Quantum Group Particles and Non-Archimedean Geometry

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ABSTRACT

Classical and quantum mechanics on the quantum plane are discussed. They correspond to q -deformation and $\hbar$ - q -deformations of the usual mechanics respectively. A particle on the quantum line as the simplest example of such a kind of system is investigated in more detail. A new phenomena occurs even for the free particle, namely, the mass for a free particle on the quantum line is not a number but a non-commutative element. Some relations between q -deformed quantum mechanics and p -adic quantum mechanics are also presented.

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1 Introduction

Non-commutative geometry [1] and quantum groups [2, 3] are of relevance to the problem of the quantization of space-time. Although in recent physics papers a discussion of this possibility was avoided rather carefully, it seems that the mathematical formalism itself pushes us to this direction. In particular, Manin [4] discussed quantum vector spaces and his consideration looks rather appropriate in this context. Wess and Zumino [3] considered the deformed Schrodinger equation in quantum space, see also [5]. Gomes and Sierra [6] found the quantum Lorentz group in strong coupling string theory.

The idea of quantization of space-time using non-commuting coordinates x_μ is an old one [7]. In this context relations like

$$x_\mu x_\nu - x_\nu x_\mu = i\hbar g_{\mu\nu} \quad (1)$$

or the anisotropic one

$$x_\mu x_\nu = q x_\nu x_\mu \quad (2)$$

were considered.

It is natural to attempt to relate the non-commutativity of coordinates to the uncertainty δx in the length measurement

$$\delta x > l_{Pl}, \quad (3)$$

where l_{Pl} is the Planck length. For a discussion (3) and corresponding references, see [10]. More recently the relation (3) was considered in string theory [8, 9]. However trying to derive the uncertainty relation (3) from (1) or (2) by analogy with the Heisenberg uncertainty relation we get a problem because (3) holds even in one dimension, i.e. on the line. It is unclear what is a "quantum line" because a coordinate always commutes with itself. Therefore in [11, 10] it was suggested that at the Planck scale one should use non-Archimedean geometry and p-adic numbers. The adelic picture which connects p-adic numbers with real numbers was discussed in [12]. The approach with non-commuting coordinates over real or complex numbers looks very different from the p-adic approach with commuting coordinates. It is a goal of this paper to discuss the similarity of these two approaches. We will see that actually some formulas in these two approaches coincide, so one can hope that a unified picture will emerge.

In this paper we start with a discussion of the distinction between q-deformation and $\hbar$ -deformation. Then we discuss a quantum line which is a reduction of q-deformed phase space and is a simple model of non-commutative geometry. In the consideration of the free particle on the quantum line an interesting property is pointed out. The mass of the particle cannot be a number but should be a non-commuting element of an algebra. Quantum mechanics on the quantum line admits two formulations one of which deals with complex valued functions and the second one with wave functions taking values in an algebra.

After that we discuss the relation of ideas of non-commutative geometry with the range of ideas on non-Archimedean geometry of the space-time as well as relations between q-analysis and p-adic analysis.

Quantum deformation of the Schrodinger equation for the harmonic oscillator was considered in [3, 5]. An important point in [3, 5] is that the corresponding wave functions take values in a non-commutative algebra. A problem appears with physical interpretation because there is not the standard Hilbert space and probability interpretation. It is

interesting to note that one has analogous problems in p-adic quantum mechanics [13]. There is a formulation of p-adic quantum mechanics with complex wave functions [14] and with p-adic wave functions [15]. Note that the quaternionic quantum mechanics also deals with non-commutative wave functions [16].

Another interesting aspect of the to quantum group is related to the idea that symmetry is not necessarily realized as a symmetry of the group transformation [21]. For instance, symmetries under q-deformed groups or algebras (in the sense of the invariance under coaction) can be considered as the underlying principles for constructing sensitive theories. For these models fields have to take values on q-deformed "manifolds". The q-deformed free particle and q-deformed harmonic oscillator are the simplest examples of such kinds of model. As one will see what we call the q-deformed harmonic oscillator is different from the so-called $\tilde{q}$ -oscillator, although in some aspects it is very similar.

