S16-91/33

SISSA-67/91/FM
KA-THEP-1991-3

Reduction of the Rational Spin $sl(2, \mathbb{C})$ WZNW Conformal Theory

P. Furlan

*Dipartimento di Fisica Teorica dell'Università di Trieste, Italy
Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Trieste, Italy*

A.Ch. Ganchev*

Theory Division, CERN CH-1211 Geneva 23, Switzerland

R. Paunov*

International School for Advanced Studies, 34014 Trieste, Italy

V.B. Petkova*

*Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Trieste, Italy
Institut für Theoretische Physik, Universität Karlsruhe (TH), D-7500 Karlsruhe, Germany*

Abstract

We construct explicitly the correlation functions on the plane of the rational spin and level $A_1^{(1)}$ WZNW conformal model. The corresponding Virasoro minimal correlations are recovered in the limit when the isospin $sl(2, \mathbb{C})$ variables x_i approach the coordinates z_i . The Knizhnik - Zamolodchikov equation reduces to a differential equation for the minimal conformal blocks, reflecting the decoupling of the Virasoro null vectors.

* Permanent address: Institute for Nuclear Research and Nuclear Energy, Sofia 1784, Bulgaria.

May 1991

1. The WZNW models play a fundamental role in the 2-dim conformal world. Starting with the work of Knizhnik, Polyakov, Zamolodchikov on the 2-dim induced gravity [1,2] it has become clear that through a kind of quantum hamiltonian reduction procedure [3-6] the rational level WZNW theories [7] generate many of the known RCFT. The simplest example is provided by the theory based on the algebra $g = A_1^{(1)}$. The representations are labelled by a rational value of the level k and a set $\{J\}$ of rational spins

$$J = j - j'(k+2), \quad k+2 = p/p',$$

$$1 \leq n = 2j+1 \leq p-1, \quad 0 \leq n' = 2j'+1 \leq p'-1, \quad (1.1)$$

so that the Sugawara conformal dimensions Δ_J are given by

$$\Delta_J = \frac{J(J+1)}{k+2} = \Delta_{-J-1}. \quad (1.2)$$

This theory is related to the minimal Virasoro theories of central charge c_k and dimensions $\{h_J \equiv h_{n'n} = h_{\underline{J}}; \underline{J} = k+2-J-1\}$ [8] according to [2]

$$c_k = 13 - 6(k+2 + \frac{1}{k+2}),$$

$$h_J = h_{\underline{J}} = \Delta_J - J. \quad (1.3)$$

The map $(J, \underline{J}) \rightarrow h_J$ produces also representations beyond the minimal set, so it gives rather a correspondence between the reducible (Verma, or Fock space) modules of the two theories. Choosing in (1.3) $k+2 = -p/p'$ one recovers instead the rational $c > 25$ set of dimensions.

While the reduction of the chiral algebra and the representations are by now well understood, this is not the case for the complete theory. In this paper, and in a detailed version [9] to follow, we initiate the study of the reduction of the correlation functions (at genus zero) and the equation governing them - the Knizhnik - Zamolodchikov (KZ) equation [10]. A complete and consistent construction of the general rational level and spin WZNW correlations is lacking. The existing results [11 - 18] cover basically the "thermal" case $\{J = j\}$. An example with a rational spin has been considered recently [19], exploiting the generalized Coulomb gas representation, based on the Wakimoto bosonization of the algebra $A_1^{(1)}$. It seems to indicate that this technique produces in general functions with modified braid properties if compared with the corresponding minimal conformal blocks. This makes the relation between the correlations of the two theories rather obscure. We propose here an extension beyond the WZNW Coulomb gas expressions which avoids this problem. The correlation functions satisfy the rational level and spin KZ equation and reproduce in a simple limit their minimal $c < 1$ (or $c > 25$) counterparts. We show furthermore that the KZ equation reduces for thermal spins to a differential equation for the minimal functions. This reduction takes place as well for the rational "quasithermal" spins $\{J = -j'(k+2)\}$, though now it is less straightforward - one has to take into account the interplay between the model defined by (1.1) and its dual, obtained by replacing $(k+2, J)$ with $(1/(k+2), -J/(k+2))$.

A different type of relation for the characters of the two theories has been established in [20].

