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ANYONS

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Abstract

An elementary introduction is given to the concept of anyon-quantum particles whose "statistics" interpolates smoothly between that of bosons and fermions. Such particles - or quasi-particle excitations - can occur in two-dimensional systems in which the particles experience strong short range mutual repulsions. These circumstances effectively mean that one has to apply quantum mechanics to a configuration space in which some points are excluded (i.e. it has "holes") - namely, those points where two particles would coincide. The Anharinov-Bohm effect provides the simplest physical example of quantum mechanics in such a non-simply connected space, and is also fundamental to a simple model for anyons. A quantitative introduction is given to two currently proposed physical applications of anyons: high temperature superconductivity, and the (fractional) quantum Hall effect.

1. Introduction

At first sight it is hard to see what might be wrong with the following argument⁽¹⁾:

"Suppose we have to multiply the amplitudes by a certain phase factor [say $e^{i\delta}$] when we exchange the roles of the two [identical] particles. If we exchange them again we should get the same factor again. But then we are back to the first process. The phase factor taken twice must bring us back where we started - its square must be equal to 1. There are only two possibilities: $e^{i\delta}$ is equal to +1 or is equal to -1. Either the exchanged case contributes with the same sign, or it contributes with the opposite sign. Both cases exist in nature, each for a different class of particles. Particles which interfere with a positive sign are called Bose Particles and those which interfere with a negative sign are called Fermi particles."

Feynman's argument (and similar ones in many text books) seems to show that quantum-mechanical amplitudes - or wavefunctions - for two identical particles must necessarily be either even or odd under exchange of the "roles" of the two particles, and that these two possibilities are satisfactorily exhausted by the familiar kinds of identical particles actually found in nature, namely bosons and fermions. Nevertheless, it was pointed out some time ago ⁽²⁾⁽³⁾ (though not followed up at the time) that there are conceivable physical circumstances in which what is asserted above is not necessarily true - most interestingly, in the case of identical particles constrained to move in only two spatial dimensions, and with the proviso that no two particles may occupy the same position in space (ie they are "impenetrable", or in perhaps more physical terms, there are strong short range repulsive forces between the particles). In this case, far from there being only two possible ways for a quantum amplitude to behave when the roles of two identical particles are changed - viz. either to remain the same ($e^{i\delta} = +1$, "even") or to change sign ($e^{i\delta} = -1$, "odd") - there are actually infinitely many possibilities. The amplitude may change by a phase factor $e^{i\delta}$ which can be anything - hence the name

anyons⁽⁴⁾ to describe generically such (two-dimensional, non-penetrating) identical particles. This is interesting enough in itself, both because it is a new concept in physics, and because the reasoning behind it, which we shall outline in the course of Sections 2 and 3, helps us to gain a deeper understanding of why, after all, anyons can not occur in three dimensions. However, the subject has become less academic in recent years, with the observation⁽⁵⁾ (admittedly, somewhat indirect) of certain types of anyon in the fractional quantum Hall effect (FQHE)⁽⁶⁾⁽⁷⁾; and as a consequence of the suggestion ⁽⁸⁾⁽⁹⁾ that the "high-temperature" superconductivity observed in, for example, the Cu-O compounds (which have a markedly planar structure) might be due to a novel kind of mechanism involving bound systems of specific types of anyons.

The purpose of this article, then, is to provide an elementary introduction to the idea of anyons, and to those (probable and possible) applications of it already mentioned. Nothing in the article is believed to be original; nor is it intended as anything like a full technical review, so that the references are very far from complete⁽¹⁰⁾.

2. Quantum mechanics and "sums over paths"

In order to understand where there might be a loophole in Feynman's argument for the (apparent) necessity of having only bosons and fermions, we have to go back to some of the first principles of quantum mechanics. As is well known, there are various equivalent formulations of quantum mechanics, but for the present purpose by far the most suitable is the one particularly developed by Feynman himself, often called the "path-integral", or "sum-over-paths", approach. We shall only need a few simple ideas, as expounded so clearly in the Feynman lectures ⁽¹⁾. First, consider the quantum mechanics of a single particle - in particular, the description of the famous two-slit interference experiment for particles such as electrons (Figure 1). The probability that a particle, starting from some known source point S, will be detected at any "downstream" point P with coordinate x is given by $|\psi(x)|^2$, where the x -dependent complex number $\psi(x)$ is called the "probability amplitude" (or wavefunction). When the particle can reach

the point x by two possible routes, or paths, with amplitudes $\psi_1(x)$ and $\psi_2(x)$ respectively, the amplitude for finding it at x is given by the sum $\psi_1(x) + \psi_2(x)$ of the amplitudes for the separate paths. In the two-slit case, for example, $\psi_1(x)$ is the amplitude for the particle to reach x via slit 1 (S_1), and $\psi_2(x)$ is the amplitude for it to reach x via slit 2 (S_2). This is the simplest example of a "sum over paths". The crucial point here, of course, is that the probability of finding the particle at x is now $|\psi_1(x) + \psi_2(x)|^2$, which is not equal to the sum of the separate probabilities $|\psi_1|^2$ and $|\psi_2|^2$; rather, characteristic interference effects in $|\psi_1 + \psi_2|^2$ can arise due to phase differences between the two complex amplitudes ψ_1 and ψ_2 . Note also, for future reference, that if we had "kept tabs on" the electrons so as to detect which slit each one passed through, the correct procedure would have been to add the probabilities, so as to obtain $|\psi_1|^2 + |\psi_2|^2$. Amplitudes are added for all paths that are not operationally distinguishable, given the announced experimental set up (eg "counters by the slits" or "no counters by the slits").

The mathematical forms of the ψ amplitudes are determined by the relevant dynamical equations (eg Schrodinger's equation): in the simple two-slit case, the ψ 's are essentially plane waves, and the phase difference between ψ_1 and ψ_2 at the screen point P arises - as in the similar calculation in optics - from the difference in path lengths SS_1P and SS_2P .

A remarkable variation on this two-slit experiment was proposed many years ago by Aharonov and Bohm⁽¹¹⁾; although seemingly a diversion at this point, their (now famous) "effect" is conceptually crucial for much of what follows. Imagine, they said, doing the two-slit experiment for electrons, but with the addition of a long thin solenoid just downstream of the two slits (figure 2). The solenoid produces a vertical tube of magnetic flux which is essentially localised to the core of the solenoid, from which the electrons are totally excluded by some form of shielding. Thus the electrons pass through field-free regions of space as in the unmodified experiment. Nevertheless, the interference pattern is laterally shifted by an amount proportional to the flux produced by the solenoid! This

is the A-B effect. It was first observed by Chambers⁽¹²⁾; a recent definitive experiment is described by Tonomura et al⁽¹³⁾.