The paper is organized as follows. We start with a general discussion of q-deformation and $\hbar$ -deformation. After that we present the dynamics of the simplest q-deformed system, a free movement on the quantum line. We discuss also its quantization in the sense of $\hbar$ -deformation. We finish this section with a brief presentation of a generalization of the notion of quantum plane for an infinite number of degrees of freedom. In sect.5 we present some relations between q-deformed systems and p-adic models. In the conclusion we point out some questions for future investigations.

2 q-deformation and $\hbar$ -deformation

Our goal is to try to consider dynamical systems on q-deformed "manifolds" and their quantization. We will consider the simplest examples of of such dynamical systems. Generally speaking there are the following possibilities for a "classical" system:

1) Space-time is standard. "Classical" variables in mechanics and field theory take values in a q-vector space or quantum group or quantum algebra [21].

2) "Classical" mechanics or field theory on q-deformed space-time, i.e. field dynamical variables are defined on quantum space-time and take values in the usual groups or algebras.

3) "Classical" variables in mechanics and field theory take values in a q-deformed "manifold" (q-vector space, or quantum group, etc.) and space-time is also q-deformed .

Let us stress that we distinguish q-deformation and quantization ($\hbar$ -deformation) [21]. For example, dealing with standard space-time and "classical" variables taken values in the q-deformed vector space (or the quantum group), one has to perform an additional " $\hbar$ -deformation" of the system to construct a quantum theory.

Schematically one can present the situation in the following diagram (which in

general is not commutative)

$\begin{aligned} x_i x_j &= x_j x_i, \\ p_i p_j &= p_j p_i, \\ \underbrace{p_i x_j = x_j p_i}_{\text{classical system}} \end{aligned}$	$\xrightarrow{q\text{-deformation}}$	$\begin{aligned} x_i x_j &= q x_j x_i, \quad i < j, \\ p_i p_j &= q p_j p_i, \\ \underbrace{p_i x_j = q x_j p_i + \dots}_{q\text{-deformed system}} \end{aligned}$
$\downarrow \hbar\text{-deformation}$		
$\downarrow \hbar\text{-deformation}$		$\underbrace{p_i x_j - q x_j p_i = i\hbar \Delta_{ij}(q) + \dots}_{\hbar\text{-}q\text{-deformed system}}$
$[p_i, p_j] = [x_i, x_j] = 0,$		$?$
$\underbrace{[p_i, x_j] = i\hbar \delta_{ij}}_{\text{quantum system}}$	$\xrightarrow{q\text{-deformation}}$	$\underbrace{p_i x_j - q x_j p_i = i\hbar \tilde{\Delta}_{ij}(q) + \dots}_{q\text{-}\hbar\text{ deformed system}}$

The dots in this table mean a special quadratic form in p and x (see eq.(67)).

There are also different possibilities for dealing with quantization of the above mentioned systems. We can deal with the usual complex valued wave function or a q -valued wave function (compare with the different possibilities in p -adic quantum mechanics [10]). So dealing with q -deformed mechanics and field theory we have a variety of possibilities.

3 Classical mechanics on the quantum line

Quantum phase space R_q^2 is formed by the generators of position x and momentum p satisfying the relation

$$px = qxp \quad (4)$$

This is what we mean by saying that x is the coordinate on the quantum line. So the quantum line is a reduction of the quantum phase space. Here the term "quantum space" for R_q^2 is a little misleading. In fact we are considering R_q^2 as a "classical" but q -deformed space and we will quantize the corresponding dynamical system in the sense of $\hbar$ -deformations.

More exactly, one can assume two realizations of the space R_q^2 . Working over the field of real numbers R , we consider an algebra $\mathcal{A}$ with elements p and x belonging to $\mathcal{A}$ and satisfying the relation (4). In this case q is an arbitrary real number. If one works over the field of complex numbers C , then one considers an algebra $\mathcal{A}$ with involution and the elements p and x are Hermitean. In this case it follows from (4) that q is a complex number satisfying the relation $|q| = 1$.

