CERN-TH.6066/91

Quantum Field Theory at Finite Energy

Masud CHAICHIAN *

Theory Division, CERN
CH-1211 Geneva 23, Switzerland
and

Ikuo SENDA

International Center for Theoretical Physics
34100 Trieste, Italy

Abstract

We give the general formulation of statistical quantum field theory at finite energy in the functional integral approach using the microcanonical distribution. Definitions of thermodynamical quantities and the effective potential are given. The Feynman rules for the real scalar field with ϕ^4 interaction and some computations of physically interesting quantities are presented. We discuss how a structure which is the same as the statistical bootstrap comes into the theory in a natural way. The entropy in ϕ^4 theory is calculated perturbatively up to two loops. The behaviour of symmetry at finite energy is investigated using the effective action and symmetry restoration at high energy is shown in a theory which has spontaneous symmetry breaking at zero energy. Finally, applications and the perspectives of the formalism are discussed.

*Permanent address: Department of High Energy Physics, University of Helsinki, Siltaavuorenpenger 20 C, SF-00170 Helsinki, Finland

CERN-TH.6066/91
March 1991

1 Introduction

In this paper, we are going to discuss the path-integral formulation of relativistic quantum field theory at finite energy using the microcanonical distribution. Our formalism is a natural extension of the so-called 'time-path formalism', which enables us to investigate the quantum field theory in Minkowski space-time.

In nature, we often encounter the situation that finite energy/temperature effects are important. Quantum field theory (QFT) at finite temperature has been developed by many people.^[1, 2, 3, 4, 5, 6] It was initiated by Matsubara^[7] in the imaginary-time formulation and was extended to the real-time method by Takahashi and Umezawa^[8], using the operator approach, and by Niemi and Semenoff^[9], using the path integral.

In statistical mechanics, several ensembles which allow us to investigate the equilibrium properties of the system are known. Among them, the microcanonical ensemble and the canonical ensemble have special importance.[†] The canonical ensemble is used most extensively in the literature. In this ensemble, one considers the situation, where the system of interest with volume V is surrounded by a large thermal reservoir such that the system is kept at a certain temperature T . This is the situation in which quantum field theory at finite temperature is formulated. The microcanonical ensemble, on the other hand, is appropriate for investigating an isolated system with a finite energy E and a volume V . In the microcanonical distribution, each microscopic state has the same probability, which is based on the ergodic hypothesis. This microcanonical ensemble is considered to be more fundamental than the canonical one and it is well known that in the thermodynamical limit, $E, V \gg 1$ with E/V finite, the canonical and microcanonical ensembles are equivalent. However, for a system at finite E or V , there are finite differences between these two ensembles and we should use the microcanonical ensemble to investigate such a system.^[10] Recently the behaviour of the relativistic quantum field theory at finite energy was investigated by Weldon.^[10] In this paper, we are going to develop the path integral formulation of the relativistic quantum field theory at finite energy using the microcanonical distribution and discuss the general formulation of the effective action.[‡] The field theory is formulated in Minkowski space-time so that it can also be used to investigate the time evolution of the system.^[3, 12, 13]

[†]In the context of quantum field theory, we should call them the grand-microcanonical ensemble and grand-canonical ensemble, respectively, since we are not fixing the number of particles.

[‡]One can find a similar attempt, but in a different context, in Ref. [11]

The physical applications we have in mind are the early universe and hadron physics. For example, the universe does not have a thermal reservoir. At a late stage in the evolution of the universe, the use of the canonical ensemble is justified, but at an early stage the microcanonical investigation would be preferred. In hadron physics, there are lots of situations where the microcanonical treatments should be used, namely the central rapidity region in the relativistic nucleus-nucleus collision^[14, 15], quark-gluon confinement^[16] and so on. In the 1970's, the statistical bootstrap model was investigated extensively in order to understand the properties of hadrons at high density.^[17, 18] In the text of this paper, we will show that the structure of the bootstrap appears naturally in quantum field theory at finite energy, which gives us some hope for the application of the present formulation to hadron physics.

In section 2, the path integral formulation is presented using the microcanonical distribution. In section 3, the Green functions and Feynman rules are given for the real scalar theory with a ϕ^4 interaction and fermions. Some examples of computations, such as the expectation value of the number operator and the perturbative calculation of the full propagator, are given. In section 4, we will discuss the general definition of the thermal quantities and the effective action in the finite-energy quantum field theory. Sections 5 and 6 are devoted to explicit calculations. In section 5, we perform the calculation of the entropy of the real scalar theory with a ϕ^4 interaction in the two-loop approximation. Section 6 is used for the one-loop calculation of the effective action. As an example, we consider a theory with spontaneous symmetry breaking at zero-energy. We will show that symmetry restoration occurs at high energy density and that the critical energy density agrees with the one obtained in the finite-temperature field theory in the thermodynamical limit.^[4] Section 7 is used to discuss the perspective of our formulation of quantum field theory. In the Appendix, the detailed calculations are collected.

2 The Path Integral and the Microcanonical Distribution

In this section, we will discuss the path integral formulation of quantum field theory at finite energy, where the ensemble sum is evaluated by the use of the microcanonical distribution. We will define the path integral such that the ensemble average of the time-dependent operator is evaluated. For this purpose, we use the 'time-path' method, which is used extensively in finite-temperature QFT.^[2, 3] In this section, explicit calculations are shown for real scalar boson. The extension

to the other fields is straightforward. The relation of the generating functional defined here to the thermal functions will be discussed in section 4.

Let us start by defining the generating functional at finite energy E

$$Z_E[j] = \int d\phi < \phi, t | \delta(\hat{H} - E) \mathcal{T} \left(e^{i \int d^4x j(x) \phi(x)} \right) | \phi, t >, \quad (2.1)$$

where $\mathcal{T}$ means the time-ordered product and $j(x)$ is an external source introduced to develop the perturbation expansion. The sum over the complete set of field configurations is taken at a certain time t . Using the integral representation of the delta function, $\delta(\hat{H} - E) = \int \frac{d\alpha}{2\pi} \exp(-i\alpha(\hat{H} - E))$, eq.(2.1) is written as

$$\begin{aligned} Z_E[j] &= \int \frac{d\alpha}{2\pi} e^{i\alpha E} \int d\phi < \phi, t + \alpha | \mathcal{T} \left(e^{i \int d^4x j(x) \phi(x)} \right) | \phi, t > \\ &= \int \frac{d\alpha}{2\pi} e^{i\alpha E} \mathcal{N}_\alpha \int D\phi D\Pi e^{i \int d^4x (L + j\phi)}, \end{aligned}$$

where we have used $e^{i\alpha\hat{H}} | \phi, t > = | \phi, t + \alpha >$. The second line is the path integral representation of the first line and Π is the conjugate momentum of ϕ . The symbol " $\mathcal{T}$ " in the exponent in the second line means that the time path in the integration over x_0 should be chosen such that it connects t to $t + \alpha$. The α -dependent normalization factor is represented by $\mathcal{N}_\alpha$. Following the usual procedure, the generating functional for the perturbative calculation becomes

$$Z_E[j] = \int \frac{d\alpha}{2\pi} e^{i\alpha E} e^{i \int d^4x L_I} Z_\alpha^0 e^{-\frac{i}{2} \int d^4x d^4x' j(x) D_\alpha(x-x') j(x')}, \quad (2.2)$$

where, as usual, L_I is the interaction Lagrangian and $D_\alpha(x-x')$ is a propagator. The factor Z_α^0 is the contribution of the normalization factor and one-loop partition function. In general, one has to be careful about the ordering of operators and derivatives. In the case of the real scalar boson, the propagator is defined by the differential equation:

$$\begin{aligned} -(\partial^2 + m^2) D_\alpha(x-x') &= \delta^4(x-x') \\ \phi(t+\alpha, \mathbf{x}) &= \phi(t, \mathbf{x}), \quad D_\alpha(t+\alpha, \mathbf{x}) = D_\alpha(t, \mathbf{x}). \end{aligned} \quad (2.3)$$