2. We shall use the standard functional realization of the representations of the isospin $sl(2, \mathcal{C})$, used also in [11], so that the fields $\Psi^J(x, z)$ depend on two complex variables.

The n -point functions

$$W^{(n)}(\{x_i, z_i; J_i\}) = \langle \Psi^{J_1}(x_1, z_1) \dots \Psi^{J_n}(x_n, z_n) \rangle \quad (2.1)$$

are invariant under the projective coordinate and isospin transformations with generators represented by the differential operators

$$L_n = z^{1+n} \partial_z + (1+n) \Delta z, \quad S^n = x^{1+n} \partial_x - (1+n) J x; \quad n = -1, 0, 1. \quad (2.2)$$

The 2- and 3- point invariants are determined up to constants

$$\langle \Psi^J(x_1, z_1) \Psi^J(x_2, z_2) \rangle = \frac{x_{12}^{2J}}{z_{12}^{2\Delta_J}},$$

$$\langle \Psi^{J_1}(x_1, z_1) \Psi^{J_2}(x_2, z_2) \Psi^{J_3}(x_3, z_3) \rangle = C_{J_1 J_2 J_3} \frac{x_{12}^{J_1+J_2-J_3} x_{13}^{J_1+J_3-J_2} x_{23}^{J_2+J_3-J_1}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3} z_{13}^{\Delta_1+\Delta_3-\Delta_2} z_{23}^{\Delta_2+\Delta_3-\Delta_1}}. \quad (2.3)$$

In the limit $x_a \rightarrow z_a$ one recovers from (2.3) the 2- and 3-point functions of the Vir primary fields $\Psi_{h_J}(z)$ of dimension $h_J = \Delta_J - J$. For $n > 3$ the WZNW primary fields correlation functions are determined as solutions of the KZ system of differential equations

$$[(k+2)\partial_{z_a} - \sum_{b \neq a} \frac{\Omega_{ab}}{z_{ab}}] W^{(n)}(\{x_i, z_i; J_i\}) = 0, \quad (2.4a)$$

$$a = 1, 2, \dots, n,$$

$$\Omega_{ab} = -x_{ab}^2 \frac{\partial^2}{\partial x_a \partial x_b} - 2x_{ab}(J_a \partial_{x_b} - J_b \partial_{x_a}) + 2J_a J_b. \quad (2.4b)$$

Taking into account the projective invariance the n -point functions can be written as

$$W^{(n)}(\{x_a, z_a; J_a\}) = f(\{z_a, \Delta_a\}) f(\{x_a, -J_a\}) \prod_{1 \leq a < b \leq n-1} \frac{z_{ab}^{\frac{2J_a J_b}{k+2}}}{z_{ab}^{\frac{2J_a J_b}{k+2}}} U(\{\underline{x}_a, \underline{z}_a; J_a\}), \quad (2.5a)$$

$$f(\{z_a, \Delta_a\}) = \frac{z_{13}^{\Delta_n - \sum_{a=1}^{n-1} \Delta_a} z_{1n}^{\Delta_2 + \Delta_3 - \Delta_1 - \Delta_n + \sum_{a=1}^{n-1} \Delta_a} z_{3n}^{\Delta_1 + \Delta_2 - \Delta_3 - \Delta_n + \sum_{a=1}^{n-1} \Delta_a}}{z_{2n}^{2\Delta_2} \prod_{a=1}^{n-1} z_{an}^{2\Delta_a}}, \quad (2.5b)$$

$$\underline{z}_a = \frac{z_{1a} z_{3n}}{z_{13} z_{an}}, a = 1, 2, \dots, n. \quad (2.5c)$$

For reasons of convenience we have left in (2.5a) a factor depending on $\underline{z}_a, a = 2, 4, 5 \dots n-1$. The prefactor in (2.5a) recovers in the limit $x_a \rightarrow z_a, a = 1, 2, \dots, n$ the standard prefactor in the corresponding minimal correlation.

To make things more explicit, let us consider the case $n = 4$. We shall look for a solution of (2.4) given by the power series

$$U(x, z) = \sum_{s=0} (x - z)^s C_s(z). \quad (2.6a)$$

where $x = \underline{x}_2, z = \underline{z}_2$; and $s = 0, 1, 2, \dots$. The first term in (2.6a) will turn out to coincide with the corresponding Dotsenko - Fateev (DF) integral [21] so that the limit to the minimal correlations will be straightforward.