In order to understand the A-B effect, we need to add one more quantum-mechanical rule to the two introduced so far, describing how quantum amplitudes for charged particles are modified by the introduction of magnetic fields. This rule (which follows from the Schrodinger equation) is most simply stated in terms of the magnetic vector potential $\underline{A}$, which is related to the magnetic field by $\underline{B} = \nabla \times \underline{A}$. The rule states that the amplitude for a particle of charge q travel along a certain path through a region of non-zero $\underline{A}$ field has the same modulus as the amplitude for the $\underline{A} = 0$ case, but its phase is changed by an amount δ_A where

$$\delta_A = \frac{q}{\hbar} \int_{\text{path}} \underline{A} \cdot d\underline{s} \quad (1)$$

the integral being along the particular path considered. Now it happens that, in the case of the solenoidal magnetic field considered in the A-B effect, the corresponding $\underline{A}$ is non-zero in the region where the electrons are free to move, even though $\underline{B}$ is zero there. Thus we see that a shift in the interference pattern may occur, since on paths such as SS_1P and SS_2P in Figure 2 the respective amplitudes will acquire different δ_A 's according to (1).

In fact, we can easily calculate the extra phase difference $\delta_{A_1} - \delta_{A_2}$ between these two paths (ie the difference additional to the previous "optical" one) which is the net observable effect of the non-zero $\underline{A}$ field. We have

$$\delta_{A_1} - \delta_{A_2} = \frac{q}{\hbar} \int_{SS_1P} \underline{A} \cdot d\underline{s} - \frac{q}{\hbar} \int_{SS_2P} \underline{A} \cdot d\underline{s} \quad (2)$$

However, the line integral in the sense " $S \rightarrow S_2 \rightarrow P$ " can always be written as minus the line integral in the opposite sense, ie " $P \rightarrow S_2 \rightarrow S$ ". Thus

$$\delta A_1^- \delta A_2 = \frac{q}{\hbar} \int_{SS_1P} \underline{A} \cdot d\underline{s} + \frac{q}{\hbar} \int_{PS_2S} \underline{A} \cdot d\underline{s} \quad (3)$$

In this form the two integrals in (3) can be immediately added together to give

$$\delta A_1^- \delta A_2 = \frac{q}{\hbar} \int_{\text{closed path } SS_1PS_2S} \underline{A} \cdot d\underline{s} \quad (4)$$

Stokes' theorem in vector calculus now enables us to convert the integral over a closed path into a surface integral of $\nabla \times \underline{A}$ over any surface bounding the closed loop. Since $\nabla \times \underline{A}$ is just the magnetic field $\underline{B}$, whose integral over such a surface gives the total flux enclosed, Φ , we finally get the result

$$\delta A_1^- \delta A_2 = \frac{q\Phi}{\hbar} \quad (5)$$

a fundamental relation. We note, of course, that all mention of the potential $\underline{A}$ has disappeared from the observable quantity (5), which is expressed in terms of the physical field (or flux) Φ . Nevertheless, the paths of the charged particles pass only through $\Phi=0$ regions, and so if we were to use only a " Φ -description" in the intermediate stages of the discussion we would be forced to contemplate something like an action-at-distance effect. The potential $\underline{A}$ allows a local description.

But what has all this to do with anyons? Well, as an appetizer, consider the following thought experiment. Two electrons are constrained to move into a plane, which is pierced at a certain point by an A-B type flux tube. Suppose electron 1 is initially (at $t=t_i$) at some known point $\underline{r}_1$, and electron 2 is at some known point $\underline{r}_2$, while at some later time $t=t_f$ they are detected at points $\underline{r}_1'$ and $\underline{r}_2'$. How would we form the probability amplitude for this process? Certainly there is some amplitude which we might write as⁽¹⁴⁾ $\psi(\underline{r}_1, \underline{r}_2, t_i \rightarrow \underline{r}_1', \underline{r}_2', t_f)$ for the process in which electron 1 at point $\underline{r}_1$ at time t_i ends up at point $\underline{r}_1'$ at time t_f , while electron 2 goes from $\underline{r}_2$ at t_i to $\underline{r}_2'$ at t_f as shown in figure 3, where we suppressed the flux tube, however. But wait a minute - these electrons are "identical!" How do we know that it is "number 1" which arrived at $\underline{r}_1'$ rather than "number 2", and vice versa? The answer, of course, is that if we do not "keep tabs" on the electrons all the time between t_i and t_f , we do not know which has

arrived where. The possibilities " $r_1 \rightarrow r_1', r_2 \rightarrow r_2'$ " and " $r_1 \rightarrow r_2', r_2 \rightarrow r_1'$ " cannot be operationally distinguished, and the amplitudes for each possibility must in some way be added together, just as in the case of the two possible paths in the electron two-slit experiment, when we do not modify the set up to include detectors at the slits. There are, however, two complications in the present case. The first arises if there is no flux through the solenoid. For, of course, our electrons are fermions, and we know that any quantum mechanical amplitude involving two electrons has to be odd under exchange of their "roles" - which, here⁽¹⁴⁾, means their arrival points i.e. the exchange $r_2' \leftrightarrow r_1'$. Thus in the $\Phi=0$ case the correct prescription for electrons is that the amplitude for the "exchange" process (shown in Fig. 3b) has to be subtracted from that for the "direct" process (Fig. 3a), giving the total amplitude

$$\psi(r_1, r_2, t_i \rightarrow r_1', r_2', t_f) - \psi(r_1, r_2, t_i \rightarrow r_2', r_1', t_f)$$

which is indeed odd under $r_1' \leftrightarrow r_2'$. As Feynman put it in our original quotation, they "interfere with a negative sign".

The second complication arises if $\Phi \neq 0$, and is rather unsettling when first encountered. To understand it, we begin by rephrasing the prescription of the previous paragraph. Consider the amplitude $\psi(r_1, r_2, t_i \rightarrow r_1', r_2', t_f)$, the time-slice $t=t_f$ being shown again in Figure 4, but with the addition of the flux tube. Now imagine generating an amplitude $\psi(r_1, r_2, t_i \rightarrow r_2', r_1', t_f)$ by physically exchanging the positions r_1' and r_2' of the two particles, following the particular two paths shown, which "embrace" the flux tube. Particle 1 ends up at r_2' , and particle 2 at r_1' , and so their "roles" have been exchanged. In order to satisfy the "negative interference for fermions" rule in the case $\Phi=0$, the exchange amplitude $\psi(r_1, r_2, t_i \rightarrow r_2', r_1', t_f)$ must be judged to have acquired a minus sign in this process, before it can be added to (interfere with) the direct amplitude $\psi(r_1, r_2, t_i \rightarrow r_1', r_2', t_f)$. In Feynman's language, the amplitude has to be multiplied by the phase factor $e^{i\delta}$ with $\delta = \pi$, and we can say that the "interchange phase" is π . But now comes the point: for $\Phi \neq 0$, there is an additional interchange

phase due to the A-B effect! Electron 1 acquires a phase

$$\frac{e}{\hbar} \int_{C_1} \underline{A} \cdot d\underline{s} \quad \text{and electron 2 a phase} \quad \frac{e}{\hbar} \int_{C_2} \underline{A} \cdot d\underline{s}$$

so that the overall A-B phase is just the sum of these two: since the electrons are identical, it is of no significance that we called "1" the electron travelling along C_1 , and "2" the one along C_2 . The total A-B phase is therefore

$$\frac{e}{\hbar} \int_{C_1+C_2} \underline{A} \cdot d\underline{s}$$

which is, by Stokes' theorem, just $e \Phi/\hbar$. So the interchange phase is now not just the fermionic value π , but also includes an A-B piece involving Φ .

