CERN-TH.6125/91

REGULAR GRAVITATIONAL LAGRANGIANS

Norbert Dragon ¹⁾

Theoretical Physics Division - CERN
1211 - Geneva 23, Switzerland

Abstract

The Einstein action with vanishing cosmological constant is for appropriate field content the unique local action which is regular at the fixed point of affine coordinate transformations. Imposing this regularity requirement one excludes also Wess-Zumino counterterms which trade gravitational anomalies for Lorentz anomalies. One has to expect dilatational and $SL(D)$ anomalies. If these anomalies are absent and if the regularity of the quantum vertex functional can be controlled then Einstein gravity is renormalizable.

CERN-TH.6125/91
June 1991

¹⁾ On leave of absence from Institut für Theoretische Physik, Universität Hannover.

Symmetries restrict very effectively the arbitrariness of Lagrangians. It is not always appreciated that also the domain of regularity of the Lagrangian is essential for the resulting restrictions. For example if scale invariance leads to the condition

$$(\Phi_1 \frac{\partial}{\partial \Phi_1} + \Phi_2 \frac{\partial}{\partial \Phi_2}) \mathcal{L}(\Phi_1, \Phi_2) = 0, \quad (1)$$

then the general solution

$$\mathcal{L}(\Phi_1, \Phi_2) = f(\Phi_1/\Phi_2) \quad (2)$$

contains infinitely many parameters, while the solution which is regular at the fixed point $(\Phi_1, \Phi_2) = (0, 0)$ of the differential operator in (1)

$$\mathcal{L}(\Phi_1, \Phi_2) = \text{const} \quad (3)$$

contains only one parameter.

For a compact internal symmetry group G it is well known [1] how the requirements that the Lagrangian $\mathcal{L}$ be regular at a fixed point of a subgroup $H \subset G$ affect $\mathcal{L}$: the fields have to consist of Goldstone fields which parametrize the coset G/H and of H -multiplets, H is realized linearly and the H -multiplets have to couple to H -invariants. The tightest conditions on $\mathcal{L}$ follow if H is chosen maximal $H = G$ [1].

Imposing the requirement that the Lagrangian be regular at the fixed point $g_{mn} = \eta_{mn}$ of Poincaré - transformations all local actions which are invariant under general coordinate transformations and their anomalies have been determined [2]. The result is unaltered if one requires regularity at the fixed point of the DeSitter group [3] which is also a maximal symmetry group for an invertible metric.

Here I investigate the restrictions which follow for gravitational Lagrangians if one requires them to be regular at the fixed point of the affine coordinate transformations

$$x'^m(x) = M^m_n x^n + a^m, \quad M \in GL(D), \quad a \in R^D. \quad (4)$$

This is the highest dimensional subgroup of general coordinate transformations (unique up to the choice of coordinates) which can act linearly on an affine connection $\Gamma_{kl}^m = \Gamma_{lk}^m$. I assume such a connection to be one of the gravitational fields. Γ_{kl}^m transforms under infinitesimal coordinate shifts ξ as

$$s\Gamma_{kl}^m = \partial_k \partial_l \xi^m + \xi^r \partial_r \Gamma_{kl}^m + \partial_k \xi^r \Gamma_{rl}^m + \partial_l \xi^r \Gamma_{kr}^m - \partial_r \xi^m \Gamma_{kl}^r \quad (5)$$

with the characteristic inhomogeneous piece $\partial_k \partial_l \xi^m$. The conditions for a fixed point

$$s\partial_{n_1} \dots \partial_{n_i} \Gamma_{kl}^m = \partial_{n_1} \dots \partial_{n_i} s\Gamma_{kl}^m = 0, \quad i = 0, 1, \dots \quad (6)$$

show that the second and higher Taylor coefficients of ξ^m are determined in terms of the $D(D+1)$ coefficients $\xi^m, \partial_s \xi^m$. There are further restrictions on the parameters of the fixed point group if the Riemann curvature R_{klm}^n