The KZ equation (2.4) implies a system of differential equations for the coefficients $C_s(z)$ in (2.6), which is of the same type as in the standard WZNW theory [11]

$$[(k+2)\partial_z + a_{ss}]C_s + a_{s\ s+1}C_{s\ s+1} + a_{s\ s-1}C_{s\ s-1} = 0, \quad (2.7)$$

only now in general s is not restricted from above. In (2.7) $a_{0\ -1} = 0$ and

$$a_{ss} = \frac{A_s(J_1 + J_2)}{z} + \frac{A_s(J_3 + J_2)}{z-1}, \quad A_s(J) = s(2J - s + 1),$$

$$a_{s\ s-1} = \frac{(M - s + 1)(2J_2 - s + 1)}{z(1 - z)}, \quad a_{s\ s+1} = (s + 1)c_s, \quad (2.8)$$

$$c_s = 1 + \sum_{a=1}^4 J_a - s - (k + 2); \quad M = J_1 + J_2 + J_3 - J_4 = m - m'(k + 2). \quad (2.9)$$

To describe the solutions of (2.7) let us start with the simplest, thermal case, $J_a = j_a, M = m$, which parallels the integer level WZNW theory. A solution of (2.7) is provided by

$$C_s(z) = B_s I_{(0,s,0)}^{(m)}(z), \quad (2.6b)$$

$$B_s = (-1)^s \binom{2J_2}{s} \frac{c_s!}{(M - s)! c_M!}, \quad (2.10)$$

and $I_{(\mu_1, \mu_2, \mu_3)}^{(m)}(z)$ are the multiple contour integrals considered in general in [22]:

$$I_{\mu}^{(m)}(\{z_a\}) = \int_{C_1} \dots \int_{C_m} du_1 \dots du_m \Phi^{(\nu)}(\{u_i, z_a; j_a\}) \varphi_{\mu}(\{u_i, z_a\}), \quad (2.11)$$

$$\Phi^{(\nu)}(\{u_i, \underline{z}_a; j_a\}) = \prod_{a=1}^{n-1} \prod_{i=1}^m (u_i - \underline{z}_a)^{-2j_a \nu} \prod_{i < j} u_{ij}^{2\nu} ; \quad \nu = 1/(k+2) , \quad (2.12a)$$

$$\varphi_\mu(\{u_i, \underline{z}_a\}) = \frac{(m - |\mu|)!}{m!} \sum_{i_1, i_2, \dots, i_s} \prod_{k=1}^s \frac{1}{u_{i_k} - \underline{z}_{a_k}} , \quad (2.12b) ,$$

where the sum is over noncoinciding $i_k \in \{1, \dots, m\}$ and

$$\mu = (\mu_1, \mu_2, \dots, \mu_{n-1}) = \epsilon_{a_1} + \epsilon_{a_2} + \dots + \epsilon_{a_s} ; \quad |\mu| = \sum_{a=1}^{n-1} \mu_a = s. \quad (2.12c)$$

Here $\epsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$ is a $n-1$ - vector with unity at the i 'th place. In (2.6b) we use (2.11,12) for $n = 4$, and $a_k = 2$ for all k so that $\mu = (0, s, 0)$.

The contours $C_1, \dots, C_m$ can be chosen closed, encircling the points $\underline{z}_a$, or deformed to contours connecting them. The dependence on $C_1, \dots, C_m$ becomes important when one is interested in the monodromy (or braiding) properties of the chiral blocks. When $n = 4$ we have $m+1$ independent sets $\{C_i\}_p$, $p = 1, 2, \dots, m+1$ of contours, providing for generic J_a , $m+1$ linearly independent solutions. We shall suppress the dependence on p in the integrals (2.11), assuming that one of the fundamental solutions is chosen.

In the thermal case under consideration the sum in (2.6a) is finite, the bound being given by $m_0 = \min(2j_2, m)$. The first term in (2.6a,b), provided by $I_0^{(m)}(z)$, coincides with the integral entering the corresponding Vir 4-point block [21]. Thus we recover the minimal correlation functions as boundary values of the WZNW ones for $x_a \rightarrow z_a$.