Actually, it is worse than that. For suppose we interchanged the particles in the "opposite sense", as indicated in Figure 5: then the sense of the line integrals is reversed, and consequently so is the sign of the A-B interchange phase. So the A-B interchange phase even depends on how the interchange is done! Furthermore, we could exchange them twice, and we do not get back to where we started from. Instead, the acquired A-B phase is just twice what it is on one interchange - and so on. Following the rule "add amplitudes for all paths which are indistinguishable for the announced experimental set-up" we should then presumably have to add the amplitudes for all these various ways that particles might (without us watching) have got interchanged, each with the appropriate Φ -dependent phase factor; indeed, this is the case ⁽¹⁵⁾. Thus, in a real sense, the presence of the flux tube has altered the way in which the complete amplitude behaves under exchange of the two particles.

Perhaps by now, the reader may have a good idea as to where that loophole in the argument we started with may lie, and how "anyons" may be possible. Before moving on to that, however, there is one important piece of good news to be derived from the foregoing thought experiment. This is that the details of the actual paths followed by the interchanging particles are not important for the Φ -dependent (and path dependent) phase:

what matters is merely whether the "total" path (as in the $C_1 + C_2$ case above) does or does not encircle the flux, how many times it does so, and in what sense. This follows again from Stokes' theorem: the enclosed flux is always just Φ , whatever the shape of the loop encircling the solenoid may be; and the sense has to be specified. For any path that encircles it twice (in the same sense), the flux threaded is 2Φ , and so on. Thus one can say that this interchange phase is a kind of topological effect, depending only upon the "degree of linkage" between a given encircling path and the flux; in a self-explanatory phrase, this is called the "winding number" of that path. We do not have to worry about separate phases for all possible interchange paths: we can lump paths into topologically distinct classes, such that each class contains all paths of a given winding number i.e. all paths which can be smoothly distorted into each other without crossing the flux tube; all paths of a given class, in this sense, have the same Φ -dependent interchange phase in our thought experiment. (Paths which do not encircle the flux tube at all of course acquire only the usual fermionic phase factor, since for them $\Phi=0$).

Reflecting on the foregoing, we see that the crucial feature of the A-B flux tube is that it has altered the topology of the configuration space of the particles. By assumption, the particles are not allowed to penetrate into the region $\Phi \neq 0$; thus the point or points where $\Phi \neq 0$ are effectively removed from the space. This means that the space is no longer "simply connected": if it were, all closed loops could be smoothly shrunk to a trivial point, but as we have seen this is precisely what is not possible for paths enclosing non-zero Φ . The sum-over paths approach is particularly well suited to discussing the generalization of quantum mechanics to non-simply connected space. It is also appropriate to note, in concluding this section, that in three dimensions the configuration space for two identical free particles is doubly connected ⁽³⁾, and this is the fundamental reason why in that case only bosons and fermions are possible⁽³⁾.

3. Anyons

The main message from the foregoing is that interchange phases can be path-dependent, but they do not depend on the details of the interchange path, only on its topological class, or winding number. However, in the case discussed, the complications of course arose from considering not free particles, but particles moving in the A-field due to a solenoid. With all this in mind, let us now consider quantum mechanics for two free identical impenetrable particles in two dimensions.

It should by now be amply clear that the crucial question is this: with what relative phase should amplitudes for two such particles, which differ by one or more particle exchanges, be "added"? Or in other words, what is the appropriate interchange phase? To answer this, we consider a thought experiment exactly the same as the one we have just been discussing, but without the flux tube Φ ! (Since, after all, we want to know about the behaviour of free particles, at least in the first instance). As before, we begin with the process $\mathbf{r}_1, \mathbf{r}_2, t_i \rightarrow \mathbf{r}_1', \mathbf{r}_2', t_f$ which has amplitude $\psi(\mathbf{r}_1, \mathbf{r}_2, t_i \rightarrow \mathbf{r}_1', \mathbf{r}_2', t_f)$. Consider exchanging the particles along the paths C_1 and C_2 as indicated in Figure 6a: nothing untoward so far. But now examine what happens on a second interchange as shown in Figure 6b (recall "If we exchange then again we should get the same factor again. But then we are back to the first process"). The interchange paths followed by each particle are now closed loops. If, and only if, these loops can be smoothly distorted to points (i.e. no loop at all), can we really say that "we are back to the first process". But if the particles are constrained to be in the two-dimensional plane, the two loops cannot both be shrunk to points without intersecting each other (at least twice); and if the particles are impenetrable, such intersections - which would allow the possibility of simultaneous occupancy of the same point by the two particles - are forbidden. The particles cannot "pass through" each other, and so this kind of smooth distortion of the two loops to points can't be done. We have to conclude that the amplitude corresponding to two interchanges belongs to a different topological class from the original amplitude we started from - and we then have no reason at all to say that the two

amplitudes are the same. However, it is surely true that they can only differ by a phase, since the physical probability (the modulus squared of the amplitude) should certainly be the same. In other words, we have here a situation such that if we call the phase factor for one interchange $e^{i\delta}$, then $(e^{i\delta})^2 \neq 1$, and so $e^{i\delta} \neq \pm 1$; such particles are then neither fermions nor bosons, but anyons, since δ can be anything. Here is the advertised loophole, and the essential reason why anyons are conceivable. However, there are further useful observations to be made.

First, it is obvious that if the restriction to two dimensions is relaxed, the loops we have just been considering can be smoothly deformed to points, simply by lifting one of them slightly out of the plane, say, at the "intersection points": the particles would then not have to collide, and "non-colliding double-exchange" paths, which could be smoothly shrunk to points, would exist naturally. In this case we would indeed have $(e^{i\delta})^2 = 1$, $e^{i\delta} = \pm 1$. This immediately shows us that the reduced dimensionality is crucial. Actually it can be shown quite generally that in three dimensions double exchanges are topologically equivalent to no exchange at all⁽³⁾, and hence (for two particles) only the two possibilities $e^{i\delta} = \pm 1$ exist (c.f. the remark at the end of Section 2.)

