeed only slightly larger than the zero current RMS bunch length at the observed threshold current of 100 μ A.

7 Generalization to arbitrary Impedances

The method was developed above for a resonator impedance which has the particularity of a purely exponential step-function response.

11) D. Boussard, CERN Report Lab II/RF/Int/75-2 (1975)

12) F. Sacherer, Trans. IEEE-NS 24 (1977) p.1393

13) A. Hofmann, J. Maidment, CERN-LEP Note 168 (1978)

14) P. Channell, Proc. 12-th Intern. Conf. on High-Energy Accelerators (1985) p.276

15) K. Oide, K. Yokoya, KEK-Preprint 90-10 (1990)

16) R. Meller, Proc. Part. Accel. Conf. (1987) p. 1145

This facilitates the separation of variables in the integrand of the Haissinski equation, and permits its conversion to a Volterra equation by expanding the step-function into a Fourier series.

It is, however, possible to generalize this method to any impedance function as long as its Fourier transform exists. Then the wake function is given by Eq.(4), and the step-function response is obtained from Eq.(3) as

$$S(\tau) = \int_{-\infty}^{\infty} d\omega Z(\omega) [\exp(\omega\tau) - 1]/(2\pi j\omega) \quad (44)$$

If the complex poles ω_k of $Z(\omega)$ are known, we can evaluate this integral with the residue theorem to obtain the expression

$$S(\tau) = \sum_{n=1}^N Z(\omega_n) [\exp(\omega_n \tau) - 1]/\omega_n \quad (45)$$

where N is the number of poles. E.g., for a resonator impedance $N=2$ and (for $Q > 1/2$) $\omega_2 = -\omega_1^*$. One can thus get directly the step-function response of a resonator, shown in Eq.(12), with this approach. Substitution of the last expressions into the Haissinski equation yields for $\lambda(\tau) = K.\exp\{-U_0(\tau) +$

$$-\xi \sum_{n=1}^N [Z(\omega_n)/\omega_n] \left\{ \exp(j\omega_n \tau) \int_{-\infty}^{\tau} dt.\exp(j\omega_n t).\lambda(t) - \int dt \lambda(t) \right\} \quad (46)$$

We can again extend the limits of integration over t to infinity by multiplying the integrands by step-functions $H(\tau - t)$. After expansion of these into Fourier series (Eq.16), we obtain

$$\begin{aligned} \lambda(\tau) = & K.\exp[-U_0(\tau) - (\xi/2) \sum_{n=1}^N [Z(\omega_n)/\omega_n] \{ [\exp(j\omega_n \tau) p(\omega_n) - 1] + \\ & + (2i/\pi) \sum_{k=-\infty}^{\infty} [\exp(j(\omega_n + \Omega_k)\tau) p((\omega_n + \Omega_k) - \exp(j\Omega_k \tau) p(\Omega_k)]/(2k-1) \} \end{aligned} \quad (47)$$

$$\text{where } \Omega_k = (k-1)\Omega_0 \text{ and } p(\omega) = \int_{-\infty}^{\infty} dt.\exp(-j\omega t).\lambda(t) \quad (48)$$

is the Fourier transform of the line-density. Since it does not contain the independent variable, it can be considered a constant in the integral equation which thus becomes again an infinite set of (nonlinear) algebraic equations as for the resonator impedance.

Alternatively, it is possible to develop the product of the step-function response and the Heavyside step-function directly into a Fourier series and proceed in the same manner as for the resonator impedance. Depending on the response function, this may be even simpler than the expansion in the poles in the complex plane.

8 Conclusions

The Haissinski equation describes the line-density of a bunched particle beam in a storage ring in terms of the "step-function response", i.e. the integral over the wake function. It is a nonlinear, "Fredholm" integral equation of second kind i.e. the (upper) limit of integration is the independent variable and the unknown function occurs also outside the integral. Its conversion into a "Volterra" integral equation, - with constant (in our case infinite) integration limits - is possible by multiplying the integrand with a step-function, which we then expand into an exponential Fourier series. For a resonator impedance, also the step-function response is (the imaginary part of) an exponential, and hence all terms depending on the independent variable can be taken out of the integrals which can then be considered constants.

The same technique can be used for arbitrary impedances if their complex poles are known: then the step-function response can again be expressed as a (finite) series of exponentials, and the integral equation is converted into an infinite set of (nonlinear) algebraic equations. Also, the direct Fourier expansion of the product of the response and a step-function leads to a set of algebraic equations.

Approximate expressions for the solution can be obtained by a perturbation method. It turns out that they converge very well for typical storage ring parameters, and are independent of the period chosen for the Fourier expansion of the Heavyside step-function. Basing the perturbation on shifted and lengthened bunches (first and second moments of the line-density), the analytic solution shows excellent agreement with the results of direct integration for the parameters of LEP at injection energy with currents below the turbulent threshold ("potential well renion").

A rather simple expression for the phase shift permits also a rough, analytic estimate of the "turbulent" threshold, assumed to occur when the shift becomes about equal to the RMS bunch length. As next

step, we intend to use the analytic expressions obtained with this approach to improve the estimates for the turbulent threshold by using it as a basis for a time-dependent solution of the Fokker-Planck equation.

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Fig. 1