Having related the solution of the KZ equation to the min conformal blocks, it is natural to relate the equation itself to the "null vector decoupling" equation in the min theory. In terms of the integrals I_s the KZ - ZF (Zamolodchikov - Fateev) system (2.7) reads

$$(2j_2 - s)(m - s)I_{s+1} = L_{s+1}I_s + a_{s-1}(z)I_{s-1} , \quad (2.13)$$

where $s = 0, 1, 2, \dots, m_0$, and

$$a_{s-1} = \frac{s c_{s-1}}{z(z-1)} , \quad (2.14a)$$

$$L_{s+1} = [(k+2)\partial_z + a_{ss}] . \quad (2.14b)$$

Using (2.13) one may express recursively I_{s+1} by the lower ones and finally represent it as a differential operator of order $s+1$ acting on I_0 :

$$\frac{(2j_2)!}{(2j_2 - s - 1)!} \frac{m!}{(m - s - 1)!} I_{s+1} =$$

$$= \sum_{k=0}^{\lfloor \frac{s+1}{2} \rfloor} \sum_{\{i_k, \dots, i_1\}} L_{s+1} L_s \dots \underbrace{L_{i_k+1} L_{i_k}}_{L_{i_k-1} \dots L_{i_1+1} L_{i_1}} \dots L_2 L_1 I_0 \quad (2.15a)$$

with

$$\underbrace{L_{i+1} L_i}_{L_{i-1} \dots L_{i_1+1} L_{i_1}} \equiv (2j_2 + 1 - i)(m + 1 - i) a_{i-1}(z) . \quad (2.15b)$$

The second sum in (2.15a) is over all subsets $\{i_k, \dots, i_1\}$ of $\{s, \dots, 2, 1\}$ with $i_{j+1} - i_j \geq 2$ and $\lfloor (s+1)/2 \rfloor$ is the integer part of $(s+1)/2$.

Setting $s = m_0$ in (2.15a) we get a differential equation of Fuchs type for I_0 ,

$$\sum_{t=0}^{m_0+1} \sum_{l+u=t} \frac{F_{lu}^{m_0+1}}{z^l (z-1)^u} \partial_z^{2j+1-t} I_0 = 0 \quad (2.15c)$$

with coefficients $F_{lu}^{m_0+1}$ which can be computed from (2.15a).

The equation (2.15a) could have been recovered in principle starting from the explicit expression for the Vir singular vectors, written in the thermal case in [23].

A similar reduction of the KZ integer level equation has been discussed in [13], where the solution is described by a series, reflecting the isospin tensor product decomposition, the summation running over the spins $j = j_1 + j_2, j_1 + j_2 - 1, \dots, j_1 - j_2$. The counterpart of the min integral $I_0^{(m)}(z)$ in (2.15c) is given by an integral of the type defined in (2.11,12), $I_\mu^{(m)}(z)$, $\mu = (0, 0, m)$, which is the coefficient at the contribution of the spin $j = j_1 + j_2$.

3. Before turning to the general case let us relate the expansion (2.6) to some alternative representations of the solution of the KZ equation in the thermal case.

The 4-point block (2.6) can be written equivalently as a power series in x , namely

$$U(x, z) = (-1)^M \sum_{|\mu| = M} \prod_{a=1}^3 \binom{2J_a}{\mu_a} \underline{x}_a^{\mu_a} I_\mu^{(m)}(z) , \quad (3.1)$$

or,

$$U(x, z) = \sum_{\substack{\{\hat{\mu}_a\} \\ |\hat{\mu}| = 2J_4 + M}} \prod_{a=1}^3 \binom{2J_a}{\hat{\mu}_a} \underline{x}_a^{2J_a - \mu_a} \sum_{|\mu| = M} \prod_{a=1}^3 \binom{\hat{\mu}_a}{\mu_a} I_\mu^{(m)}(z) . \quad (3.2)$$

Since $\underline{x}_1 = 0$, in (3.1) $\mu_1 = 0$, and in (3.2) $\hat{\mu}_1 = 2J_1$.

The representation (3.1) corresponds to the one considered in [14], while its "dual" (3.2) originates from the bosonization technique of [16-19]. The integrals (2.11,12) have

the structure of the minimal integrals with one type of screening charge operators. now they are modified by the factor (2.12b). It accounts for the contribution of the free bosonic pair (γ, β) . In particular the second sum in (3.2) is reproduced using the WZNW Coulomb gas vertex operators (see (4.4) below).