Secondly, note that this would certainly be true in two dimensions also, were it not for the "no-intersection" condition. As we have seen, this prevented us from smoothly deforming the two loops to points. Such a condition is called a "topological obstruction". In the A-B effect, the same role is played by the flux tube, whose existence defined classes of path according to how many times they encircled the flux - the essential point being that paths could not be pulled through the flux (since the electrons were excluded from that region). Indeed, if we think of the tube as being a line, in the "very thin" limit, and of the particles in the A-B effect as being restricted to move in a plane which the flux line punctures at just one point, this point constitutes precisely a topological obstruction: it is "removed" from the plane, and ceases to be an allowed point for a path to pass through. This is very similar to the no-intersection rule for anyons.

In fact the analogy can be made very much closer. We have been considering amplitudes of the form $\psi(\underline{r}_1, \underline{r}_2, t_i \rightarrow \underline{r}_2', \underline{r}_1', t_f)$ which describe the propagation of two (identical) particles between times t_i and t_f . In quantum mechanics, as in classical, the coordinate variable that really matters is the relative position vector of the two particles, since the centre of mass moves uniformly and we can choose a frame in which it is at rest, so that the position vector of the centre of mass disappears from the problem. This we need really only consider amplitudes like $\psi(\underline{r}, t_i \rightarrow \underline{r}', t_f)$ where $\underline{r} = \underline{r}_1 - \underline{r}_2$, $\underline{r}' = \underline{r}_1' - \underline{r}_2'$. In this notation, an exchange process such as the one shown in Figure 6a can be represented in the $\underline{r}'$ plane as some path C leading from the initial value $\underline{r}'$ and ending at the point $-\underline{r}'$ (since $\underline{r}_1'$ and $\underline{r}_2'$ have been exchanged), as shown in Figure 7. However, since the particles are identical, these two relative coordinate values, $\underline{r}'$ and $-\underline{r}'$, must be identified as being really the same point, since the physical situation each represents is the same. Thus the exchange path C is actually a closed path! Furthermore, there is an excluded point in the $\underline{r}'$ plane through which such paths may not pass - namely $\underline{r}' = \underline{0}$, where the particle positions coincide. So $\underline{r}' = \underline{0}$ is a topological obstruction, just like the point at which the flux line punctured the plane, and it divides exchange paths - which are loops in this "identified" $\underline{r}'$ -space - into distinct topological classes according to the number of times, and the sense in which, they wind around it.

There is an amusingly concrete way to construct the $\underline{r}'$ plane with points $\underline{r}'$ and $-\underline{r}'$ identified⁽³⁾. Draw Figure 7 on an overhead transparency sheet, and then cut it along the dotted line with scissors. Next, fold the sheet around, twice, so as to form the 30° cone shown in Figure 8. The marked points $\underline{r}'$ and $-\underline{r}'$ will now fall one under the other, and the interchange path C will be a closed loop encircling the excluded point $\underline{r}' = \underline{0}$, which is the vertex of the cone.

With this modest apparatus, a number of things become very clear. First, the double exchange previously considered corresponds to two loops around the vertex - and clearly cannot be "undone" without passing twice through the illegal point $\underline{r}' = \underline{0}$. Next, suppose

we consider the "reflected" interchange process shown in Figure 9. This translates into a loop on the cone executed in the other sense - which cannot be smoothly deformed into the originally chosen interchange loop. Thus if such funny particles exist in nature, one expects to see evidence of the violation of such reflection symmetry (which is the 2-D version of "parity"). Also, one can imagine reversing the sense of time on the first interchange process, and this too results in an inequivalent loop (because the sense is reversed); so one expects time-reversal symmetry to be violated. These are simple but phenomenologically important consequences of the very idea of "anyons".

In the preceeding discussion we have emphasized the importance of the "impenetrability" of the particles, as leading to the exclusion of the point $\underline{r}' = \underline{0}$. In practice, however, one should probably not take this quite so literally. Consider, for instance, the case of superfluid helium. The helium atoms behave just like bosons, provided that the kinematic conditions are such that the atoms do not approach too close to each other; if they are forced into close proximity, the fermionic nature of their constituents will have to be considered. Indeed, these very constituents lead to strong short range repulsions between the atoms, so that at all energies relevant to low temperature physics, it is an excellent approximation to regard the atoms as effectively impenetrable. The fermionic substructure plays no role in the analysis of superfluidity, in which the atoms are treated as bosons with a strong short range repulsion. Thus, by analogy, one might well imagine that for reasonable interparticle separations it might be a good approximation to treat certain (composite?) quantum-mechanical systems as "anyonic", while at shorter distances their "real" nature might reassert itself, though without detriment to the physics associated with their anyonic character at longer distances.

Thus far we have considered only systems of two identical particles. Yet we know very well that the properties of many identical quantum particles, in the boson and fermion cases, are quite extraordinarily affected by the behaviour of their amplitudes under particle exchanges - which in this context is usually called the "statistics" of the particles. One thinks of the superfluidity of bosons, as just discussed, and of the exclusion principle

for fermions as manifested in the periodic system. What are the corresponding phenomena for, say, a gas of anyons? The general answer to this is not known, for essentially the following reason. Let us include just one more anyon, making now three in all, and consider the interchange process of Figure 10a, with time going upwards from t_i to t_f . Particles 1 and 2 have certainly been exchanged. But the trajectory of particle 3, though quietly minding its own business, has actually been "looped" by the trajectories of both particles 1 and 2 (in this example). Thus if these anyons have an interchange phase factor $e^{i\delta}$, the result of the apparently pairwise exchange in Figure 10a is not $e^{i\delta}$ but $e^{3i\delta}$. On the other hand, if the trajectory of particle 3 had missed that of particles 1 and 2, as in Figure 10b, the answer would have been the standard $e^{i\delta}$. Thus in general the actual factor for one pair will depend on the motions of all the other particles as well. This complication will not arise in three dimensions because, as we have seen, in that case $e^{i\delta} = \pm 1$ only, so that $e^{3i\delta} = e^{i\delta}$; for standard Bose and Fermi "statistics" in three dimensions all that matters is the behaviour of amplitudes (or states) under independent pairwise exchanges.

There is, however, one more simple (but at first sight surprising) thing to be said about the behaviour of several anyons, which concerns the statistics of bound systems of anyons. For bosons and fermions it is easy to deduce the behaviour of bound systems - any number of bosons make a boson, an even number of fermions make a boson (as in the α -particle nucleus, unless we probe inside it), while an odd number of fermions make a fermion (like the ${}^3\text{He}$ nucleus). All this follows from the elementary fact that $(-1)^n = +1$ if n is even and -1 if n is odd. But consider a bound state of two anyons whose interchange phase δ equals $\pi/2$. Such anyons might be thought of as "half-fermions" (whose corresponding phase is π), and have been christened "semions". Consider interchanging two such semionic bound states (Figure 11). Each particle must see each other particle looping around it, and so the total interchange phase is four times δ , which is 2π : thus a bound state of two half-fermions behaves like a boson when exchanged with another such bound state! Similarly, if $\delta = 2\pi/3$, we could form a bosonic system out of

three of them ($9 \times 2\pi/3 = 6\pi$). As we shall remark again in Section 5, these considerations may have some relevance to "anyonic superconductivity": after all, a key point in the BCS theory of conventional superconductors is the "Cooper Pairing Mechanism", whereby pairs of electrons are bound together to form bosons, so that a very special state (the "condensate") can be created which takes full advantage of the bose (symmetrical) statistics. If two semions do bind naturally, we would have the possibility of a similar phenomenon.