The q -deformed symplectic form on R_q^2 is

$$B_q(z, z') = px' - qxp'. \quad (5)$$

Here $z = (x, p)$, $z' = (x', p') \in R_q^2$. The symplectic form B_q is invariant under transformations $z \rightarrow gz$ with matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belonging $SL_q(2)$ and with entries commuting with p and x , i.e.

$$B_q(gz, gz') = B_q(z, z') \quad (6)$$

Note that there are transformations which leave invariant the relation (4) but violate the relation (6). For example, the transformation

$$x(t) = e^{tH(p,x)} x e^{-tH(p,x)}, \quad p(t) = e^{tH(p,x)} p e^{-tH(p,x)}, \quad (7)$$

where t is real variable and H is for example a polynomial function in p and q is, generally speaking, one of such kind of transformation.

For the case when $|q| = 1$ it is more convenient to use the symplectic form $\mathcal{B}$ which differs from B by the factor $q^{1/2}$, i.e.

$$\mathcal{B}_q(z, z') = q^{-1/2} p x' - q^{1/2} x p' \quad (8)$$

3.1 Free particle

For the motion of a free particle on the quantum line our coordinates x and p will depend on the real variable t and one has

$$p(t)x(t) = qx(t)p(t) \quad (9)$$

The equation of motion has the form

$$\ddot{x} = 0 \quad (10)$$

or in the canonical form

$$p = m\dot{x} \quad (11)$$

$$\dot{p} = 0 \quad (12)$$

It is an important property of this system that the *the mass m cannot be a number* because from (9), (11) and (12) one gets

$$mp = qpm \quad (13)$$

The solution of these equations of motion (10-12) is

$$x(t) = x + m^{-1}pt$$

$$p(t) = p \quad (14)$$

This solution satisfies the requirement (9). Under the additional relations $x'm^{-1} = qm^{-1}x'$, $p'm^{-1} = qm^{-1}p'$, the q -symplectic form (6) is also left invariant under the evolution (14). Note that the evolution (14) is not a transformation of the form $z(t) = gz$ with the entries of g commuting with the components of z .

Equations (11) and (12) and their solution (14) are written for the case when one deals with real numbers. For the case when one deals with an algebra with involution, i.e. $|q| = 1$ and x and p are Hermitian, it is more convenient to deal with a Hermitian operator m and a slightly modified equation (14)

$$\dot{x} = q^{1/2}m^{-1}p. \quad (15)$$

This equation has the following solution

$$x(t) = x + q^{1/2}m^{-1}pt \quad (16)$$

Equations of motion (12) and (15) can be written in the Hamiltonian form

$$\dot{x} = \{H, x\}_q, \quad \dot{p} = \{H, p\}_q \quad (17)$$

with the following Hamiltonian

$$H = \frac{q}{1+q^2} p m^{-1} p. \quad (18)$$

Here we introduce q-Poisson brackets. These brackets must satisfy a deformed Leibnitz rule because, assuming linearity of the Poisson bracket, one has from (4)

$$\{x, xp\}_q = q^{-1} x \{x, p\}_q. \quad (19)$$

The q-deformed Poisson bracket is connected with the q-deformed symplectic form

$$\{f, g\}_q = \frac{\partial f}{\partial z^A} \Omega(q)^{AB} \frac{\partial g}{\partial z^B}, \quad z^A = (x, p), \quad (20)$$

where

$$\Omega(q)^{AB} = \begin{pmatrix} 0 & -q^{1/2} \\ -q^{1/2} & 0 \end{pmatrix}, \quad (21)$$

$$\mathcal{B}_q(z, z') = z^A \Omega(q)_{AB} z'^A, \quad \Omega(q)_{AB} = \begin{pmatrix} 0 & q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix}.$$

The derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial p}$ have to satisfy the following rules

$$\frac{\partial}{\partial x} p^n x^m = q^n \frac{1 - q^{2m}}{1 - q^2} p^n x^{m-1}, \quad (22)$$

$$\frac{\partial}{\partial p} p^n x^m = \frac{1 - q^{2n}}{1 - q^n} p^{n-1} x^m, \quad (23)$$

which are the same as in [3], and we also assume

$$\partial_p m = q m \partial_p. \quad (24)$$

The Poisson bracket for the "canonical" variables p and q is

$$\{p, x\}_q = q^{1/2}. \quad (25)$$

Let us stress that the most appealing feature of the quantum line analog of free motion is that the theory is aimed at yielding a mass operator m which is a non-commuting object.

3.2 q-harmonic oscillator.