$$R_{klm}^n = \partial_k \Gamma_{lm}^n - \partial_l \Gamma_{km}^n - \Gamma_{km}^r \Gamma_{lr}^n + \Gamma_{lm}^r \Gamma_{kr}^n \quad (7)$$

does not vanish. R_{klm}^n transforms as a consequence of (5) as a tensor, i.e. by its Lie-derivative

$$\begin{aligned} sR_{klm}^n &= \mathcal{L}_\xi R_{klm}^n \\ &= \xi^r \partial_r R_{klm}^n + \partial_k \xi^r R_{rlm}^n + \partial_l \xi^r R_{krm}^n + \\ &\quad \partial_m \xi^r R_{klr}^n - \partial_r \xi^n R_{klm}^r. \end{aligned} \quad (8)$$

The fixed point equation $s R_{klm}{}^n = 0$ imposes a linear relation among ξ^m and $\partial_s \xi^m$, leaving a lower dimensional subgroup H , unless

$$\begin{aligned} \partial_r R_{klm}{}^n &= 0 \\ \delta_k{}^s R_{rlm}{}^n + \delta_l{}^s R_{krm}{}^n + \delta_m{}^s R_{klr}{}^n - \delta_r{}^n R_{klm}{}^s &= 0. \end{aligned} \quad (9)$$

Contracting with $\delta_s{}^k$ one obtains

$$D \cdot R_{rlm}{}^n - R_{rml}{}^n + \delta_r{}^n R_{lsm}{}^s = 0 \quad (10)$$

if one uses the first Bianchi identity

$$R_{klm}{}^n + R_{lmk}{}^n + R_{mkl}{}^n = 0. \quad (11)$$

Summing the cyclic permutations of r, l, m in (10) one finds that the Ricci tensor

$$R_{mn} = 1/2(R_{mkn}{}^k + R_{nkm}{}^k) \quad (12)$$

and the dilatational curvature

$$r_{mn} = R_{mnk}{}^k = R_{mkn}{}^k - R_{nkm}{}^k \quad (13)$$

have to vanish at a fixed point of a $D(D+1)$ dimensional subgroup of the general coordinate transformations acting on $\Gamma_{kl}{}^m$. But then (10) implies $D \cdot R_{klm}{}^n = R_{kml}{}^n$ so the Riemann curvature vanishes at the fixed point. Therefore $\Gamma_{kl}{}^m$ is a pure gauge which we can choose such that $\Gamma_{kl}{}^m$ vanishes. Inserted into $s \Gamma_{kl}{}^m = 0$ this means that the fixed point transformations are generated by linear inhomogeneous transformations, i.e. they are the affine transformations (4). So we have shown that the affine group is the highest dimensional fixed point group of the affine connection and the fixed point is $\Gamma_{kl}{}^m = 0$.

From $\Gamma_{kl}{}^m$ alone one can only construct Chern-Simons forms as Lagrangians in odd dimensions [2]. I do not know [4] whether some of these Lagrangians are acceptable i.e. yield $D > 4$ -dimensional theories which allow for a 4-dimensional Poincaré invariant solution and which, moreover, restrict near-by solutions to obey the 4-dimensional gravitationally self-coupled wave equation.

Rather, I want to investigate theories with an additional tensor density $G^{mn} = G^{nm}$, transforming according to

$$sG^{mn} = \mathcal{L}_\xi G^{mn} + d\partial_r \xi^r G^{mn}. \quad (14)$$

Let us investigate the weight assignment $d(G^{mn}) = 1$. If $D > 1$ then the only fixed point of $(\partial_{k_1} \dots \partial_{k_i} G^{mn}, i = 0, 1, \dots)$ under the affine transformations (4) is a vanishing G^{mn} . It is a matter of taste whether one still calls G^{mn} a metric even if one allows G^{mn} to vanish. If G^{mn} is invertible and $D > 2$ one can construct a metric g_{mn} from G^{mn} :

$$\begin{aligned} \text{if } \det G^{mn} \neq 0 \quad G^{mn} &= \sqrt{g} g^{mn} \\ g_{mn} &= (G^{-1})_{mn} (\det G^{kl})^{\frac{1}{D-2}}. \end{aligned} \quad (15)$$

However (15) should only be understood as an appeal to the reader's background in differential geometry: G^{mn} is defined by its transformation law (14). It is an argument of the local action and may have any value including values with $\det(G^{mn}) = 0$.