4. A model for anyons

We have seen that the existence of anyons is a logical possibility, but can we come up with any conceivable physical system, even an idealised one, which behaves like an anyon? The reader may already have guessed the answer which was provided by Wilczek in his early pioneering papers on this subject⁽⁴⁾⁽¹⁶⁾. All we need is to create a situation in which the effect of one particle circling round another produces a phase factor $e^{i\delta}$, say. But this is exactly like what happens when a charged particle moves around a flux line! If the charge is q and the flux carried is Φ , one complete circuit of the particle round the line produces an A-B phase of $q\Phi/\hbar$ (see Eq. (5)). Here, one of the "particles" is the point where the flux line punctures the plane, and the other is the charged particle. But this is obviously not symmetrical - what we need is to attach both a charge and a flux line to the coordinate of each particle. So Wilczek's model of an anyon is that it can be thought of as a particle carrying both charge and flux - i.e. as a charge-flux composite, such that $q\Phi/\hbar = \delta$. (Note that since each charge moves in the field of each flux tube, the phase acquired is twice the value given by Eq. (1); but it is only necessary to do half a complete 2π rotation in the relative coordinate - see Fig. 7 - in order to interchange the two particles).

Note that since we only want the A-B phase effect, and no other consequence of electrodynamics, we have to imagine $q \rightarrow 0$ with $q\Phi$ fixed. Indeed, it is worth stressing that we need not - and to avoid confusion usually should not - think of the "charge"

and "flux" as being real electromagnetic attributes. They are simply to be imagined as pertaining to some vector potential field $\underline{a}$, whose "magnetic field" is $\underline{b} = \nabla \times \underline{a}$, where $\underline{a}$ and $\underline{b}$ behave like their electromagnetic analogues $\underline{A}$ and $\underline{B}$. These fields (introduced purely to provide a working model of the anyonic interchange phase) are often called - a bit misleadingly - "fictitious" gauge fields; the interchange phase is of course not at all fictitious.

The charge-flux composite model immediately yields a beautiful dividend. Consider⁽⁴⁾ a single such composite but imagine the charge slightly displaced from the flux line, for clarity (Figure 12), and with the flux initially zero. Now slowly increase the flux by turning on the "current" in the thin solenoid. By Faraday's law, the charged particle will feel a tangential electric field tending to make it circle the flux line, so it will acquire angular momentum L_z about the line, which we take to be along the z-axis. It is an elementary exercise to show⁽¹⁶⁾⁽¹⁷⁾ that when the final value Φ is reached, the total angular momentum of the particle is $q\Phi/2\pi$. But this is precisely $\delta\hbar/2\pi$, where δ is the interchange phase for two such composites. Thus if the interchange phase is π (Fermionic) the "stored" angular momentum in one such composite is $\frac{1}{2}\hbar$, just as would be required for a fermion; likewise if $\delta = 0, 2\pi, \dots$ as for a boson, the angular momentum is (integer) $\times \hbar$. This is a working model of the famous "spin-statistics" connection - the spin of a single particle and the interchange phase of two identical ones are uniquely interrelated. This relation is generalized to the case of arbitrary δ via the model identifications $\delta = q\Phi/\hbar, L_z = q\Phi/2\pi = \frac{\hbar\delta}{2\pi}$.

The foregoing may alarm some readers familiar with the quantum-mechanical theory of angular momentum. It is a well-known result that the possible values of any one component of angular momentum are quantized in units of (integer) $\times\hbar$ or (integer + $\frac{1}{2}$) $\times\hbar$ - corresponding, of course, to bosons and fermions, respectively. Yet " $q\Phi$ " has not been in any way constrained, and our L_z need not have the familiar allowed values. Once again, the short answer to the difficulty is that it is the reduced dimensionality that makes the novel result possible. The crucial ingredient in the standard result is the form of the commutation relations between the three angular momentum operators $\hat{L}_x, \hat{L}_y$ and $\hat{L}_z$.

But of course in our two-dimensional anyonic world, there is only $\hat{L}_z$! Nevertheless, this point has excited considerable controversy⁽¹⁸⁾ - especially as, after all, the flux line in the model seemed precisely to go, in some sense at least, into the third dimension⁽¹⁹⁾....

Apart from providing a tangible realization of the abstract concept of an anyon, the charge-flux composite model has been, and is, enormously fruitful in suggesting more physical ways in which the seemingly hopeless complexity of the many-anyon-problems can be tackled - as we now indicate.

5. The mean field theory of anyons, and anyonic superconductivity

Using the charge-flux composite model, a gas of N anyons in an area A can be thought of as a system of charged particles, each one "transfixed" by a flux line as in Figure 13. We must now note, however, that in presenting this picture we tacitly assumed that the only interchange phase was that due to the "fictitious" A-B effect i.e. we assumed the particles carrying the charge and flux were actually bosons. Attaching charge and flux to bosons in this way converts them into anyons. But there is no need to assume this - we could just as well assume the "core" particles are fermions⁽²⁰⁾. In this case the interchange phase δ would be $\pi + q\Phi/\hbar$, but the product $q\Phi$, which is arbitrary anyway, can still be adjusted to give any desired δ . The advantage of this move will be apparent in what follows, but we can guess why it is likely to be useful. The fermionic phase factor π means that, before we add the flux, amplitudes are antisymmetric under interchange of any pair, and this implies that they must vanish when the relative separation of any pair is zero - so the requirement that no two particles can occupy the same place is already built in.

So consider again Figure 13, where we think of the points as charged fermions, bearing flux lines as shown. If the density of anyons $\rho = N/A$ is large, we might perhaps imagine that the effect on any single anyon of the flux lines on all the others could be, at least roughly, represented by the total flux smoothed out over the given area.

The N individual flux lines would then be lumped into a mean flux or, in other words, a particular anyon would feel an effective mean ("fictitious") magnetic field b . It should not pass un-noticed that this is a non-trivial step: it was pointed out earlier that the sole purpose of the localised flux lines, each of which intersects the plane at a point, was to give the A-B phase effect, but no electrodynamic effects in the regions between lines; yet once we have introduced the smoothed out field b , Lorentz-type forces should presumably come into play. Actually, numerical simulations for a few (~ 8) anyons seem to show⁽²¹⁾ that the mean field approximation is not at all bad. So let us pass on.

What is the value of the mean field b ? At first sight it would seem, obviously, to be given by $bA = N\Phi$. However, there is a subtlety. Recall that the crucial parameter is the quantity $\delta = q\Phi/\hbar$, where $\delta = \pi + \delta'$, and note the comment in parentheses at the end of the first paragraph of Section 4. The mean field approximation turns a many-body problem into many one-body problems. To obtain the correct "interchange phase", which now has to be generated by moving one anyon round a half-circuit about any chosen point, we shall need to double the effective flux since we have lost the contribution from the "partner" anyon circling the same point in the opposite direction. That is, our effective mean field is given by $b = 2\rho\Phi$.