To relate the three representations (2.6a,b), (3.1) and (3.2) one has to exploit two basic identities satisfied by the integrals $I_\mu^{(m)}(z)$ [22],

$$\sum_{a=1}^{n-1} (2J_a - \mu_a) I_{\mu+\epsilon_a}^{(m)}(\{z_a\}) = 0, \quad (3.3)$$

$$\sum_{a=1}^{n-1} (2J_a - \mu_a) z_a I_{\mu+\epsilon_a}^{(m)}(\{z_a\}) + c_{|\mu|} I_\mu^{(m)}(\{z_a\}) = 0. \quad (3.4)$$

Here $c_{|\mu|}$ is the constant in (2.9).

Indeed, iterating (3.3) μ_1 times one expresses $I_\mu^{(m)}$ as a linear combination of integrals with $\mu_1 = 0$. Then the summation over $\mu_1, \mu_1 + \mu_3$ in (3.2) can be taken using standard formulae for the hypergeometric function and one recovers (3.1). Note that the integrals $I_\mu^{(m)}$ in (3.1), (3.2) satisfy the condition $|\mu| = m$, while in (2.6b) $0 \leq |\mu| \leq m$. Iterating $m - |\mu|$ times the second identity (3.4) one can express any integral $I_\mu^{(m)}$ with $0 \leq |\mu| \leq m$ as a z -dependent linear combination of integrals with fixed $|\mu| = m$. This allows to recover (2.6) starting from (3.1) and expanding $x = (x - z) + z$. In particular the minimal integral $I_0^{(m)}$ is represented by the r.h.s of (3.1) when $x = z$ [22].

4. Let us now turn to the general case of arbitrary nonthermal spins $\{J = j - j'(k+2)\}$. To find the counterpart of (3.2) one can exploit the rational WZW Coulomb gas representation with two types of screening charges [5,19,24]. One obtains the same representation as (3.2), only the integral $I_\mu^{(m)}$ is replaced by

$$I_\mu^{(m',m)}(\{z_a\}) = \int_{C_1} \dots \int_{C_{m+m'}} du_1 \dots du_m dv_1 \dots dv_{m'} \\ \Phi^{(\nu)}(\{u_i, z_a, J_a\}) \Phi^{(1/\nu)}(\{v_i, z_a, J'_a\}) \prod_{i,j} \frac{1}{(u_i - v_j)^2} \varphi_\mu(\{u_i, v_j, z_a\}), \quad (4.1)$$

$$\varphi_\mu(\{u_i, v_i, z_a\}) = \frac{(K!)^{m'}}{M!} \sum_{\{\alpha_a + \beta_a = \mu_a\}} \prod_a \frac{\mu_a!}{\alpha_a! \beta_a!} \varphi_\alpha(\{u_i, z_a\}; 1) \varphi_\beta(\{v_i, z_a\}; K), \quad (4.2)$$

Here $K = -(k+2) = -1/\nu$, $J'_a = J_a/K = j'_a + j'_a/K$ and

$$\varphi_\mu(\{u_i, z_a\}, L) = \sum_{\{A_{ia}\}} \frac{\prod_a \mu_a!}{\prod_{a,i} A_{ia}!} \prod_{a,i} \frac{1}{(u_i - z_a)^{A_{ia}}}, \quad (4.3a)$$

where the sum in (4.3a) is over A_{ia} such that

$$\sum_i A_{ia} = \mu_a, \quad \sum_a A_{ia} = L. \quad (4.3b)$$

In (4.2) we have used (4.3) for $L = 1$, or $L = K$, so that $|\alpha| = m$, or $|\beta| = m'K$, and hence $|\mu| = M = m + m'K$.