A field G^{mn} is suited to discuss questions such as whether the dynamics can enforce a change in the signature of the metric g_{mn} if one starts from an invertible G^{mn} with signature $(-1, 1, 1, 1, \dots)$.

Just as the ordinary invertible metric g_{mn} , the tensor density G^{mn} measures in a Lagrangian the kinetic energy such as

$$\mathcal{L}_{kin}(\Phi) = G^{mn} \mathcal{D}_m \Phi \mathcal{D}_n \Phi \quad (16)$$

of a scalar field. If G^{mn} becomes noninvertible this means that components of the gradient of Φ can be nonzero without a kinetic energy being required for that fluctuation. This makes perturbation theory hard, but the world has not necessarily come to an end where $\det(G^{mn}) = 0$.

For the weight assignment $d(G^{mn}) = 1$ and for even D with $D \geq 4$ the s-invariant local action which is regular for vanishing G^{mn} and Γ_{kl}^m is unique: it is given by the Lagrangian

$$\mathcal{L} = c_1 G^{mn} R_{mn}(\Gamma) + c_2 + dY \quad (17)$$

(where the value of c_1 is relevant only together with a non-vanishing vacuum expectation value of G^{mn}).

This follows along the lines of [2] because, up to the subtlety of Chern-Simons forms which occur in odd dimensions, the Lagrangian can be constructed from tensor densities.

In the case at hand the tensors are the covariant derivatives

$$\mathcal{D}_{k_1} \dots \mathcal{D}_{k_i} G^{mn}, \quad \mathcal{D}_{k_1} \dots \mathcal{D}_{k_i} R_{jlm}^n \quad i = 0, 1, \dots \quad (18)$$

where the covariant derivative $\mathcal{D}_m$ of a tensor density T of weight d contains a piece $(-d \cdot \Gamma_{kl}^l \cdot T)$ e.g.

$$\mathcal{D}_k G^{mn} = \partial_k G^{mn} + \Gamma_{kl}^m G^{ln} + \Gamma_{kl}^n G^{ml} - d \Gamma_{kl}^l G^{mn}. \quad (19)$$

No inverse of G^{mn} can appear in the Lagrangian $\mathcal{L}$ because G_{mn}^{-1} is not regular at the fixed point of the affine coordinate transformation.

The variables (18) have to be coupled to $SL(D)$ -singlets which must have weight $d = 1$ - apart from a possible constant contribution to $\mathcal{L}$. This means that - possibly by help of the numerical tensors $\underline{\varepsilon}_{n_1 \dots n_D}$ and $\bar{\varepsilon}^{m_1 \dots m_D}$ which are completely antisymmetric and have values $(0, 1, -1)$ - all indices have to be contracted. $\bar{\varepsilon}^{m_1 \dots m_D}$ is a tensor density of weight $d = 1$ and $\underline{\varepsilon}_{n_1 \dots n_D}$ has $d = -1$

$$\begin{aligned} s\bar{\varepsilon}^{m_1 \dots m_D} &= 0 = \mathcal{L}_\xi \bar{\varepsilon}^{m_1 \dots m_D} + \partial_n \xi^n \bar{\varepsilon}^{m_1 \dots m_D} \\ s\underline{\varepsilon}_{m_1 \dots m_D} &= 0 = \mathcal{L}_\xi \underline{\varepsilon}_{m_1 \dots m_D} - \partial_n \xi^n \underline{\varepsilon}_{m_1 \dots m_D} \end{aligned} \quad (20)$$

because antisymmetrization of $D + 1$ indices vanishes in D dimensions ($\bar{m}_i$ denotes omission of m_i)

$$\begin{aligned} \sum_{i=0}^D (-)^i \partial_{m_0} \xi^{m_i} \bar{\varepsilon}^{m_0 \dots \bar{m}_i \dots m_D} &= 0 \\ \sum_{i=0}^D (-)^i \partial_{m_i} \xi^{m_0} \underline{\varepsilon}_{m_0 \dots \bar{m}_i \dots m_D} &= 0. \end{aligned} \quad (21)$$