In the derivation of eq.(2.2), periodic boundary conditions for the fields and the propagators are assumed, namely for the scalar boson:

In the real-time formulation of finite-temperature QFT, different choice of the time path defines field theories, which are physically equivalent but different, in the sense that quantities such as propagators are different.^[19] We find the same situation in finite-energy QFT. The only restriction in choosing the time path is

that it should connect a certain time t and $t + \alpha$. The choice of the time path which we find most convenient for real-time formulation is " $\mathcal{C}$ " shown in Fig.1. It consists of three parts: C_1 connecting $-T$ to T , C_2 connecting T to $-T$ backward in time and C_3 connecting $-T$ to $-T + \alpha$. Because of the requirement of causality and the well-definedness of the path integral, the path monotonically decreases in the imaginary time direction, by the infinitesimal amount ' ϵ '. We will take the limit T goes to infinity. Like in finite-temperature QFT, physical operators, which appear as external lines in Green functions, are assumed to have support on the path C_1 and fields having support on C_2 and C_3 are considered to be ghosts. Since in the path integral the operator ordering is given by the T^* ordering, we have to modify the operator ordering to the so-called path-ordering, namely the operator placed at the time closer to the end of the path has to be on the left.

The great simplification resulting from the choice of this path is that the operators on the C_3 -path decouple from those on the C_1 and C_2 paths. This is because, in the limit $T \rightarrow \infty$, the propagator connecting paths C_a ($a = 1, 2$) and C_3 vanishes due to the Riemann-Lebesgue theorem and the adiabatic switching-off of the external source $j(x)$. The proof of this decoupling is illustrated in Appendix B. Therefore, the real-time part $C_1 + C_2$, which is used to evaluate the ensemble average of the time-dependent operators, and the thermodynamical part C_3 are separated. Thus one can write eq. (2.2) as

$$\begin{aligned} Z_E[j] &= \int \frac{d\alpha}{2\pi} e^{i\alpha E} Z_\alpha[j], \\ Z_\alpha[j] &= Z_\alpha^{(1+2)}[j] \cdot Z_\alpha^{(3)}[j], \\ Z_\alpha^{(3)}[j] &= \mathcal{N}_\alpha^{(3)} \int D\phi D\Pi \exp\left\{i \int_{C_3} d^4x (\mathcal{L} + j\phi)\right\}. \end{aligned} \quad (2.4)$$

In general, one can choose any path connecting t and $t + \alpha$ within the requirement of the consistency of the theory and they give physically equivalent results when the system is in equilibrium. The equivalence of the different choices of path is a simple consequence of the completeness of the set of states inserted at each moment in the path integration,^[12] which is the same as the situation in finite-temperature QFT.⁵ An example of a calculation showing the equivalence of physical quantities obtained by choosing different paths is given in Appendix E.

The reader might have noticed that the α -integration in eq.(2.2) has the meaning of inverse-Laplace transformation. Therefore, one can consider our definition of the generating functional as the inverse-Laplace transformation of the canonical generating functional. However this is not always a correct understanding.

⁵ We would like to thank A. Niemi for bringing this point to our attention.

In the presence of phase transitions, this α -integration is not the same as the simple inverse-Laplace transformation. If the system has a phase transition, the canonical generating function has singularities in the complex temperature plane. Therefore the path of temperature integration in the inverse-Laplace transformation is not deformed to our choice of α -integration analytically. An example of such a singular canonical partition function is the one for the system of hadrons with exponentially increasing mass level density.^[20]

An important property of the generating functional in the microcanonical formulation of QFT is the convolution. For example, if one takes derivatives of $Z_E[j]$ in eq. (2.2) with respect to the external source and performs a perturbative calculation, as a general form one obtains

$$\prod_k \left(\frac{\delta}{i\delta j(x_k)} \right) Z_E[j] \Big|_{j=0} = \int \frac{d\alpha}{2\pi} e^{i\alpha E} Z_\alpha^0 F_\alpha^{(1)} F_\alpha^{(2)} \dots F_\alpha^{(n)}, \quad (2.5)$$

where the $F_\alpha^{(i)}$'s are functions of α as well as of the coordinates of external fields. After α -integration, the right-hand side of eq. (2.5) becomes

$$\int_0^E dE' \prod_{i=1}^n dE_i \delta(E - E' - E_1 - \dots - E_n) Z_E^0 F_{E'}^{(1)} F_{E_1}^{(2)} \dots F_{E_n}^{(n)},$$

$$Z_E^{(i)} = \int \frac{d\alpha}{2\pi} e^{i\alpha E'} Z_\alpha^0, \quad F_{E_n}^{(i)} = \int \frac{d\alpha}{2\pi} e^{i\alpha E_i} F_\alpha^{(i)}.$$

Therefore, the Green functions in the microcanonical formulation of QFT are obtained as convolutions of the subsystems under the constraint that the sum of energies in the subsystems is E . In section 3-2, we will show that the same structure as the statistical bootstrap comes into the theory due to the convolution property.

Because of the convolution property, we define the N-point Green function in a medium of energy E by

$$G_E(x_1, \dots, x_N) = \int \frac{d\alpha}{2\pi} e^{i\alpha E} \left(\prod_{i=1}^N \frac{\delta}{i\delta j(x_i)} \right) e^{i \int d^4x d^4x' j(x) D_\alpha(x-x') j(x')} \Big|_{j=0}, \quad (2.6)$$

where we have separated the contribution of Z_E^0 .

3 The Green Function and Feynman Rules

In this section, we are going to discuss the Feynman rules. In section 3-1, the real scalar bosonic field theory of mass m with a ϕ^4 interaction is discussed. In

section 3-2, the expectation value of the number operator and the full propagator as insertions of one-loop self-energy are investigated using ϕ^4 theory in order to illustrate how the structure of the bootstrap comes into our formalism. Section 3-3 is used to discuss the fermionic fields.

3-1. Real scalar boson

Since the $C_1 + C_2$ part decouples from the C_3 part in the limit $T \rightarrow \infty$, let us concentrate on $C_1 + C_2$ part first. The generating functional of the Green functions of fields on $C_1 + C_2$ is given from eq. (2.6):

$$Z_E^{(1+2)}[j] = \int \frac{d\alpha}{2\pi} e^{i\alpha E} Z_\alpha^{(1+2)}[j]$$

$$Z_\alpha^{(1+2)}[j] = \exp(i \int d^4x \mathcal{L}_I)$$

$$\times \exp\left(-\frac{i}{2} \sum_{a,b} \int_{C_a} d^4x \int_{C_b} d^4x' j_a(x) D_\alpha^{ab}(x-x') j_b(x')\right), \quad (3.1)$$

where j_a is an external source having support on C_a and summation is over $a, b = 1, 2$. In eq. (3.1), D^{ab} represents a propagator between the fields on C_a and C_b

$$D_\alpha(x-x') \equiv D_\alpha^{ab}(x-x'); \quad x \in C_a, x' \in C_b, \quad a = 1, 2.$$

The integration over the path C_a is given by

$$\int_{C_a} d^4x = \epsilon_a \int_{-T}^T d^4x |T \rightarrow \infty|, \quad (3.2)$$

where ϵ_a equals 1 for $a = 1$ and -1 for $a = 2$. Therefore, it is convenient to introduce a notation

$$\bar{D}_\alpha^{ab}(x-x') \equiv \epsilon_a \epsilon_b D_\alpha^{ab}(x-x'). \quad (3.3)$$