To clarify the relation of these expressions with the bosonization technique let us write down one more formula. We have

$$M!(2J_4)! \sum_{|\mu|=M} \prod_{a=1}^3 \frac{\hat{\mu}_a!}{(\hat{\mu}_a - \mu_a)! \mu_a!} \varphi_\mu(\{u_i, v_i, z_a\}) = \lim_{z_4 \rightarrow \infty} z_4^{2J_4} \\ < \gamma^{\mu_1}(z_1) \gamma^{\mu_2}(z_2) \gamma^{\mu_3}(z_3) \beta^{2J_4}(z_4) \beta(u_1) \dots \beta(u_m) \beta^K(v_1) \dots \beta^K(v_{m'}) >. \quad (4.4)$$

Here $\gamma(z)$ and $\beta(u)$ are (generalized) free bosonic fields with propagator $< \gamma(z) \beta(u) >$ given by the inverse power of $(u - z)$.

The computation is first done assuming that $\hat{\mu}_a, K$ are integers and then continuing analytically the result. Thus all factorials in (4.2,3) are understood as the corresponding Γ - functions. Furthermore we can choose in the sums in (4.3) as independent variables the nonnegative integers A_{ia} , $a = 1, 2, \dots, n - 2$; $i = 1, 2, \dots, m - 1$; the rest parameters are expressed using the restrictions in (4.3b).

One can show (see [9] for details) that the Aomoto relation (3.3) holds true for $J_a = j_a + j'_a K$, if the thermal integrals are replaced by the integrals $I_{\mu+\epsilon_a}^{(m',m)}$ in (4.1). Hence following the same procedure of analytic continuation we can repeat the resummation done in the thermal case and get a generalization of the DJMM type formula (3.1). On the other hand in order to generalize the expansion (2.6) we have to extend the integrals (4.1), generated by the bosonization, for integer values of μ_a . There is one more reason for such an extension. The noninteger powers of $(u_i - z_a)$ which appear in (4.1,2) in general modify the parameters of the fusion (braid) matrix. This can change drastically the fusion rules of the rational WZW theory if compared with the corresponding minimal one, as already seen in the example considered in [19]. This analysis might be incomplete since we deal with multivalued functions and changing the contours, one can produce compensating factors. The extension with integer powers we are looking for will not produce such ambiguities.

As a first step we shall extend the integrals in (4.1) for $|\mu| \leq M$. We shall start with the case $J_a = j'_a K$, to be referred to as the "quasithermal" case. Since we shall cover actually both the thermal and quasithermal cases we shall omit the primes in m', j' . As a first step we generalize (4.3a,b) by replacing the second constraint in (4.3b) with

$$\sum_a A_{ia} = l_i, \quad (4.5a)$$

so that

$$|l| = \sum_{i=1}^m l_i = |\mu|. \quad (4.5b)$$

Denote by φ^l the resulting expression where l is the m -vector $l = (l_1, l_2, \dots, l_m)$. In this notation (4.3) is recovered for $l = \rho L$, $\rho = (1, 1, \dots, 1)$.

Now we shall choose $l = L\rho - \epsilon_{i_1} \dots - \epsilon_{i_r}$ and define

$$\varphi_\mu^{(L)} = \frac{(L!)^m}{(mL)!} \sum_{\{i_1, \dots, i_r\}} \varphi_\mu^{L\rho - \epsilon_{i_1} \dots - \epsilon_{i_r}}, \quad (4.6)$$

so that $r = mL - |\mu|$.

One can show [9] that the integrals $I_\mu^{(0,m)}(\nu, L)$ defined as in (4.1), but with the factor (4.3) replaced by (4.6), satisfy an Aomoto type identity

$$\sum_{a=1}^{n-1} (2j_a L - \mu_a) I_{\mu + \epsilon_a}^{(0,m)}(\nu, L) = 0 \quad (4.7)$$

iff either $L = 1$, or $L = -\nu$. The first choice recovers the standard Aomoto relation (3.3) satisfied by the standard thermal integrals defined in (2.11, 12) for arbitrary μ . The second choice defines new (thermal) integrals satisfying a new Aomoto- type identity. Choosing furthermore $\nu = -K$ in the second case, i.e., $L = K = -\nu$, we obtain an extension of the quasithermal integrals for $|\mu| \leq M$. Then (4.7) looks exactly as (3.3), covering both the thermal and the quasithermal cases. The integrals so defined satisfy as well the second relation (3.4), in which $M = m$, or $M = m'K$, respectively.