We can always arrange the contributions to $\mathcal{L}$ to contain either only $\bar{\varepsilon}^{m_1 \dots m_D}$ or only $\varepsilon_{n_1 \dots n_D}$ because each product of these tensors can be written in terms of Kronecker δ 's. The condition that all indices be contracted yields

$$\begin{aligned} \text{if } N_{\underline{\varepsilon}} \geq 0 \quad N_{\mathcal{D}} + 2N_R + DN_{\underline{\varepsilon}} &= 2N_G \\ \text{if } N_{\bar{\varepsilon}} \geq 0 \quad N_{\mathcal{D}} + 2N_R &= 2N_G + DN_{\bar{\varepsilon}} \end{aligned} \quad (22)$$

where $N_{\mathcal{D}}, N_R, N_{\bar{\varepsilon}}, N_{\underline{\varepsilon}}, N_G$ denote the numbers of derivatives, curvatures, $\bar{\varepsilon}^{m_1 \dots m_D}$, $\varepsilon_{n_1 \dots n_D}$ and metrics G^{mn} . The condition that the resulting $SL(D)$ invariant has weight $d = 1$ requires

$$\text{if } N_{\underline{\varepsilon}} \geq 0 \quad N_G - N_{\underline{\varepsilon}} = 1 \quad \text{or} \quad \text{if } N_{\bar{\varepsilon}} \geq 0 \quad N_G + N_{\bar{\varepsilon}} = 1 \quad \text{respectively.} \quad (23)$$

These equations have to be satisfied with nonnegative integers.

If $N_{\bar{\varepsilon}} \neq 0$ then $N_{\bar{\varepsilon}} = 1, N_G = 0$ by (23) and $N_{\mathcal{D}} + 2N_R = D$. $N_{\mathcal{D}} = 0$, however, for the following argument: no more than two indices of each factor $R_{klm}{}^n$ can be contracted to $\bar{\varepsilon}^{m_1 \dots m_D}$ because of the first Bianchi identity (11). Therefore all covariant derivatives and exactly two indices of each $R_{klm}{}^n$ have to be contracted to $\bar{\varepsilon}^{m_1 \dots m_D}$ because $N_{\mathcal{D}} + 2N_R = D$. But then the second Bianchi identity

$$\mathcal{D}_k R_{lmr}{}^s + \mathcal{D}_l R_{mkr}{}^s + \mathcal{D}_m R_{klr}{}^s = 0 \quad (24)$$

makes each contribution with covariant derivatives vanish. So if $N_{\bar{\varepsilon}} = 1$ then $N_{\mathcal{D}} = 0$ and $N_R = D/2$, i.e. only products of Chern forms for $SL(D) \times$ dilatations remain. They are topological densities of the form dY (17). Therefore no term with $\bar{\varepsilon}$ contributes to the equations of motion.

If $N_{\bar{\varepsilon}} = N_{\underline{\varepsilon}} = 0$ then $N_G = 1$ and $N_{\mathcal{D}} + 2N_R = 2$ has the two solutions $(N_{\mathcal{D}}, N_R) = (2, 0)$ or $(0, 1)$. The first solution leads only to a derivative term

$$\mathcal{D}_m \mathcal{D}_n G^{mn} = \partial_m (\mathcal{D}_n G^{mn}) \quad (25)$$

because $\mathcal{D}_n G^{mn}$ is a vector density of weight $d = 1$ and because $\Gamma_{kl}{}^m = \Gamma_{lk}{}^m$.