Then the generating functional in eq. (3.1) becomes

$$Z_E^{(1+2)}[j_1, j_2] = \exp\left(i \sum_a \epsilon_a \int d^4x \mathcal{L}_I(\delta/i\delta j_a)\right)$$

$$\times \exp\left(-\frac{i}{2} \sum_{a,b} \int d^4x d^4x' j_a(x) \bar{D}_\alpha^{ab}(x-x') j_b(x')\right). \quad (3.4)$$

Then the Feynman rules of the $C_1 + C_2$ part in ϕ^4 theory become:

where $\bar{D}(k)$ is the Fourier transformation of $\bar{D}(x)$ and τ_3 is the third component of the Pauli matrix. In Appendix A, we collect the explicit calculation of the

propagators and their periodicity properties. The 2×2 -matrix propagator $\bar{D}^{ab}$ is given by

$$i\bar{D}_\alpha^{ab}(k) = \begin{pmatrix} \frac{i}{k^2 - m^2 + i\epsilon} + 2\pi\delta(k^2 - m^2)f_\alpha^{(-)}(\omega_k), & -2\pi\epsilon(k_0)\delta(k^2 - m^2)f_\alpha^{(-)}(k_0) \\ -2\pi\epsilon(k_0)\delta(k^2 - m^2)f_\alpha^{(+)}(k_0), & \frac{-i}{k^2 - m^2 - i\epsilon} + 2\pi\delta(k^2 - m^2)f_\alpha^{(-)}(\omega_k) \end{pmatrix} \quad (3.5)$$

$$f_\alpha^{(-)}(u) = \sum_{n=1}^{\infty} e^{-inu}, \quad f_\alpha^{(+)}(u) = 1 + f_\alpha^{(-)}(u), \quad (3.5)$$

where $\omega_k = \sqrt{k^2 + m^2}$. Notice that α has a small negative imaginary part, $\alpha := \alpha - i\epsilon$, due to our choice of the path C . The propagator given in eq. (3.5) has the property:

$$\bar{D}_\alpha^{ab}(x) = \bar{D}_\alpha^{ba}(-x), \text{ and } \bar{D}_\alpha^{ab}(k) = \bar{D}_\alpha^{ba}(-k).$$

For example, the two-point Green function of fields on C_1 at the tree level is obtained from eq. (3.5) [10]:

$$G_E^{11}(k) = \int \frac{d\alpha}{2\pi} e^{i\alpha E} (iD_\alpha^{11}(k)) \\ = \frac{i\delta(E)}{k^2 - m^2 + i\epsilon} + 2\pi\delta(k^2 - m^2) \sum_{n=1}^{\infty} \delta(E - n\omega_k). \quad (3.6)$$

It is useful to rewrite the matrix form of the propagator in the following form

$$i\bar{D}_\alpha^{ab}(k) = [\theta(k_0)\Phi_{R,\alpha}^{(+)}(\omega_k) + \theta(-k_0)\Phi_{A,\alpha}^{(-)}(\omega_k)] \frac{i}{k^2 - m^2 + i\epsilon} \\ + [\theta(k_0)\Phi_{A,\alpha}^{(+)}(\omega_k) + \theta(-k_0)\Phi_{R,\alpha}^{(-)}(\omega_k)] \frac{i}{k^2 - m^2 - i\epsilon}, \quad (3.7)$$

where A and R stand for advanced and retarded, respectively, and $(\pm)$ for positive and negative k_0 parts. The 2×2 -matrix Φ 's are defined by:

$$\Phi_{R,\alpha}^{(+)}(\omega_k) = \begin{pmatrix} f_\alpha^{(+)}(\omega_k) & -f_\alpha^{(-)}(\omega_k) \\ -f_\alpha^{(+)}(\omega_k) & f_\alpha^{(-)}(\omega_k) \end{pmatrix}, \quad \Phi_{R,\alpha}^{(-)}(\omega_k) = \begin{pmatrix} -f_\alpha^{(-)}(\omega_k) & f_\alpha^{(+)}(\omega_k) \\ f_\alpha^{(+)}(\omega_k) & -f_\alpha^{(-)}(\omega_k) \end{pmatrix}, \\ \Phi_{A,\alpha}^{(+)}(\omega_k) = \begin{pmatrix} -f_\alpha^{(-)}(\omega_k) & f_\alpha^{(+)}(\omega_k) \\ f_\alpha^{(+)}(\omega_k) & -f_\alpha^{(-)}(\omega_k) \end{pmatrix}, \quad \Phi_{A,\alpha}^{(-)}(\omega_k) = \begin{pmatrix} f_\alpha^{(+)}(\omega_k) & -f_\alpha^{(-)}(\omega_k) \\ -f_\alpha^{(+)}(\omega_k) & f_\alpha^{(-)}(\omega_k) \end{pmatrix}.$$

Another form that is useful in calculating the entropy and the one-loop effective action is found in Appendix C. The matrix Φ 's have simple product relations

$$\Phi \cdot \Phi \equiv \Phi_{\tau_3} \Phi, \quad (3.8) \\ \Phi_R^{(\pm)} \cdot \Phi_R^{(\pm)} = \Phi_R^{(\pm)}, \quad \Phi_R^{(\pm)} \cdot \Phi_A^{(\pm)} = 0, \quad \Phi_A^{(\pm)} \cdot \Phi_R^{(\pm)} = \Phi_A^{(\pm)}.$$

The important property of the generating functional of fields on $C_1 + C_2$ given in eq. (3.4) is the Z_2 -symmetry with respect to the interchange between j_1 and

j_2 . This symmetry was discussed by Feynman and Vernon as a property of the influence function [12] and was extended to the finite-temperature QFT by Niemi and Semenoff [9]. As a consequence of the properties of the interaction Lagrangian and the propagators

$$\mathcal{L}_I[j_1, j_2] = -\mathcal{L}_I[j_2, j_1] \\ j_a (i\bar{D}_\alpha^{ab}) j_b = [j'_a (i\bar{D}_{-\alpha}^{ab}) j'_b]^*,$$

where $(j'_1, j'_2) = (j_2, j_1)$, the generating functional has the property:

$$(Z_E[j_1, j_2])^* = Z_E[j_2, j_1].$$

This symmetry guarantees that the generating functional is real when we take the limit $j_a \rightarrow 0$.

Let us turn to the C_3 -part. The propagator of fields on the C_3 path depends on the sign of α . Using the same notation as before:

$$i\bar{D}_\alpha^{33}(x) = \theta(\alpha) \left\{ \theta(x_0) \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} 2\pi\delta(k^2 - m^2) f_\alpha^{(+)}(k_0) \right. \\ \left. + \theta(-x_0) \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} 2\pi\delta(k^2 - m^2) f_\alpha^{(-)}(k_0) \right\} \\ + \theta(-\alpha) \left\{ \theta(x_0) \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} 2\pi\delta(k^2 - m^2) f_\alpha^{(-)}(k_0) \right. \\ \left. + \theta(-x_0) \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} 2\pi\delta(k^2 - m^2) f_\alpha^{(+)}(k_0) \right\}. \quad (3.9)$$

Because of the finiteness of the time interval on the C_3 path, the propagator in eq. (3.9) is not Fourier transformed like those in eq. (3.5).

3-2. Examples of Calculations in ϕ^4 theory

This section contains some examples of calculations in order to understand the formulations introduced in previous sections as well as how a structure like the bootstrap comes into finite-energy QFT.