For the q-harmonic oscillator one has the equation

$$\ddot{x}(t) = -\omega^2 x(t). \quad (26)$$

or

$$\dot{x}(t) = q^{1/2} \omega m^{-1} p(t) \quad (27)$$

$$\dot{p}(t) = -\frac{\omega^2}{q^{1/2}}mx(t) \quad (28)$$

with the following solutions

$$x(t) = x \cos \omega t + q^{1/2}p(\omega m)^{-1} \sin \omega t, \quad (29)$$

$$p(t) = p \cos \omega t - q^{-1/2}\omega m x \sin \omega t, \quad (30)$$

Here ω is a number, but we cannot assume that m is number; rather we must assume that it is subject to the following relations

$$xm = q^{-1}mx, \quad pm = q^{-1}mp. \quad (31)$$

For derivatives, the relations (22) and (23), as well as

$$\frac{\partial}{\partial x}m = qm \frac{\partial}{\partial x}, \quad (32)$$

are assumed. Under the conditions (4) and (31) the evolution (30),(29) leaves invariant the relation $p(t)x(t) = qx(t)p(t)$ and also the q -symplectic form.

With the Poisson bracket (20) the equations of motion (27) and (28) can be written in the form (17) with the following Hamiltonian

$$H = \frac{q}{1+q^2}pm^{-1}p + \frac{1}{q} \frac{\omega^2}{1+q^2}xmx. \quad (33)$$

4 Quantum mechanics on the quantum line

Now let us quantize our relations (16), substituting a q -commutator instead of the q -Poisson bracket and $-i\hbar$ in the left hand side. So we have

$$px - qxp = -i\hbar q^{1/2}. \quad (34)$$

We call this relation the $\hbar$ - q -deformed relation. We have $q^{1/2}$ in the right hand side of (34) in agreement with hermiticity (note that here $|q| = 1$).

One can consider a more general relation

$$px - qxp = -i\hbar \Delta \quad (35)$$

If Δ is a c -number, we come back to (34) with $|q| = 1$. If Δ is an operator one gets a relation

$$px(1 - |q^2|) = i\hbar(q\Delta^* - \Delta).$$

In this case q may be a real number.

An explicit connection between p and x in the relation (34) and the usual operators P and X , satisfying the relation

$$[P, X] = -i\hbar, \quad (36)$$

is the following

$$p = -\frac{\hbar}{2 \sin \theta/2} X^{-1} [e^{-\frac{\theta}{\hbar} XP} - 1] \quad (37)$$

with $x = X$ and $q = e^{i\theta}$ (see for example [20]).

In the usual coordinate representation the formula gives

$$pf(x) = -i\hbar q^{1/2} \frac{f(qx) - f(x)}{(q-1)x}. \quad (38)$$

This operator can be represented in terms of operators K and D by

$$pf(x) = -i\hbar q^{1/2} q^{K/2} D q^{1/2}, \quad (39)$$

where

$$D_q f(x) = \frac{f(qx) - f(x/q)}{(q - 1/q)x}, \quad (40)$$

$$q^K f(x) = f(qx). \quad (41)$$

The definition (38) has a meaning for example for functions of the form

$$\mathcal{P}(x) \exp(-x^2/2), \quad (42)$$

with $\mathcal{P}$ being a polynomial, if $\text{Re } q < \pi/2$. With the usual definition of scalar product the operators D and K have the following properties under hermitian conjugation

$$(D)^* = -D, \quad (K)^* = -K - 1. \quad (43)$$

Thus it is natural to introduce the following hermitian operator

$$M = q^{-\tilde{K}}, \quad (44)$$

where

$$\tilde{K} = K + 1/2, \quad (\tilde{K})^* = -\tilde{K}. \quad (45)$$

The commutations relations of the operator $\tilde{K}$ with p and x have the form

$$[\tilde{K}, x] = x, \quad [\tilde{K}, p] = -p \quad (46)$$

and therefore

$$pM^\nu = q^{-\nu} M^\nu p, \quad (47)$$

$$xM^\nu = q^\nu M^\nu x. \quad (48)$$

We see that the algebra of operators p and M is the same as the algebra (31) of p and m while the algebra of x and M is different from the algebra of x and m . Note that the operators p and x in (34) are hermitian. This is in contrast with the so called q -oscillator algebra [17, 18, 19], where creation and annihilation operators satisfying analogous relations are reciprocally-conjugate operators.