The second solution yields the Einstein term

$$G^{mn} R_{mn}{}^k{}_k. \quad (26)$$

Consider now $N_{\underline{\varepsilon}} > 0$: combining (22,23) one derives

$$N_{\mathcal{D}} + 2N_R + (D - 2)(N_{\underline{\varepsilon}} - 1) = 4 - D \quad (27)$$

which has no solution for $D \geq 4$ and $N_{\underline{\varepsilon}} \geq 1$ – the case $D = 4$, $N_{\underline{\varepsilon}} = 1, N_{\mathcal{D}} = N_R = 0$ yields a vanishing contraction.

Consider the low dimensions $D = 2$ and $D = 3$: $D = 2$ is special because the weight-0 field

$$\Phi_2 = \varepsilon_{kl} \varepsilon_{mn} G^{km} G^{ln} \quad (28)$$

can appear arbitrarily in the Lagrangian which therefore contains infinitely many arbitrary parameters. For $D = 3$, (27) reads

$$N_{\mathcal{D}} + 2N_R + (N_{\underline{\varepsilon}} - 1) = 1 \quad (29)$$

with the solutions

$$D = 3 \quad (N_D, N_R, N_{\underline{e}}, N_G) = (1, 0, 1, 2) \quad \text{or} \quad (0, 0, 2, 3) \quad (30)$$

The first solution gives only vanishing contractions, the second one gives the three dimensional cosmological constant

$$\Phi_3 = \underline{\varepsilon}_{klm} \underline{\varepsilon}_{k'l'm'} G^{kk'} G^{ll'} G^{mm'} \quad (31)$$

(it coincides with $\sqrt{g}$ if G^{mn} is invertible) . This concludes the proof of the uniqueness of $\mathcal{L}$ as given by (17).

It is easy to see that (17) describes matter-free Einstein gravity wherever G^{mn} is invertible. Just use (15) as a definition for the metric; then (17) is the Palatini form of the Einstein action. In the form (17) the action can, however, also be used at space-time points where g_{mn} (15) is not defined.

If one wants to include matter in a regular Lagrangian one has either to derive it from higher dimensional G^{mn} and spontaneous compactification, or one has to choose different gravitational fields with different charge assignments. No such charge assignment, however, will make a $D = 4$ dimensional model unique up to finitely many parameters if Yang-Mills fields are to be included with a usual kinetic energy

$$G^{kl} G^{mn} F_{km}^i F_{ln}^j c_{ij} . \quad (32)$$

For this term to be regular G^{mn} can have at most $d = 1/2$. It could have $d < 1/2$ if scalar fields with positive weight provide the necessary weight. But then the scalar field

$$\Phi_D = \underline{\varepsilon}_{n_1 \dots n_D} \underline{\varepsilon}_{m_1 \dots m_D} G^{n_1 m_1} \dots G^{n_D m_D} \quad (33)$$

has at most weight $d = \frac{1}{2}(D - 4)$, in particular for $D = 4$ it can appear arbitrarily in the Lagrangian.

If $D \geq 9$ then the standard Lagrangian below is unique up to finitely many parameters and has no cosmological constant. One introduces scalars Φ with weight $d = 1/4$, chiral spinors with weight $d = 3/8$, the affine connection Γ_{kl}^m and Yang-Mills-fields for $G_{Gauge} \times SO(1, D - 1)_{Lorentz}$ with weight $d = 0$ (they cannot have a weight $d \neq 0$ and a nilpotent s -transformation) and a vielbein E_a^m with weight $d = 1/4$ (which need not be invertible as opposed to the usually employed vielbein e_a^m with weight $d = 0$) . Up to a finite number of terms which contain covariant derivatives of the vielbein and up to Chern-Simons terms in odd dimensions, the resulting regular Lagrangian is in $D \geq 9$

$$\begin{aligned} \mathcal{L} = & \eta^{ab} E_a^m E_b^n R_{msn}^s(\Gamma) V_2(\Phi) + \eta^{bc} E_a^m E_b^n R_{nmc}^a(\Omega) V_2'(\Phi) \\ & - \frac{1}{4} \eta^{ab} \eta^{cd} E_a^k E_b^l E_c^m E_d^n F_{km}^i F_{ln}^j c_{ij} + \\ & + \frac{i}{2} E_a^m \bar{\Psi} \gamma^a \overleftrightarrow{\mathcal{D}}_m \Psi + \bar{\Psi} V_1(\Phi) \Psi + \\ & + \frac{1}{2} \eta^{ab} E_a^m E_b^n \mathcal{D}_m \Phi \mathcal{D}_n \Phi - V_4(\Phi) \end{aligned} \quad (34)$$

($V_i(\Phi)$ are homogeneous polynomials of order i).