First let us consider the expectation value of the number operator at tree level. The number operator of the state with momentum $\mathbf{k}$ is $\hat{n}_\mathbf{k}^{(1)} = a_\mathbf{k}^{(1)\dagger} a_\mathbf{k}^{(1)}$, where $a_\mathbf{k}^{(1)}$ is a Fourier component of the field on C_1 defined in Appendix A. The expectation value of $\hat{n}_\mathbf{k}^{(1)}$ in a medium of energy E is

Here, we impose the renormalization condition that $m^2 + \Sigma^{(0)}$ be the physical mass. For a more precise renormalization procedure, see section 6. For example, the positive k_0 part of (3.12) is

$$i\Delta_\alpha(k)|_{k_0>0} = \left[\frac{i}{k^2 - m^2 + i\epsilon} \sum_{n=0}^{\infty} \left(\frac{\Sigma_\alpha^{(1)}}{k^2 - m^2 + i\epsilon} \right)^n \Phi_{R,\alpha}^{(+)}(\omega_k) \right. \\ \left. + \frac{i}{k^2 - m^2 - i\epsilon} \sum_{n=0}^{\infty} \left(\frac{\Sigma_\alpha^{(1)}}{k^2 - m^2 - i\epsilon} \right)^n \Phi_{A,\alpha}^{(+)}(\omega_k) \right], \quad (3.13)$$

where m is replaced by the physical mass. Integrating over α , eq.(3.13) becomes

$$i\Delta_E(k)|_{k_0>0} = \\ = \frac{i}{k^2 - m^2 + i\epsilon} \int_0^E dE' [\delta(E') + \sum_{n=0}^{\infty} \left(\frac{1}{k^2 - m^2 + i\epsilon} \right)^n \Omega^n(E')] \Phi_{R,E-E'}^{(+)} \\ + \frac{i}{k^2 - m^2 - i\epsilon} \int_0^E dE' [\delta(E') + \sum_{n=0}^{\infty} \left(\frac{1}{k^2 - m^2 - i\epsilon} \right)^n \Omega^n(E')] \Phi_{A,E-E'}^{(+)}, \quad (3.14)$$

where $\Omega^n(E')$ is an n -body phase space integral defined by

$$\Omega^n(E') \equiv \int dE_1 dE_2 \cdots dE_n \delta(E' - E_1 - \cdots - E_n) \prod_{i=1}^n \Sigma_{E_i}^{(1)} \\ = \left(\frac{\lambda}{2} \right)^n \prod_{i=1}^n \left(\sum_{k_i=1}^{\infty} \int \frac{d^3 k_i}{(2\pi)^3 \omega_{k_i}} \right) \delta(E' - n_1 \omega_{k_1} - \cdots - n_n \omega_{k_n}), \quad (3.15)$$

where $\Sigma_{E_i}^{(1)}$ is the α integration of $\Sigma_\alpha^{(1)}$,

$$\Sigma_{E_i}^{(1)} = \int \frac{d\alpha}{2\pi} e^{i\alpha E_i} \Sigma_\alpha^{(1)}$$

Therefore, the full propagator for a finite energy E is a convolution of the n -body phase space integral. The physical meaning of the n -body phase space volume becomes clear if one compares eq. (3.15) with eq. (3.10). From eq. (3.10):

$$\Sigma_E^{(1)} = \lambda \frac{(2\pi)^3}{V} < \dot{N} >_E$$

Thus $\Sigma_{E_i}^{(1)}$ in eq. (3.15) is the number density of particles in the subsystem of energy E_i , and $\Omega^n(E')$ sums up the product of the particle number densities over all possible ways of dividing the system of energy E into n subsystems.

One might have noticed that this convolution structure is similar to the concept which we encounter in the statistical assumption in the discussion of the bootstrap.

$$< \hat{n}_k^{(1)} >_E = \lim_{\epsilon \rightarrow 0} \int \frac{d^3 x d^3 y}{(2\pi)^3 2\omega_k} e^{-ik(x-y)} (\partial_{x_0} + i\omega_k) (\partial_{y_0} - i\omega_k) \\ \times < T(\phi_1(x) \phi_2(y)) >_E |_{x_0=y_0=t} \\ = \frac{V}{(2\pi)^3} \sum_{n=1}^{\infty} \delta(E - n\omega_k),$$

where V is the spatial volume: $V = \int d^3 x$. To obtain the last line, we have used the explicit form of the two-point Green function, $G_E^{11} = i\bar{D}_E^{11}$, where $\bar{D}_E^{11}$ is given in eq. (3.5). The total number operator is given by

$$\hat{N} = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \hat{n}_k^{(1)}.$$

Thus its expectation value becomes

$$< \hat{N} >_E = \frac{V}{(2\pi)^3} \int \frac{d^3 k}{2\omega_k} \sum_{n=1}^{\infty} \delta(E - n\omega_k) \\ = \frac{V}{(2\pi)^3} \int d\alpha e^{i\alpha E} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} f_\alpha^{(-)}(\omega_k). \quad (3.10)$$

Next let us consider the full propagator as a sum of self-energies. In the one-loop approximation, the self-energy of the scalar boson is given by the diagram in Fig. 2. We denote a contribution of this diagram by Σ_α^{ab} . After a straightforward calculation, noticing the positions of poles of propagators, one obtains

$$\Sigma_\alpha^{ab} = (\tau_3)^{ab} (-i\Sigma_\alpha^{(0)} - i\Sigma_\alpha^{(1)}) \equiv (\tau_3)^{ab} \Sigma_\alpha, \quad (3.11) \\ \Sigma_\alpha^{(0)} = \frac{\lambda}{2} \frac{2\pi^{n/2} \Gamma(1 - n/2)}{32\pi^4 (m^2)^{1-n/2}}, \\ \Sigma_\alpha^{(1)} = \frac{\lambda}{2} \int \frac{d^3 k}{(2\pi)^3 \omega_k} f_\alpha^{(-)}(\omega_k),$$

where we are using dimensional regularization, $n \rightarrow 4$.

Using the propagator in eq. (3.7) and the product property among $\Phi_{A,R}^{(\pm)}$ in eq. (3.8), the sum of self-energies illustrated in Fig. 3 is performed. Let us denote the right-hand side of Fig. 3 by $i\Delta_\alpha^{ab}(k)$. Then we find

$$i\Delta_\alpha(k) = \left\{ \theta(k_0) \left[\frac{i}{k^2 - m^2 - i\Sigma_\alpha + i\epsilon} \Phi_{R,\alpha}^{(+)} + \frac{i}{k^2 - m^2 - i\Sigma_\alpha - i\epsilon} \Phi_{A,\alpha}^{(+)} \right] \right. \\ \left. + \theta(-k_0) \left[\frac{i}{k^2 - m^2 - i\Sigma_\alpha + i\epsilon} \Phi_{A,\alpha}^{(-)} + \frac{i}{k^2 - m^2 - i\Sigma_\alpha - i\epsilon} \Phi_{R,\alpha}^{(-)} \right] \right\}. \quad (3.12)$$

To understand this claim more clearly, let us examine the quantity appearing on the right-hand side of eq. (3.14)

$$\Omega(E) \equiv \sum_{n=1}^{\infty} \left(\frac{1}{k^2 - m^2 + i\epsilon} \right)^n \Omega^n(E) .$$

Using the explicit form of $\Omega^n(E)$ in eq. (3.15), one finds that $\Omega(E)$ satisfies the relation:

$$\Omega(E) = \sum_{N=1}^{\infty} \frac{1}{N!} \int \prod_{i=1}^N (dE_i \Omega(E_i)) \delta(E - \sum_{i=1}^N E_i) .$$

This is the bootstrap equation with respect to $\Omega(E)$, discussed extensively in hadron physics.[17, 18] At the present stage, we do not know if our formalism can be applied to hadron physics directly. However, this speculation gives us some hope of applying our formulation in that direction.