The problem is now reduced to that of N charged fermions moving in two dimensions under the influence of a uniform transverse "magnetic" field b - a problem which has a very well known exact solution. Classically, the particles would move in closed circular orbits, the inward acceleration being provided by the Lorentz force via $mv^2/r = qvb$, for orbit speed v and radius r . Hence the ("cyclotron") radius is $r_c = mv/qb$, and the orbiting angular velocity $\omega (= v/r_c)$ is just the "cyclotron frequency" $\omega_c = qb/m$. Now we know that circular motion is closely related to simple harmonic motion. This is the feature that carries over to the quantum treatment. In quantum mechanics, the allowed energy values of a particle in a harmonic oscillator potential are quantised, and have the values $(n + \frac{1}{2})\hbar\omega$, where ω is the oscillator frequency. In the present case, the allowed energy values for our particles in the field b are just $(n + \frac{1}{2})\hbar\omega_c$.

So we must start filling these energy levels (which are called Landau levels) with our particles, remembering that we are choosing them to be (spin-polarised) fermions, and hence that no two of them can be in the same spatial state. The vital question is therefore: how many states are there in each energy level? i.e. how many particles will each level hold? The answer to this is simple to obtain, but we shall just quote it here: the number of available states per Landau level per unit area A is equal to $qb/2\pi\hbar$. But this can be rewritten, quite remarkably, as

$$q b/2\pi\hbar = q 2\rho\Phi/2\pi\hbar = \rho(\delta'/\pi) \quad (6)$$

Hence the number of available states per level per unit area is directly determined by the anyonic phase parameter δ' .

The result (6) has dramatic consequences. Suppose, for example, that $\delta' = \pi/2$ (semions). In this case, since each level could then accommodate a density $\rho/2$, the actual density ρ will fill exactly two levels. More generally, an integral number of levels will be full whenever $\delta' = \pi/(\text{integer})$. We must remember, however, that we have already "included" π of the total phase δ , in thinking of our anyons as fermions (rather than bosons) modified by flux tubes. So for example $\delta' = \pi/3$ really means an anyon interchange phase of $4\pi/3$, which is equivalent (modulo 2π) to $-2\pi/3$. These anyon phases $\pm\pi/2, \pm2\pi/3, \dots$ are precisely the ones which, as we saw in Section 3, allow integral numbers of them (2, 3, ...) to form bosonic bound states.

Why is this coincidence interesting? For two reasons⁽⁹⁾. First, when a Landau level is fully occupied, there is then a definite energy gap (of size $\hbar\omega_c$) before the next lot of available states, which must be in the next level. This means that the system cannot be "excited", by promoting particles into higher energy states, unless it receives at least as much energy as $\hbar\omega_c$: anything less will not disturb it. Just this kind of energy gap for fermion excitations is a key feature in the BCS theory of superconductivity. Secondly, the

bosonic bound states which can be formed at these magic δ values might conceivably be able to play a role similar to the Cooper pairs of BCS theory, as we noted earlier, in Section 3. They could "condense" into a highly symmetrical correlated state, which is another essential feature of superconductivity. One attractive idea is that the value $\delta = \pi/2$ must be preferred, since then a pair forms a boson, and - assuming each carries a real charge of e - this would imply that (real) magnetic flux would be quantised in units of $2\pi\hbar/(2e)$, as in the BCS case, and as has been observed in the high- T_c case also.

What has been said so far is, of course, merely suggestive. However, calculations have been reported ⁽²²⁾⁽²³⁾ which go beyond the mean field approximation, and seem to show that, at zero temperature, a gas of anyons with statistics parameter $\delta' = (\pi/\text{integer})$ is a superfluid if the anyons are electromagnetically neutral, and is a superconductor if they are charged. It is noteworthy that these results do not appear to depend on any specific pairing mechanism.

However, it seems at present rather unlikely that such a phenomenon, by itself, can be the essential physics behind high- T_c superconductivity. The main reason is that it has been argued more recently⁽²⁴⁾ that the zero temperature superconductivity of the charged anyon gas does not survive at finite temperature, at least according to a perturbative calculation. Perhaps more detailed pairing mechanisms do have to be considered. Another difficulty relates to the point made in Section 3 in connection with Figure 9 - namely, that anyonic systems tend generically to violate both parity and time-reversal invariance. In the present context, this seems to imply that the high- T_c ground state should break these symmetries "spontaneously". Evidence for this has been sought, with conflicting results as yet⁽²⁵⁾⁽²⁶⁾.

It has also to be admitted that, even if some form of anyon-based superconductivity does prove to be sufficiently robust, one would still be a long way from demonstrating that it had anything to do with the physics of the observed high- T_c phenomenon. Many questions remain unanswered. Perhaps most obviously, one has to ask how much of the

anyon physics, which is predicated upon literal confinement of the particles to two dimensions, can survive in the real world of material samples which are undoubtedly three-dimensional. Even if this can be understood (in terms, perhaps, of the relevant correlation lengths), there is no clear explanation available of how anyons might arise, starting from a material consisting of old-fashioned bosons and fermions. What is the origin of the statistical (or "fictitious") gauge field? And even if anyons can arise somehow, why should they be the "magic" ones with $\delta' = \pi/2$ (or $2\pi/3$, ...)? Also (cf. Section 3), would parity and time-reversal invariance have to be violated by the superconducting ground state? Work continues on these and related questions, for discussion of which we must refer to some recent compilations ⁽¹⁰⁾⁽²⁷⁾⁽²⁸⁾.

6. Anyons in the fractional quantum Hall effect (FQHE)

The quantum Hall effect⁽⁷⁾⁽²⁹⁾ has to do with the behaviour of electrons ingeniously confined to two dimensions, and subjected to a strong (1 - 10 T) magnetic field B at very low temperatures. Before proceeding further it is as well to be absolutely clear that this time B is a real, not "fictitious" or "statistical", field. Adopting the most simple-minded approach, we shall think of the electrons as free⁽³⁰⁾, except for the influence of the B field. This is then precisely a "Landau level" situation again, but this time due to the real B .

The standard Hall geometry is shown in Figure 14, and the essential features of the data are sketched in Figure 15. The fundamental observation is that at certain definite values of B , for fixed electron density, "plateaus" are seen in the transverse resistivity at exactly the values $2\pi\hbar/\nu e^2$, for integer ν , while at the same B values the longitudinal resistivity vanishes altogether. Now according to the elementary treatment of the Hall effect, the transverse resistivity should be given by $B/\rho e$, where ρ is the electron density. If we equate this to the plateau value, we find the remarkable relation

$$\nu = \rho / \left(\frac{eB}{2\pi\hbar} \right) \quad (7)$$

Eq. (7) can be easily interpreted: since $eB/2\pi\hbar$ is precisely the number of states available per Landau level per unit area, ν is nothing but the "filling factor" of the Landau levels. For $\nu = 1, 2, 3, \dots$ the situation (in this simple model!) is that an integer number of levels are filled. This is the integer quantum Hall effect, or IQHE.