$\mathcal{L}$ has a vanishing cosmological constant and remarkably coincides with a dilatationally invariant Lagrangian in $D = 4$. For $D = 8$ a cosmological constant $\Lambda \cdot \det E_a^m$ can occur. The analogue of (27) now reads

$$N_\Phi + \frac{3}{2} N_\Psi + N_D + 2N_R + (D - 4)(N_{\underline{e}} - 1) = 8 - D \quad (35)$$

which excludes $N_\epsilon \geq 1$ in $D \geq 9$ dimensions. For $N_\epsilon = 0$ it exhibits the reason for the dilatational invariance : the equation

$$N_\Phi + \frac{3}{2}N_\Psi + N_D + 2N_R = 4 \quad (36)$$

coincides with counting the dimensions of fields in $D = 4$.

The quantization of (17,34) has some uncommon features: the Lagrangians are polynomial and (17) has only a Φ^3 -interaction. The propagators, however, can only be defined for the spontaneously broken situation $\det(< G^{mn} >) \neq 0$. This makes it hard to control perturbatively the regularity requirement for the quantum vertex functional at vanishing fields. If this regularity can be maintained then the vertex functional is unique up to local contributions of the form (17,34) and therefore renormalizable in the sense that only a finite number of parameters need be specified.

One has to expect, however, that anomalies conflict with the invariance under general coordinate transformations. These anomalies can no longer be exchanged for Lorentz anomalies by regular Wess-Zumino counterterms [5]. This is different from the situation [2, 6] when G^{mn} is taken to be invertible. One has to expect the non-abelian $SL(D)$ anomaly, a mixed dilatational and $SL(D)$ anomaly, and a purely dilatational anomaly.

The restrictions which the absence of these anomalies imposes on acceptable models need a separate investigation. Perhaps one can live with dilatational anomalies because unitarity does not require this part of the gauge group [7].

Another obvious direction of investigation will determine regular supergravity Lagrangians and their properties. In particular R-invariance could possibly make regular $D = 4$ -dimensional theories with Yang-Mills fields unique.

I thank W. Bernreuther, F. Brandt, U. Ellwanger, R. Grimm, O. Piguet and R. Stora for stimulating discussions and T. Schwander for help in writing this paper.

REFERENCES

- [1] S. Coleman, J. Wess and B. Zumino; *Phys. Rev.* **177** (1969) 2239
- [2] F. Brandt, N. Dragon and M. Kreuzer; *Nucl. Phys.* **B 340** (1990) 187
- [3] M. Maischak; Diplomarbeit, Hannover, to be published
- [4] N. Dragon; *Proceedings of the XXIV International Symposium Ahrenschoop* ed. G. Weigt, Inst. f. Hochenergiephysik, Zeuthen, Germany (1991) 204
- [5] J. Wess and B. Zumino; *Phys. Lett.* **37 B** (1971) 95
- [6] L. Alvarez-Gaumé and E. Witten; *Nucl. Phys.* **B 234** (1984) 269
W. Bardeen and B. Zumino; *Nucl. Phys.* **B 244** (1984) 421
- [7] J. J. van der Bij, H. van Dam and Y. J. Ng; *Physica* **116 A** (1982) 307
N. Dragon and M. Kreuzer; *Z. Phys.* **C 41** (1988) 485
W. Buchmüller and N. Dragon; *Phys. Lett.* **223 B** (1989) 313
M. Kreuzer; *Class. Quantum Grav.* **7** (1990) 1303