3-3. Fermion Fields

In this subsection, we will consider fermionic fields. The free Lagrangian of a fermionic field with mass m is given by

$$\mathcal{L}_0 = \bar{\psi}(\gamma^\mu \partial_\mu - m)\psi .$$

The generating functional is obtained similarly to the one in eq. (3.4)

$$Z_E[\eta, \bar{\eta}] = \frac{1}{\mathcal{N}} \int \prod_{\alpha} d\psi_{\alpha} d\bar{\psi}_{\alpha} \exp \left[i \int d^4x \bar{\psi}_{\alpha} (\gamma^\mu \partial_\mu - m) \psi_{\alpha} + i \int d^4x \bar{\eta}_{\alpha} \psi_{\alpha} + i \int d^4x \bar{\psi}_{\alpha} \eta_{\alpha} \right] ,$$

where η 's are Grassmann external sources and the summation over $\alpha = 1, 2$ is understood. The propagator $S_{\alpha\beta}$ is derived similarly to the bosonic case. In this section, we will show only results for fields on $C_1 + C_2$. The 2×2 matrix form of the propagator of fields on $C_1 + C_2$ is given by

$$\begin{aligned} \tilde{S}_{\alpha}(p) &= \frac{p + m}{p^2 - m^2 + i\epsilon} [\theta(p_0) \Psi_{R\alpha}^{(+)}(\omega_p) + \theta(-p_0) \Psi_{A\alpha}^{(-)}(\omega_p)] \\ &\quad + \frac{p + m}{p^2 - m^2 - i\epsilon} [\theta(p_0) \Psi_{A\alpha}^{(+)}(\omega_p) + \theta(-p_0) \Psi_{R\alpha}^{(-)}(\omega_p)] , \\ \Psi_{R\alpha}^{(+)}(\omega_p) &= \begin{pmatrix} g_{\alpha}^{(+)}(\omega_p) & g_{\alpha}^{(-)}(\omega_p) \\ -g_{\alpha}^{(+)}(\omega_p) & -g_{\alpha}^{(-)}(\omega_p) \end{pmatrix} , \quad \Psi_{A\alpha}^{(-)}(\omega_p) = \begin{pmatrix} g_{\alpha}^{(+)}(\omega_p) & -g_{\alpha}^{(+)}(\omega_p) \\ -g_{\alpha}^{(-)}(\omega_p) & -g_{\alpha}^{(-)}(\omega_p) \end{pmatrix} , \\ \Psi_{A\alpha}^{(+)}(\omega_p) &= \begin{pmatrix} g_{\alpha}^{(-)}(\omega_p) & -g_{\alpha}^{(-)}(\omega_p) \\ g_{\alpha}^{(+)}(\omega_p) & g_{\alpha}^{(+)}(\omega_p) \end{pmatrix} , \quad \Psi_{R\alpha}^{(-)}(\omega_p) = \begin{pmatrix} g_{\alpha}^{(-)}(\omega_p) & g_{\alpha}^{(+)}(\omega_p) \\ -g_{\alpha}^{(+)}(\omega_p) & -g_{\alpha}^{(-)}(\omega_p) \end{pmatrix} , \end{aligned}$$

where $g^{(\pm)}$ is defined by

$$g_{\alpha}^{(-)}(\omega_p) = \frac{1}{e^{i\omega_p} + 1} , \quad g_{\alpha}^{(+)}(\omega_p) = 1 - g_{\alpha}^{(-)}(\omega_p) . \quad (3.16)$$

Because of the small negative imaginary part of α , the right-hand side of the above expression is expanded well-definedly in terms of $\exp(-i\alpha\omega)$. The matrices $\Psi_{A,R}^{(\pm)}$ have similar product relations as before

$$\begin{aligned} \Psi \cdot \Psi &\equiv \Psi_{\tau_3} \Psi , \\ \Psi_R^{(\pm)} \cdot \Psi_R^{(\pm)} &= \Psi_R^{(\pm)} , \quad \Psi_R^{(\pm)} \cdot \Psi_A^{(\pm)} = 0, \quad \Psi_A^{(\pm)} \cdot \Psi_R^{(\pm)} = 0, \quad \Psi_A^{(\pm)} \cdot \Psi_A^{(\pm)} = \Psi_A^{(\pm)} . \end{aligned}$$

For example, the two-point Green function of fields on C_1 of finite energy E is obtained as

$$\begin{aligned} G_E^{11}(x-y) &= iS_E^{11}(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} [i\delta(E) \frac{p+m}{p^2-m^2+i\epsilon} \\ &\quad - 2\pi i \delta(p^2-m^2) (\not{p}+m) \sum_{n=1}^{\infty} (-)^{n+1} \delta(E-n\omega_p)] . \end{aligned}$$

4 General Formulation of the Effective Action

It is an interesting problem to investigate the symmetry behaviour in finite-energy quantum field theory. For this purpose, we will develop a general formulation for the entropy and the effective action in finite-energy QFT. For simplicity, we will do this using the scalar field theory, although extension to other fields is straightforward.

Our procedure is quite standard and one can find corresponding discussions in the statistical mechanics. We start by defining the volume of the configuration space Ω_{σ} . For later convenience, we will introduce an external field j . The volume of a configuration space in a medium of finite energy E , volume V and some conserved quantities X for which there are, in the presence of the external field j is defined as the number of the field configurations whose eigenvalues of the Hamiltonian are less than E :

$$\Omega_{\sigma,E,V,X}[j] = \sum_{\{\phi\}_{V,X}} < \phi | \theta(E - \hat{H}) \mathcal{T} \left(e^{i \int d^4x \phi(x) j(x)} \right) | \phi > .$$

Next, we define the number density of the field configuration at energy E , $Z_{E,V,X}[j]$:

$$Z_{E,V,X}[j] \equiv \frac{\partial}{\partial E} \Omega_{\sigma,E,V,X}[j] .$$

This is the generating functional defined in eq. (2.1). Since we are treating the system quantum-mechanically, there is a certain width of energy, δE , such that we cannot distinguish between energy eigenvalues E and $E + \delta E$ because of the uncertainty principle. Therefore, we define a dimensionless quantity $W_{E,\delta E,V,X}[j]$

that is the number of configurations of which energy eigenvalues are between E and $E + \delta E$:

$$W_{E,SE,V,X}[j] = \int_E^{E+\delta E} dE' Z_{E',V,X}[j] \sim \delta E Z_{E,V,X}[j]. \quad (4.1)$$

From the physical point of view, the uncertainty of the energy is of the order of the momentum uncertainty, namely $\delta E \sim \mathcal{V}^{-1/3}$. In the rest of the discussion, we will not specify δE and consider it a constant with mass dimension one. However, we keep this physical view in mind. The entropy of the system, S , is defined as the logarithm of W

$$S_{E,SE,V,X}[j] = \ln\{W_{E,SE,V,X}[j]\}. \quad (4.2)$$

The variation of the entropy, which is the first law of thermodynamics, reads

$$dS_{E,SE,V,X}[j] = \beta dE + \beta P dV + \beta \int d^4x \sum_a \varphi_a^E(x) dj_a(x), \quad (4.3)$$

where on the right-hand side we define the classical field $\varphi_a^E(x)$

$$\varphi_a^E(x) = \frac{\partial S_{E,SE,V,X}[j]}{\partial j_a(x)}, \quad a = 1, 2, 3, \quad (4.4)$$

where j_a is the external source having support on the path C_a . In eq.(4.3), the quantities β and P correspond to the inverse temperature and the pressure in the thermodynamical limit. Later in the discussion, we will abbreviate the indices δE , V , X , if this does not cause confusion.