4.1 Free particle

Now we are going to consider the $\hbar$ -quantization of the q -deformed free particle, or in other words to specify the evolution of the operators $x(t)$ and $p(x)$ satisfying the commutation relations (34). There are at least two possibilities. In the first one we can try to preserve under quantization the classical equation of motion, i.e.

$$\dot{p} = 0, \quad \dot{x} = q^{1/2} M^{-1} p \quad (49)$$

and the form of the classical solution

$$x(t) = x + q^{1/2} M^{-1} p t, \quad p(t) = p. \quad (50)$$

This evolution cannot be performed by the unitarity operator $U(t) = \exp i H t$, with a Hamiltonian quadratic in p .

In the second one we can assume that there is unitary dynamics which preserves only the first equation (49)

$$i \hbar \dot{p} = [H, p] = 0. \quad (51)$$

In particular $H = \frac{1}{2\mu^2} p^2$ with μ being a number satisfies this requirement. However we have

$$i \hbar \dot{x} = \frac{1}{2\mu} [p^2, x] = i \hbar q (1 + q^2) \frac{1}{2\mu} p + (1 - q^4) \frac{1}{2\mu} x p^2, \quad (52)$$

so $\ddot{x} \neq 0$.

It is instructive to try to write down the q -deformed analog of plane-waves. Formally we have

$$p \exp_{q^2} \left(\frac{-i k x}{\hbar q^{1/2}} \right) = k \exp_{q^2} \left(\frac{-i k x}{\hbar q^{1/2}} \right) \quad (53)$$

where

$$\exp_q(x) = \sum \frac{1}{[[n]]_q!} x^n, \quad [[n]]_q = \frac{1 - q^n}{1 - q}. \quad (54)$$

However there are some problems in equation (54). The q -deformed exponent is ill-defined for $q = \exp(2\pi i \theta)$ with rational θ because the denominators have zeros.

4.2 Harmonic oscillator

To define a natural generalization of the harmonic oscillator Hamiltonian let us introduce the following creation and annihilation operators

$$\alpha^+ = \frac{1}{\sqrt{2}} (q^{\rho K/2} x + i p q^{\nu K/2}), \quad \alpha = \frac{1}{\sqrt{2}} (q^{\rho K/2} x - i p q^{\nu K/2}), \quad (55)$$

From the conjugation rules (43) it follows that

$$(\alpha^+)^* = \alpha. \quad (56)$$

The relations (34), (47) and (48) under the condition

$$\rho + \nu = -1 \quad (57)$$

lead to

$$[\alpha, \alpha^*] = \frac{1}{q^{\bar{K}}} \quad (58)$$

Note that in the left hand side of (58) there is the usual commutator. It is natural to use the following formula for the harmonic oscillator Hamiltonian

$$H = \alpha^+ M^{-1} \alpha = \frac{c}{2} [x q^{\rho \bar{K} + 1/2} x + p q^{\nu \bar{K}} p + q^{1/2} (p x - q^{-1} x p)], \quad (59)$$

One can find the first few eigenstates of this operator

$$\psi_0, \alpha^+ \psi_0, \alpha^+ \alpha^+ \psi_0, \alpha \psi_0 = 0, \quad (60)$$

with the following eigenvalues

$$E_0 = 0, \quad E_1 = 1, \quad E_2 = c\left(\frac{1}{2}(q + 1/q) + 1\right). \quad (61)$$

A formal solution of the equation $a\psi_0 = 0$,

$$\psi_0 = \exp_{1/q^2}(-qx^2), \quad \rho = -1/2 \quad (62)$$

is again ill-defined for $|q| = 1$. To avoid this difficulty we have to abandon the condition $|q| = 1$. In this case it is natural to deal with a "complex" particle, i.e. with 4-dimensional phase space z^A , $z^1 = (x, \bar{x})$, $z^2 = (p, \bar{p})$ and with the complex extension of the relations (34).