Just as in the case of anyons in mean field approximation discussed in the previous Section, these values of ν will correspond to particularly stable configurations – somewhat analogous to closed shells in atoms or nuclei – since it will cost a finite amount of energy $\hbar\omega_c$ to put an electron into a state in the next higher unfilled Landau level. Presumably this fact is also associated with the zero longitudinal resistance observed at these values of ν .

Indeed, the existence of the "plateaus" visible in figure 15 shows that even if, for fixed density ρ , the magnetic field varies somewhat around the "magic" value for which (7) would be exactly satisfied, the electrons must remain for the most part in the state satisfying (7), any excess or deficit from the "magic" densities corresponding to the different B's being associated with localized rather than bulk states. In other words, the electron system seems to be essentially incompressible, in the vicinity of the integer filling factors given by (7).

Very roughly, one can begin to understand this property in the following way. As we saw in the previous Section, the electrons move (classically) in planar orbits of radius $r_c = mv/eB$. Setting the kinetic energy $\frac{1}{2}mv^2$ equal to the lowest Landau orbit energy $\hbar eB/2m$, we find $r_c = (\hbar/eB)^{1/2}$, a quantity called the "magnetic length" – and equal to about 10^{-8}m for $B \sim 10\text{T}$. Suppose we regard the available planar area A as being covered by N such "rigid" circular orbits, so that $N\pi r_c^2 \approx A$. This yields $\rho \approx eB/(\pi\hbar)$, which is very similar to (7). This is of course only an order of magnitude calculation – nevertheless it may correctly capture the essential idea of the incompressibility of the multi-electron IQHE state at integer filling factors ν .

The first experimental results to show the IQHE⁽²⁹⁾ were quite unexpected. But, just as the theorists were beginning to feel some confidence in polished-up versions of the preceeding qualitative picture, a new surprise occurred - the fractional quantum Hall effect, or FQHE⁽⁶⁾. The plateau/zero resistance behaviour was also seen at fractional filling factors of the form $\nu = p_1/p_2$ for integers p_1 and p_2 , with p_2 odd.

The crudest possible way to begin to think about this result is probably as follows. In the previous Section, we invoked a mean field approach to produce a smooth b field from a collection of localized flux lines, and this was apparently a reasonably good approximation. Consider, now, running this idea backwards in the case of the B field in the FQHE. Suppose ν in (7) has the simplest odd-fractional form $\nu = 1/(2p+1)$ as originally considered by Laughlin⁽³⁰⁾. This means that

$$B = \frac{2\pi\hbar}{e} \cdot (2p+1) \cdot \rho \quad (8)$$

Equation (8) can be interpreted as telling us that, at this filling factor, the total field B can be regarded as being "partitioned" in bundles of $(2p+1)$ flux quanta $2\pi\hbar/e$, each such bundle being attached to one electron. Remembering the trickiness with the factor of 2 when we discussed this procedure in Section 5 we see that the electron-flux bundle composites will acquire an A-B phase of

$$\frac{1}{2} \cdot e \cdot \frac{(2p+1)}{\hbar} \frac{2\pi\hbar}{e}$$

which is exactly an odd number of π 's. But, since the "cores" were fermions in the first place, the total interchange phase is an even number of π 's - i.e. they have been converted into bosons! So the suggestion is that at these odd-fractional filling factors, the external magnetic field is effectively bosonizing the fermions, and allowing them to form a highly symmetrical correlated state, a new kind of "condensate" somewhat analogous to the pair-condensate of BCS theory. (Note, however, that again no mention is made here of the electrons' spin degree of freedom: the electrons are presumed to be fully polarised in such high B -fields at low temperatures).

At this stage the reader might well be wondering why these FQHE states (or for that matter, the IQHE ones) are not, in fact, superconducting - since much of the foregoing discussion seems to echo closely what was said previously concerning anyonic superconductivity. The QHE states are certainly not superconducting - in particular, they of course do not exclude the applied magnetic field (no Meissner effect). The essential difference between the two cases is that the anyonic superconductor is not incompressible. In thinking about this it is easier to imagine the magnetic field in the QHE as fixed, the filling fraction ν then being determined by the density ρ - a viewpoint consonant with the "N rigid orbits" picture given above. By contrast, the fixed quantity in the anyonic case is the statistics parameter $\delta' = q\Phi/\hbar$, so that the effective mean field b (given by $b = 2\rho\Phi$) is proportional to ρ for fixed δ' , and "adjusts" with ρ as it varies. Thus in the anyonic superconductor we do not have the "rigid orbit" structure which was associated with the fixed external field in the QHE case. Compressibility is an essential ingredient for superconductivity. The possibility of both superconducting and QHE states for anyon gases has been discussed by Fradkin ⁽³¹⁾, who also provides a fuller analysis of the physical mechanisms which are thought to be involved.

Thus far we have used "anyonic" ideas but have seen only fermions or bosons. Anyons proper come when we consider simple excitations from the fractionally filled states. After all, since the Landau levels are now not full, we don't have a gap of $\hbar\omega_c$ to overcome before an excited state becomes available. What will the form of the simplest excitations be? Laughlin suggested⁽³⁰⁾ that the elementary excitations are just single flux lines, each carrying flux $2\pi\hbar/e$. Now comes the key question: are we to associate a charge with this flux line, and if so what is it? Following the previous line of thought, we might imagine that since, in the FQHE ground state, $(2p+1)$ such flux lines seem to be associated with the charge e on one electron, perhaps one flux line should carry the fractional charge $e/(2p+1)$ ⁽³⁰⁾. If this were so, the elementary excitation would behave like an anyon with

$$\delta = \frac{1}{2} \cdot \frac{e}{2p+1} \cdot \frac{2\pi}{\hbar} \frac{\hbar}{e} = \frac{\pi}{2p+1} \quad , \quad (9)$$

and the anyon concept would thus have found a physical realization. It is noteworthy that in the QHE parity and time-reversal symmetries are explicitly broken by the external magnetic field.