Let us define a quantity Σ' as a Legendre transformation of the entropy

$$\Sigma'_{E,SE,V,X}[\varphi_E] = S_{E,SE,V,X}[j] - \beta \int d^4x j_a(x) \varphi_a^E(x), \quad (4.5)$$

where the summation over 'a' is understood. The variation of Σ' is

$$d\Sigma'_{E,SE,V,X}[\varphi_E] = \beta dE + \beta P dV - \beta \int d^4x j_a(x) d\varphi_a^E(x). \quad (4.6)$$

Since Σ' has the property that in the limit $j \rightarrow 0$

$$\frac{\partial \Sigma'_E[\varphi]}{\partial \varphi_a^E(x)} = -\beta j_a(x) \rightarrow 0, \quad (4.7)$$

Σ' is called an effective action. The problem with Σ' in eq. (4.5) is that the diagrammatic technique, which is known in zero-energy QFT, is not available. Thus one has to choose whether one deals with Σ' in eq. (4.5) directly or finds another consistent definition of the effective action, for which the diagrammatic technique is available. We have chosen the latter way by requiring that the effective action satisfy the following two conditions. The first one is that one should recover eq.

(4.5) if one expands the expression with respect to the external fields, $j(x)$. The second is that the derivative of the new quantity with respect to φ_E vanishes in the limit $j \rightarrow 0$, similarly to eq. (4.7).

Firstly, because of the definition of the entropy

$$\Sigma'_E[\varphi_E] \cong \ln\{W_E[j] - \int d^4x j_a(x) \frac{\partial W_E[j]}{\partial j_a(x)}\}. \quad (4.8)$$

The diagrammatic structure of $W_E[j]$ is given by

$$W_E[j] = \delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} Z_\alpha^0 B_\alpha e^{iC_\alpha[j]}, \quad (4.9)$$

where Z_α^0 is the factor in eq. (2.2), B_α is the sum of bubble diagrams, which are not connected to external sources, and $C_\alpha[j]$ is the sum of connected diagrams (see Fig. 4), which is the generating functional of connected Green functions. Using the diagrammatic expression in eq. (4.9), eq. (4.8) is rewritten as

$$\Sigma'_E[\varphi_E] \cong \ln\left\{\delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} Z_\alpha^0 B_\alpha \exp\left[i\left(C_\alpha[j] - \int d^4x j_a(x) \frac{\partial C_\alpha}{\partial j_a(x)}\right)\right]\right\}. \quad (4.10)$$

Let us define

$$\begin{aligned} \bar{\phi}_a^\alpha(x) &\equiv \epsilon_a \frac{\partial C_\alpha[j]}{\partial j_a(x)}, \quad \epsilon_a = \begin{cases} 1 & a=1 \\ -1 & a=2 \\ \epsilon(\alpha) & a=3 \end{cases}, \\ \Gamma_\alpha[\phi] &\equiv C_\alpha[j] - \int d^4x j(x) \bar{\phi}_\alpha(x), \end{aligned} \quad (4.11)$$

where $\epsilon(\alpha)$ is a step-function, $\epsilon(\alpha) = \alpha/|\alpha|$ for $\alpha \neq 0$ and 0 for $\alpha = 0$. Notice that the time integration in the definition of Γ_α is over the path C , because of the definition of ϕ^a . The quantity $\Gamma_\alpha[\phi]$, as usual, is the generating functional of the one-particle irreducible vertex depending on the parameter α . The diagrammatic structure in eq. (4.10) is clear. Thus, we define the effective action by

$$\Sigma_E[\varphi_E] \equiv \ln\left\{\delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} Z_\alpha^0 B_\alpha e^{i\Gamma_\alpha[\phi]}\right\}. \quad (4.12)$$

The relation between the classical fields φ_E defined in eq. (4.4) and $\bar{\phi}_a^\alpha$ defined in eq. (4.11) is

$$\bar{\phi}_a^\alpha(x) = \frac{\epsilon_a}{iW_\alpha[j]} \int dE e^{-i\alpha E} \partial_E W_E[j] \varphi_E^a(x), \quad (4.13)$$

where W_α is defined as an integrand of α -integral of W_E ,

$$W_E[j] = \int \frac{d\alpha}{2\pi} e^{i\alpha E} W_\alpha[j].$$

Now, we have to check if the effective action defined in eq. (4.12) satisfies the two requirements mentioned before. The first one is obvious from the above derivation. The second requirement, that Σ_E satisfy the same relation as eq. (4.7) in the limit $j \rightarrow 0$, is non-trivial. From eq. (4.13), one can obtain

$$\frac{\partial \tilde{J}_a^0(x)}{\partial \varphi_E^a(x')} = \epsilon_a \alpha \delta^4(x - x'). \quad (4.14)$$

Using eq. (4.14) and the identity

$$\frac{\partial \Gamma_a}{\partial \varphi_E^a(x)} = -j_a(x),$$

one can easily derive

$$\frac{\partial \Sigma_E[\varphi_E]}{\partial \varphi_E^a(x)} = -j_a(x) \frac{\partial}{\partial E} \Sigma_E[\varphi_E] \Big|_{j \rightarrow 0}. \quad (4.15)$$

Since eq. (4.15) has the same limit as that in eq. (4.7), eq. (4.15) is considered to be a generalization of the Euler-Lagrange equation and Σ_E defined in eq. (4.12) satisfies the requirements for being an effective action.

As far as the above two requirements on the effective action are concerned, the definition in eq. (4.12) might not be a unique one. However, in the sense that the ordinary diagrammatic method is available, it is a unique choice of the effective action.

The variation of the effective action is defined similarly to the one in eq. (4.6):

$$d\Sigma_E[\varphi_E] = \tilde{\beta} dE + \tilde{\beta} \tilde{P} dV - \tilde{\beta} \int d^4x j_a(x) d\varphi_E^a(x). \quad (4.16)$$

This variation is defined consistently with eq. (4.15). In general, the tilded quantities, $\tilde{\beta}$ and $\tilde{P}$, are different from those in eq. (4.3), defined in the variation of the entropy. However, it is not difficult to check that in the absence of the classical field, $\varphi_E = 0$, tilded quantities are equal to those in eq. (4.3). Therefore, we can consider $\tilde{\beta}$ and $\tilde{P}$ as generalizations of β and P in the presence of a non-zero classical field.

The physical meaning of eq. (4.15), which is the generalization of the Euler-Lagrange equation, is clear. Solving eq. (4.15) in the limit $j \rightarrow 0$ implies finding field configurations φ for which the effective action $\Sigma_E[\varphi]$ is stationary. As a physical solution, we have to choose the one which maximizes Σ_E . This procedure is the same as Boltzmann method in thermodynamics, where one looks for the particle distribution which maximizes the entropy to obtain an equilibrium distribution.

5 Perturbative Calculation of Entropy

In this section, we will show the perturbative calculation of the entropy defined by eqs. (4.1) and (4.2), using ϕ^4 theory. Earlier work on a similar problem was done by Kapusta.^[21] First, we will calculate the entropy up to the order of $O(\lambda^1)$ or $O(\hbar^2)$. Next, we will discuss two methods to explicitly evaluate this $O(\lambda^1)$ expression for the entropy. The first is the one discussed by Weldon^[10], which is precise but with which it is difficult to obtain an analytic expression in general. The second is the saddle point or the stationary phase method, which is rather easy to calculate and the results are suitable for analysis of the thermodynamical properties of the system.