5 Multi-dimensional generalisations

Let us define q -deformed $2n$ -dimensional "classical" phase space to be formed by generators of position x^i and momentum p_j , subject to the relations

$$x^i x^j = \frac{1}{q} \hat{R}^{ij}_{kl} x^k x^l \quad (63)$$

$$p_i p_j = \frac{1}{q} \hat{R}^{ik}_{jl} p_k p_l \quad (64)$$

$$p_j x^i = q \hat{R}^{ik}_{jl} x^l p_k \quad (65)$$

where the $\hat{R}$ -matrix is given by the usual formula

$$\hat{R}^{ij}_{kl} = \delta_k^i \delta_l^j (1 + (q - 1)\delta^{ij}) + (q - 1/q) \delta_k^i \delta_l^j \theta(i - j) \quad (66)$$

$$\theta(i - j) = 1, \quad i > j, \quad \theta(i - j) = 0, \quad i < j, \quad i = j.$$

(compare with [3]).

It is natural to assume that under $\hbar$ -deformation relations (63) and (64) do not change and the $\hbar$ -deformed version of (65) is

$$p_j x^i = -iq\hbar \delta_j^i + \hat{R}^{ik}_{jl} x^l p_k \quad (67)$$

Note that to get from this relation the commutation relations (34) one should change q to $q^{1/2}$.

The algebraic relations (63), (64) and (67) with $\hat{R}$ as in (66) are the same as for the q -deformed oscillators [17, 18, 19],

$$A_i A_j = 1/q A_j A_i, \quad A_i^+ A_j^+ = 1/q A_j^+ A_i^+ \quad i < j. \quad (68)$$

$$A_i A_j^+ - q A_j^+ A_i = \delta_j^i (1 + (q^2 - q) A_i^+ A_i + (q^2 - 1) \sum_{k>I} A_k^+ A_k).$$

However the hermiticity properties are different. For the operators p and x , the usual hermiticity properties $(p_i)^+ = p_i$, $(x_i)^+ = x_i$ are assumed.

Note also that the algebra (63), (64) and (67) admits a generalization to the continuum case. Let us write down the corresponding generalization for creation and annihilation operators. We can assume that creation and annihilation operators are labeled

by a momentum variable $k \in R^1$, the discretization of which gives a labelling by an index k , $k = 1, 2, \dots, n$, $A_k = \Delta^{1/2} a(k)$; A_k satisfies the algebra of q -deformed oscillators [17, 18, 19] and Δ is a spacing parameter. So the continuum version of the algebra (68) looks like

$$\begin{aligned} [a(k), a(l)]_q &= 0, \quad [a^+(k), a^+(l)]_q = 0, \quad k > l, \\ [a(k), a^+(l)]_q &= \delta(k-l)(1 + (q^{-2} - 1) \int_k^\infty dk' a^+(k') a(k')) \end{aligned} \quad (69)$$

6 q -analysis and p -adic analysis

We are going to turn an attention to some remarkable relations between the theory of quantum groups (q -analysis) and p -adic analysis (some of them have just been mentioned in [13]).

6.1 p -adic and q -integrals

Proposition 1 *The Haar measure on $SU_q(2)$ coincides with the Haar measure on the field of p -adic numbers Q_p if $q = 1/p$.*

It is known that the cohomology group $H^3(SU_q(2)) = C$ [24]; therefore there exists a unique linear functional

$$\int : \Gamma^3 \rightarrow C,$$

such that $\int d\zeta = 0$ for $\zeta \in \Gamma^2$, where Γ^n is the module of n -differential forms. For the coordinate ring of $SU_q(2)$ one has

$$\int f \omega_0 \omega_1 \omega_2 = m(f) \quad (70)$$

where m is the Haar measure and ω_i are differential forms on $SU_q(2)$. If f is a polynomial in cc^* , we have

$$m(f) = (1 - q^2) \sum_{n=0}^{\infty} f(q^{2n}) q^{2n}, \quad 0 < q < 1. \quad (71)$$

This is the well known Tomea-Jackson integral in q -analysis

$$\int_0^1 f(x) d_q x = (1 - q) \sum_{n=0}^{\infty} f(q^n) q^n. \quad (72)$$

Now we recall that the integral over Q_p with respect to the Haar measure is:

$$\int_{|x|_p \leq 1} f(|x|_p) dx = (1 - \frac{1}{p}) \sum_{n=0}^{\infty} f(p^{-n}) p^{-n}. \quad (73)$$