The foregoing suggestion concerning fractional charge can be related to the "incompressible system" idea, which we now extend to the states corresponding to the plateaus observed at fractional ν values. That is, we infer that these values also seem to be characteristic of specially stable configurations which the electrons will seek to preserve, as far as possible, against changes in ρ or B . Suppose then, that the total flux BA is reduced by one flux quantum $2\pi\hbar/e$ from the value satisfying (8), so that now $B' = B - 2\pi\hbar/Ae$. The electron system would like to satisfy (8) for the reduced field B' . This can be fulfilled if we imagine that the electron system, in the field B' , excludes from itself $1/(2p+1)$ of an electron - since then we would have

$$B' = B - \frac{2\pi\hbar}{Ae} = \frac{2\pi\hbar}{e} \left\{ (2p+1) \frac{N}{A} - \frac{1}{A} \right\} - \frac{2\pi\hbar}{e} (2p+1) \left\{ N - \frac{1}{2p+1} \right\} / A \quad (10)$$

Thus we are led to associate a fractional charge with a single flux quantum, the whole representing an excitation away from the incompressible state. Note, however, that the localised fractional charge is not to be thought of as some "fragment" of an electron: rather, it is a new kind of quasiparticle, providing a simple and effective phenomenology of a complicated redistribution of charge in the many-electron excited states.

It seems difficult to find direct experimental confirmation of the anyonic nature of these excitations in the FQHE. However, three groups have reported evidence⁽⁵⁾ for the predicted⁽³⁰⁾ fractional charge of the excitations, a remarkable vindication for the Laughlin theory. To this extent, then, we may say that there is at least one physical situation in which the usefulness of the concept of anyons has been established.

The preceding paragraphs have indicated an interpretation for the states with filling factors $\nu = 1/(2p+1)$, and for the elementary excitations therefrom. What about other fractional values of ν ? It is possible to extend the above picture in a consistent way. Let us begin by noting that we may regard the " $1/(2p+1)$ " series of states (1, 1/3, 1/5, 1/7, ...) as being linked to each other via adiabatic changes in the magnetic field B , for

fixed electron density ρ . Starting with B given by (8) for example (with $\nu = 1/(p+1)$), we can imagine⁽³²⁾ adiabatically reducing B by shifting flux onto the electrons, which thereby turn into anyons. However, they will re-emerge as electrons after absorbing exactly two flux quanta each, since then the change to their phase will be $\frac{1}{2} \cdot \frac{e}{\hbar} \cdot \frac{4\pi\hbar}{e} = 2\pi$. At this point B has been reduced from the value (8) to that corresponding to $\nu = 1/(2p-1)$.

Consider now starting with $\nu = \nu_0 = 1$ and $B = B_0$, and smoothly increasing B at fixed ρ . Since the evolution is assumed adiabatic, the electrons must stay in the first Landau level - which, however, is able to accommodate more electrons as B increases. This at fixed ρ , holes are created in this Landau level, a situation familiar from the elementary Hall effect. Suppose we increase B to the value $3B_0/2$, so that our final filling factor is $\nu = 2/3$. We can write

$$\rho = B_0 \cdot \frac{e}{2\pi\hbar} = \frac{3B_0}{2} \cdot \frac{e}{2\pi\hbar} - \frac{1}{3} \cdot \frac{3B_0}{2} \cdot \frac{e}{2\pi\hbar} \quad (11)$$

which we can interpret as telling us that the total density can be regarded as that appropriate to electrons with $\nu = 1$ in the final field, together with holes of charge $-e$ at a filling factor of $1/3$. Clearly for this latter filling factor the holes will "condense" in the same way as the electrons did at $\nu = 1/(2p+1)$. In this way the series $2/3, 4/5, 5/7, \dots$ can be interpreted as being associated with condensates of holes (charge $-e$) at filling factors $1/(2p'+1)$, in addition to electrons in the $\nu = 1$ state.

A more exotic possibility emerges if we consider starting from the state $\nu = 1/3$. In this case the holes would have charge $-e/3$ and would be anyons with interchange phase $\pi/3$. A natural question to ask is: what filling factor, for these anyons, would be required to bosonize them, thus causing a new condensate? A possible answer is the fraction $1/7$, since then each anyon would get seven flux quanta, leading to the additional phase $\frac{1}{2} \cdot \frac{-e}{3} \cdot \frac{7}{\hbar} \cdot \frac{2\pi\hbar}{e} = -\frac{7\pi}{3}$, which produces -2π when added to the anyons' $\pi/3$. (Other possible fractions would be $1/13, 1/19, \dots$). For the fraction $1/7$, then, the anyon density $\rho_{e/3}$ is given by $\rho_{e/3} = \frac{1}{7} \cdot \frac{eB}{2\pi\hbar}$. From charge conservation, the corresponding electron density ρ must equal $1/3 \rho_{e/3}$. Hence the electron filling factor for these holes is $1/21$, and the net filling factor is (cf. Eq. (11)) $1/3 - 1/21 = 2/7$. Thus we can interpret a $\nu = 2/7$ state as consisting of a $\nu = 1/3$ condensate of electrons together with a $\nu = 1/7$ condensate of holes in the electron condensate.

It is clearly possible to systematize this approach and arrive at a sequence of possible fractions with appropriate interpretations. A more rigorous treatment, based on an extension of Laughlin's wavefunctions⁽³⁰⁾, has been given by Haldane⁽³³⁾ and Halperin⁽³⁴⁾. These authors show how a hierarchy of fractions, all with odd denominators, can be generated. Many of the predicted ν values have now been

confirmed (in the sense of the associated plateaux in ρ_{xy} and zeros in ρ_{xx}), providing considerable further support for the anyon viewpoint.

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Figure Captions

- Figure 1 Two-Slit interface experiment.
- Figure 2 The Aharonov-Bohm set-up.
- Figure 3 Two particles propagating forwards in time, (a) without exchange (b) with exchange.
- Figure 4 Exchanging two particles around a flux tube.
- Figure 5 Exchanging in the opposite sense to Figure 4.
- Figure 6a A single exchange of two particles in two dimensions
- Figure 6b A double exchange in two dimensions
- Figure 7 Constructing the relative coordinate space for two identical particles in two dimensions.
- Figure 8 The "rolled-up" version of Figure 7, after cutting. The point $\underline{r}'$ lies on the rear "outer" sheet of the cone, and the point $-\underline{r}'$ lies on the "inner" sheet of the cone, such that $\underline{r}'$ and $-\underline{r}'$ coincide. The curve joining them is partly on the "outer" sheet (solid line) and partly on the inner (broken line).
- Figure 9 The "reflected" version of the exchange process show in Figure 7.
- Figure 10a Particles 1 and 2 are exchanged, but in the process their trajectories also link that of particle 3.
- Figure 10b A 1-2 exchange in which the trajectory of particle 3 is not linked.
- Figure 11 Exchanging two bound states each formed from two semions.
- Figure 12 Charge-flux model for an anyon.
- Figure 13 A collection of anyons (= "charges" + "flux tubes") in the planar area A.
- Figure 14 Geometry of the Quantum Hall Effect.
- Figure 15 Hall effect resistivities ρ_{xy} and ρ_{xx} versus magnetic field B.

Fig. 1

Fig. 2

Fig. 3a

Fig. 3b

Fig. 4

Fig. 5

Fig. 6a

Fig. 6b

Fig. 7

Fig. 8

Fig. 9

Fig. 10a

Fig. 10b

Fig. 11

Fig. 12

Fig. 13

Fig. 14

Fig. 15