As a next step we perform the summation in (4.6) and then continue the result to noninteger r such that $|\mu| = mL - r$ is an integer. We get finally

$$\varphi_\mu^{(L)}(u_i, z_a) = \frac{(L!)^m (mL - |\mu|)!}{(mL)!} \sum_{\{N_{ia}\}} \frac{\prod_a \mu_a! \delta_{\mu_a, \sum_i N_{ia}}}{\prod_i [(L - \sum_a N_{ia})! \prod_a N_{ia}! (u_i - z_a)^{N_{ia}}]}. \quad (4.8)$$

Since μ_a are integers, the summation in (4.8) runs over $N_{ia} = 0, 1, \dots, \mu_a$, so that, in contrast to (4.2,3), the sums are finite. It is amusing to see that the meromorphic function (4.8) can be also represented as a correlation of the bosonic (γ, β) pair. Namely it is proportional to

$$\lim_{u \rightarrow \infty} u^r < \beta^{\mu_1}(z_1) \dots \beta^{\mu_{n-1}}(z_{n-1}) \beta^r(u) \gamma^L(u_1) \dots \gamma^L(u_m) > \quad (4.9)$$

with $r = mL - |\mu|$. For $r = 0$ (4.9) recovers the factor coming from the Coulomb gas prescriptions in its "dual" form (cf. (3.1,2), (4.4) for $L = 1$).

Using the same technique as needed in order to prove the generalized Aomoto identities, one checks that the quasithermal integrals defined with the factor (4.8) satisfy as well the KZ - ZF system, i.e., we get a solution of the form (2.6a,b). The constant B_s is the same as in (2.10) with $M = m'K$.

The generalization of these results for arbitrary spins $J = j + j'K$ (hence $M = m + m'K$) is almost straightforward, combining the thermal and quasithermal cases. Let us only write down the final formula for the factor φ_μ replacing the expression (4.2)

$$\varphi_\mu(\{u_i, v_j, z_a\}) = \frac{(m'K)!m!}{(m+m'K)!} \sum_{\alpha} \prod_a \binom{\mu_a}{\alpha_a} \binom{M-|\mu|}{m-|\alpha|} \varphi_{\mu-\alpha}^{(K)}(\{v_i, z_a\}) \varphi_{\alpha}^{(1)}(\{u_i, z_a\}) . \quad (4.10)$$

For any μ the sum in (4.10) runs over $\{\alpha_a = 0, 1, \dots, m, |\alpha| \leq m\}$.

The $(x-z)$ - expansion can be recovered alternatively using (via analytic continuation) the iterations of the generalized Aomoto integrals as we did in the thermal case. All these formal resummations involve infinite sums, i.e., the integral defined using the finite factors (4.8) is expressed as an infinite linear combination of the Coulomb gas generated integrals in (4.1,2), which itself are given, according to (4.2,3), by infinite sums.

5. Having the solution (2.6a,b) in general, with $I_\mu^{(m)}$ replaced by $I_\mu^{(m',m)}$, we can take the limit $x \rightarrow z$ as before. Thus we recover the general DF minimal conformal blocks.

On the other hand the system (2.13) is now infinite and it is far from obvious how to reduce it to an equation for $I_0^{(m',m)}(z)$. Let us start again with the quasithermal case where we have spins $\{J = -j'(k+2)\}$ at level k . This case can alternatively be realized as a “true thermal” case, with spins $\{j'\}$, replacing $k+2$ with $1/(k+2)$ ($\nu = k+2$). The corresponding KZ-ZF system is finite. This suggests that there exist additional relations for the quasithermal integrals besides the generalized Aomoto relations, so that the set of these integrals is equivalent to the finite dual one. Indeed comparing the two systems of equations one can find a linear transformation relating the two sets of integrals. It intertwines the quasithermal KZ-ZF system and the dual thermal one and can be viewed as a Drinfeld Sokolov gauge transformation. It is given by a triangular matrix which furthermore degenerates due to the finiteness of the dual thermal system. Its matrix elements are recursively determined in terms of the coefficients (2.8). From this duality one immediately obtains that there is a (z -dependent) linear relation between the quasithermal integrals $I_{(0,s,0)}^{(m')}(z)$, $s = 0, 1, \dots, t$, for any $t \geq m'_0$. Taking $t = m'_0$ this effectively truncates the quasithermal KZ-ZF system to a finite one. The details will be presented in [9].