Hence we get that (72) is equal to (73) if

$$q = \frac{1}{p}. \quad (74)$$

i.e.

$$\int_0^1 f(x) d_{1/p} x = \int_{|x|_p \leq 1} f(|x|_p) dx. \quad (75)$$

6.2 Differential operators.

Let us now consider the relation between differential operators. In q -analysis one has the following operator of differentiation (which up to some factor coincides with the q -deformed momentum in the coordinate representation (37))

$$\partial_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad (76)$$

and

$$\partial_q f(x) = \sum_i f(q^{i+1/2}x)(xq^i - xq^{i+1}). \quad (77)$$

In p -adic analysis, if we consider a real valued function $f(x)$ depending on a p -adic variable x , we cannot use the standard definition of differentiation, because one cannot multiply real and p -adic numbers. In [22] the following operator was considered

$$D^\alpha f(x) = \frac{p-1}{1-p^{-1-\alpha}} \int \frac{f(x) - f(y)}{|x-y|_p^{\alpha+1}} dy, \quad (78)$$

which in terms of Fourier-transformation has the form

$$Df(x) = \int_{Q_p} |k|_p \tilde{f}(k) \chi(-kx) dk \quad (79)$$

Here χ is an additive character on Q_p . Restricting to rational x in (76) and (78), we see that for $\alpha = 1$ the expressions (76) and (78) are very similar; in particular, both are non-local operators.

6.3 Spectrum of the q -deformed oscillator and the p -adic model

Let us discuss the connections between the spectra of the q -deformed oscillator and of the p -adic model.

The spectrum of the p -adic equation [25]

$$D\psi(x) + V(|x|_p)\psi(x) = E\psi \quad (80)$$

for

$$V(|x|_p) = \frac{p}{1-p^2} \left[|x|_p - \frac{1+p+p^2}{p|x|_p} \right] \quad (81)$$

has eigenvalues

$$E_n = [n]_{1/p} = \frac{p^{-n} - p^n}{p^{-1} - p}, \quad p \in \mathbb{Z}. \quad (82)$$

For the q -deformed oscillator

$$aa^+ - qa^+a = q^{-N}, \quad [N, a] = -a, \quad [N, a^+] = a^+, \quad (83)$$

the spectrum was found by Biedenharn [18], see also [17, 19],

$$H = a^+a, \quad H|n\rangle = [n]_q|n\rangle. \quad (84)$$

If

$$q = \frac{1}{p}, \quad (85)$$

one gets the p-adic spectrum (82).

In fact in p-adic quantum mechanics it is known that in the spectrum of the more general equation

$$D^\alpha \psi(x) + V(|x|_p) \psi(x) = E \psi \quad \alpha > 0, \quad (86)$$

there is the following family of eigenvalues

$$E_{n,l} = p^{\alpha n} + V(p^{l-n}), \quad l = 1, 2, 3, \dots; \quad n = 0, \pm 1, \dots \quad (87)$$

The first three eigenvalues of the Hamiltonian (59) can be obtained from the spectrum of the p-adic Schrodinger equation if one chooses the constant c in (59) equal to some fixed value, i.e.

$$E_0 = E_{0,0}, \quad E_1 = E_{1,1}, \quad E_2 = E_{1,0}, \quad (88)$$

if $c = 1/q - 1$ for

$$V(x) = -1/2 - 1/2x^2.$$

Eigenvalues of the q-deformed Schrodinger equation for n variables have been obtained by Wess and Zumino and Carrow-Watamura et al [5]. It is an interesting problem to find a corresponding p-adic model with the same spectrum.

7 Conclusion

In this paper we point out several remarkable analogies between q-deformed systems and models over p-adic numbers. For some quantities there are even identities. Some other relations have been recently discussed by Freund [26]. Among further questions to be answered are what is the mathematical reason for these relations and what is the physical interpretation of this connection. From the physical point of view one can only speculate that at the Planck scale there are fluctuations of the metric [27] and signature [28, 29], and the number field [10, 30] undergoes fluctuations; moreover, the space-time coordinates become non-commuting operators. Quantum cosmology [31] appears to be as a more appropriate field for attempts to relate these ideas to reality.

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