The entropy defined in eq. (4.2) is given in path integral form

$$e^{S_E} = \delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} \mathcal{N}_\alpha \int D\phi D\Pi e^{i \int d^4x [L + j\phi]} \Big|_{j \rightarrow 0}.$$

After integration over fields, one gets

$$\begin{aligned} e^{S_E} &= \delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} \mathcal{N}'_\alpha \exp \left\{ -\frac{1}{2} \ln \text{Det} [\tilde{D}^{-1}] \right\} \\ &\times \exp \left(i \epsilon_a \int d^4x \mathcal{L} \left(\frac{\delta}{i\delta j_a} \right) \right) \exp \left(-\frac{i}{2} \int d^4x d^4x' j_a(x) \tilde{D}_\alpha^{ab}(x - x') j_b(x') \right) \\ &\times \exp \left(i \int_{C_3} d^4x \mathcal{L} \left(\frac{\delta}{i\delta j_3} \right) \right) \exp \left(-\frac{i}{2} \int d^4x d^4x' j_3(x) \tilde{D}_\alpha^{33}(x - x') j_3(x') \right) \Big|_{j \rightarrow 0}, \end{aligned}$$

where summation over $a = 1, 2$ is understood. The determinant is the contribution of the one-loop diagram in Fig.5(a). The calculation of the determinant is given in Appendix C. As a result, the contributions of the C_1 and C_2 paths cancel with each other and only the C_3 path contributes. The order λ^1 or two-loop contribution comes from the diagram Fig.5(b). The detailed calculation is found in Appendix C. Again, the $C_1 + C_2$ paths cancel and the $O(\lambda)$ contribution comes from the C_3 path. Since we are calculating an equilibrium quantity, it is natural that the $C_1 + C_2$ part does not contribute. As we have mentioned before, this is a consequence of the completeness of the set of states inserted at each time in the path integral. After mass renormalization and the subtraction of the zero-point energy by proper definition of the normalization factor $\mathcal{N}'$, the entropy of order λ^0 plus that of order λ^1 becomes

$$\begin{aligned} e^{S_E} &= \delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} \\ &\times \exp \left\{ \frac{V}{(2\pi)^3} \int d^3k \sum_{n=1}^{\infty} \frac{1}{n} e^{-i\alpha n \omega_k} - \frac{i}{8} \lambda \alpha V \left[\int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} f_\alpha^{(-)}(\omega_k) \right]^2 \right\}. \end{aligned} \quad (5.1)$$

The next step is to evaluate eq. (5.1). As we have mentioned before, we will evaluate this expression by two methods. Let us first discuss the more precise method of the two. The same method was discussed by Waldon.^[10] By expanding the exponent, eq. (5.1) is written as

$$W_E[0] = \delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \quad (5.2)$$

$$\times \left(\frac{\mathcal{V}}{(2\pi)^2} \int d^3k \sum_{n=1}^{\infty} \frac{1}{n} e^{-i\alpha n \omega_k} \right)^N \left(-\frac{i\lambda\alpha\mathcal{V}}{8} \left[\int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} \sum_{m=1}^{\infty} e^{-i\alpha m \omega_k} \right]^2 \right)^M.$$

Therefore, one has to evaluate the integral in the general (N, M) -term

$$I_E^{N,M} = \prod_{i=1}^N \left(\frac{\mathcal{V}}{(2\pi)^3} \int d^3k_i \sum_{n_i=1}^{\infty} \frac{1}{n_i} \right) \left(-\frac{\lambda\mathcal{V}}{8} \right)^M \prod_{j=1}^M \left(\int \frac{d^3p_j}{(2\pi)^3} \frac{1}{\omega_{p_j}} \sum_{m_j=1}^{\infty} \right)$$

$$\times \frac{d^M E}{dE^M} \delta(E - \sum_{i=1}^N n_i \omega_{k_i} - \sum_{j=1}^M m_j \omega_{p_j}).$$

For massive bosons, numerical calculations will be needed to evaluate this quantity. However, for the massless bosons, $I_E^{N,M}$ is obtained analytically

$$I_E^{N,M} = \frac{1}{(3N+3M-1)!} \left(\frac{\mathcal{V}}{E} \right)^N \left(-\frac{\lambda\mathcal{V}E^3}{8 \cdot 144} \right)^M. \quad (5.3)$$

Thus, the entropy of the massless real scalar boson is given by

$$W_E[0]|_{O(\lambda)} = \delta E \sum_{N,M=0}^{\infty} \frac{1}{N!M!(3N+3M-1)!} \left(\frac{\mathcal{V}}{E} \right)^N \left(-\frac{\lambda\mathcal{V}E^3}{8 \cdot 144} \right)^M. \quad (5.4)$$

The right-hand side of eq. (5.4) is not represented by any known simple function and the analysis of the system using (5.4) is still not easy. However, if one is willing to use the computer, the analysis of eqs. (5.2) and (5.4) is not difficult because there is a strong convergence factor $1/N!M!$ for large N and M .

Although the evaluation in eq. (5.2) is precise, the procedure is not simple. Instead of doing that, we will use the saddle point approximation to evaluate eq. (5.1).[†] For simplicity, we are going to investigate the massless case, which is also considered as a relativistic limit of the massive theory. After performing the momentum integration in the exponent in eq. (5.1), one finds

$$W_E[0]|_{O(\lambda)} = \delta E \int \frac{d\alpha}{2\pi} \exp \left\{ i\alpha E + \frac{1}{(i\alpha)^3} \left(\frac{\mathcal{V}}{90} - \frac{\lambda\mathcal{V}}{8(12)^2} \right) \right\}. \quad (5.5)$$

[†]The same calculation is found in Ref. [21]

Using the saddle point method, the right-hand side is evaluated and one obtains

$$S_E[0]|_{O(\lambda)} = \frac{4}{3} \left[\left(\frac{\pi^2}{30} - \frac{\lambda}{384} \right) \mathcal{V} E^3 \right]^{1/4} + \ln \frac{\delta E}{E} + \frac{1}{8} \ln \left[\left(\frac{\pi^2}{30} - \frac{\lambda}{384} \right) \mathcal{V} E^3 \right], \quad (5.6)$$

where we have assumed that the energy of the system E and the volume $\mathcal{V}$ are sufficiently large keeping $E/\mathcal{V}$ finite. For example, the inverse temperature is derived from eq. (5.1)

$$\beta = \frac{\partial S_E}{\partial E} = \left(\frac{\pi^2}{30} - \frac{\lambda}{384} \right)^{1/4} \left(\frac{\mathcal{V}}{E} \right)^{1/4} - \frac{5}{8} \frac{1}{E}. \quad (5.7)$$

If one neglects the second term on the right-hand side of eq. (5.7), one finds the Stefan-Boltzmann law.

Let us summarize this section. We carried out a perturbative calculation of entropy up to order λ^1 . In the general situation, entropy should be evaluated using eq. (5.4) and, for that purpose, numerical calculations might be needed. In the case where E and $\mathcal{V}$ are sufficiently large, one can evaluate the entropy using the saddle point method and find results consistent with the canonical methods in the thermodynamical limit.

6 The One-loop Effective Action and the Behaviour of Symmetry at High Energy

The effective action is known to be a very useful quantity for investigating the symmetry properties of the system.^[22, 23] In this section, we will discuss the effective action of ϕ^4 theory at finite energy in the one-loop approximation. At the end of the section, we will consider a system having spontaneous symmetry breaking at zero energy. We will discuss the fact that symmetry restoration happens at high energy density.

We consider real scalar boson field theory with a ϕ^4 interaction. This time, we will consider the counterterms explicitly. The Lagrangian reads

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 - \frac{1}{2} \delta m^2 \phi^2 - \frac{\delta \lambda}{4!} \phi^4,$$

$$\delta m^2 = \hbar \delta_1^2 + \hbar^2 \delta_2^2 \dots, \delta \lambda = \hbar \lambda_1 + \hbar^2 \lambda_2 \dots,$$

where δm^2 and $\delta \lambda$ are counterterms of mass and coupling constant renormalizations. In the definitions of δm^2 and $\delta \lambda$, δm_i^2 and $\delta \lambda_i$ mean order $\hbar^i$ counterterms.