In the general nonthermal case there are two dual sets of integrals, corresponding to $(k+2, J)$ and $(1/(k+2), J' = -J/(k+2))$. However, since both KZ-ZF systems of equations are infinite, the reduction is not straightforward. One has to find additional relations for the integrals. This is under investigation.

6. We conclude with two remarks.

The way the minimal correlations reappear in our approach makes explicit the interpretation of the quantum hamiltonian reduction in [25]: the isospin degrees of freedom are converted into orbital ones.

The differential equations for the minimal correlation functions account for the factorization of the reducible Vir modules by submodules, generated by singular vectors [8]. A general formula for the Virasoro singular vectors has appeared in a recent paper [26],

which we received when this work was in progress. Our results suggest that may be the formulae in [23] and [26] can be reformulated in a way intrinsically related to the Kac - Moody algebra $A_1^{(1)}$.

Acknowledgements

V.B.P. is acknowledging the hospitality of the Institute for Theoretical Physics of the University of Göttingen as well as the hospitality and the financial support of the National Institute for Nuclear Physics (INFN), Trieste and the Institute for Theoretical Physics of the University of Karlsruhe.

REFERENCES

- [1] A.M. Polyakov, *Mod. Phys Lett. A* **2** (1987) 893.
- [2] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, *Mod. Phys. Lett. A* **3** (1988) 819.
- [3] A.A. Belavin, "KdV type equations and W algebras", (1988), unpublished.
- [4] J.Balog, L. Fehér, L. O'Raifeartaigh, P. Forgács and A. Wipf, *Ann. Phys.* **203** (1990) 76.
- [5] M. Bershadsky and H. Ooguri, *Comm. Math. Phys.* **126** (1989) 49.
- [6] B. Feigin, E. Frenkel, *Phys. Lett. B* **246** (1990) 75.
- [7] V.G. Kac and M. Wakimoto, *Proc. Natl. Acad. Sci. USA* **85** (1988) 4956.
- [8] A. Belavin, A. Polyakov and A. Zamolodchikov, *Nucl. Phys. B* **241** (1984) 333.
- [9] P. Furlan, R. Paunov, A.Ch. Ganchev, V.B. Petkova, in preparation.
- [10] V.G. Knizhnik and A.B. Zamolodchikov, *Nucl. Phys. B* **247** (1984) 83.
- [11] A.B. Zamolodchikov and V.A. Fateev, *Sov. J. Nucl. Phys.* **43** (1986) 657).
- [12] P. Christe and R. Flume, *Nucl. Phys. B* **282** (1987) 466.
- [13] P. Christe, PhD Thesis, Bonn University (1986).
- [14] E. Date, M. Jimbo, A. Matsuo and T. Miwa, in: *Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory*, World Scientific (1989).
- [15] V.V. Schechtman and A.N. Varchenko, Bonn preprint, MPI/89-51 (1989).
- [16] B.L. Feigin, E.V. Frenkel, in V.G. Knizhnik Memorial volume, eds. L. Brink et al, World Scientific 1990.
- [17] D. Bernard and G. Felder, *Comm. Math. Phys.* **127** (1990) 145.
- [18] A.Gerasimov, A.Morozov, M. Olshanetski, A. Marshakov, S. Shatashvili, *Int. J. Mod. Phys. A* **5** (1990) 2495.

- [19] V.S. Dotsenko, Nordita preprint, 89/54 P (1989), (to appear in *Nucl. Phys. B*).
V.S. Dotsenko, Santa Barbara preprint, NSF-ITP-90-148, (1990).
- [20] S. Mukhi and S. Panda, *Nucl. Phys.* **B338** (1990) 263.
- [21] V.S. Dotsenko and V.A. Fateev, *Nucl. Phys.* **B240** (1984) 312; **B251** (1985) 691.
- [22] K. Aomoto, *J. Math. Soc. Japan* **39** (1987) 191.
- [23] L. Benoit and Y. Saint-Aubin, *Phys. Lett.* **215B** (1988) 517.
- [24] A.B. Zamolodchikov, talk given at Montreal (1988), unpublished.
- [25] A.M. Polyakov, *Int. J. Mod. Phys. A* **5** (1990) 833.
- [26] M. Bauer, Ph. Di Francesco, C. Itzykson and J.-B. Zuber, Saclay preprint, SPhT/91-020 (1991).