The complication in our present calculation is that we have to do an $n\alpha^n$ integration. The classical field $\bar{\phi}_\alpha^a$, defined in eq. (4.11) depends on α . The general procedure might be to expand $\bar{\phi}_\alpha^a$ as the sum over a complete set of some orthogonal functions of α , $\{f_{i(\alpha)}\}$, namely $\bar{\phi}_\alpha^a = \sum_i \bar{\phi}^{a,i} f_{i(\alpha)}$. Each coefficient $\bar{\phi}^{a,i}$ is determined by the condition of maximizing the effective action, eq. (4.15). For simplicity, in this paper, we will consider only the zero-mode, namely $\bar{\phi}_\alpha^a(x) = \bar{\phi}^a(x)$, which does not depend on α . In this case, eq. (4.13) gives the relation between $\bar{\phi}^{1,2}$ and $\varphi_E^{1,2}$ as

$$\varphi_E^a(x) = \epsilon_a \bar{\phi}^a(x), \quad a = 1, 2,$$

but φ_E^3 is related to $\bar{\phi}$ in a complicated way.

The quantity we have to evaluate is

$$e^{\Sigma_E[\phi]} = \delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} \mathcal{N}_\alpha \int D\phi D\Pi e^{i \int d^4x (\mathcal{L} + i(\phi - \bar{\phi}))}. \quad (6.1)$$

For the purpose of evaluation, we will use the method described in ref.[23]. By changing variables $\phi - \bar{\phi} \rightarrow \phi$ and integrating over the field, the tree and the one-loop contributions of eq. (6.1) becomes

$$e^{\Sigma_E[\phi]} = \delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} \mathcal{N}_\alpha' \times \exp \left\{ i \int d^4x \mathcal{L}(\bar{\phi}) - \frac{1}{2} \ln \text{Det} \left(\bar{D}^{-1}(\bar{\phi}) \right) \right\}, \quad (6.2)$$

where $\mathcal{L}(\bar{\phi})$ is the tree-level action of the classical field and $\bar{D}^{-1}(\bar{\phi})$ in the one-loop contribution is the same quantity defined in Appendix C, except replacing m^2 by $m^2 + \lambda \bar{\phi}^2/2$.

In order to derive a quantity corresponding to the effective potential, let us consider the case where $\bar{\phi}$ does not depend on the coordinate. As for the tree-action part, the C_1 and C_2 parts cancel each other and one obtains

$$\int d^4x \mathcal{L}(\bar{\phi}) = -\alpha \left(\frac{1}{2} (m^2 + \hbar \delta m_1^2) \bar{\phi}^2 + \frac{\lambda + \hbar \delta \lambda_1}{4!} \bar{\phi}^4 \right).$$

For one-loop contributions, again the C_1 and C_2 parts cancel. Adding tree and one-loop contributions, eq. (6.2) becomes

$$e^{\Sigma_E[\phi]} = \delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} \mathcal{N}_\alpha' \times \exp \left\{ i\alpha V \left[-\frac{m^2 + \hbar \delta m_1^2}{2} \bar{\phi}^2 - \frac{\lambda + \hbar \delta \lambda_1}{4!} \bar{\phi}^4 + \frac{\hbar}{2} \frac{\Gamma(-n/2)}{(4\pi)^{n/2}} (m^2)^{n/2} \right] \right. \\ \left. + \hbar V \int \frac{d^3k}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{1}{n} e^{-in\omega_k'} + O(\hbar^2) \right\}, \quad (6.3)$$

$$m^2 = m^2 + \frac{\lambda}{2} \bar{\phi}^2, \quad \omega_k' = \sqrt{k^2 + m^2 + \lambda \bar{\phi}^2/2},$$

where we are using dimensional regularization $n \rightarrow 4$. In the above equations, we are keeping terms up to $O(\hbar)$.

For the purpose of renormalization, let us define

$$Y(\bar{\phi}) = \frac{m^2 + \hbar \delta m_1^2}{2} \bar{\phi}^2 + \frac{\lambda + \hbar \delta \lambda_1}{4!} \bar{\phi}^4 - \frac{\hbar}{2} \frac{\Gamma(-n/2)}{(4\pi)^{n/2}} (m^2)^{n/2}, \quad (6.4)$$

and impose renormalization conditions on $Y(\bar{\phi})$, [22]

$$\frac{d^2 Y(\bar{\phi})}{d\bar{\phi}^2} \Big|_{\bar{\phi}=0} = m^2, \quad (6.5)$$

$$\frac{d^4 Y(\bar{\phi})}{d\bar{\phi}^4} \Big|_{\bar{\phi}=M} = \lambda,$$

where M is an energy scale we are interested in. After determining δm_1^2 and $\delta \lambda_1$ according to eq. (6.5) and subtracting the zero-point energy by proper definition of $\mathcal{N}_\alpha'$, one finds

$$Y(\bar{\phi}) = \frac{m^2}{2} \left(1 - \frac{\lambda}{\hbar 64\pi^2} \right) \bar{\phi}^2 + \frac{1}{4!} (\lambda + \hbar \lambda') \bar{\phi}^4 \\ + \frac{\hbar}{64\pi^2} \left\{ \left(m^2 + \frac{\lambda}{2} \bar{\phi}^2 \right)^2 \ln \frac{m^2 + \frac{\lambda}{2} \bar{\phi}^2}{m^2} + \frac{\lambda^2}{4} \bar{\phi}^4 \ln \left(\frac{m^2}{m^2 + \frac{\lambda}{2} M^2} \right) \right\},$$

where λ' is a function of λ , M and m , which becomes in the relativistic limit $M \gg m$

$$\lambda' \xrightarrow{M \gg m} -\frac{25}{64\pi^2} \lambda^2. \quad (6.6)$$

Thus we obtain the effective action

$$e^{\Sigma_E[\phi]} = \delta E \int \frac{d\alpha}{2\pi} e^{i\alpha E} \exp \left\{ -i\alpha Y(\bar{\phi}) + \hbar V \int \frac{d^3k}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{1}{n} e^{-in\omega_k'} \right\}. \quad (6.7)$$

The evaluation of $\Sigma_E[\phi]$ is still complicated. If one wants to evaluate it precisely, one has to expand the exponent on the right-hand side of eq.(6.7) and carry out the α -integration:

$$e^{\Sigma_E[\phi]} = \delta E \sum_{N=0}^{\infty} \frac{1}{N!} I_N \\ I_N = \prod_{i=1}^N \left(\int \frac{V d^3k_i}{(2\pi)^3} \sum_{n_i=1}^{\infty} \right) \frac{\delta(E - V Y(\bar{\phi}) - \sum_i n_i \omega_{k_i}')}{n_1 n_2 \cdots n_N}.$$

Probably, a time-consuming procedure would be required to analyse this equation. Some remarks on this precise evaluation are given in Appendix D. Instead of doing

it we are here going to consider the relativistic limit where $m, \bar{\phi} \ll M, E$ and use the saddle point method.

By keeping the first order in $(m^2 + \frac{\lambda}{2}\bar{\phi}^2)$, eq. (6.7) becomes

$$e^{\Sigma_E[\varphi]} = \delta E \int \frac{d\alpha}{2\pi} \exp \left\{ i\alpha (E - \nu Y(\bar{\phi})) + i \frac{\nu}{\pi^2} \frac{1}{\alpha^3} \frac{\pi^4}{90} + \frac{i\nu}{4\pi^2} (m^2 + \frac{\lambda}{2}\bar{\phi}^2) \frac{1}{\alpha} \frac{\pi^2}{6} \right\}. \quad (6.8)$$

The condition under which the saddle point method is applicable is

$$E - \nu Y(\bar{\phi}), \nu \gg 1 \text{ and } (E - \nu Y(\bar{\phi}))/\nu = \text{finite}. \quad (6.9)$$

The saddle point method gives us the effective action from eq. (6.8)

$$\begin{aligned} \Sigma_E[\varphi] = & \frac{4}{3} \left(\frac{\pi^2 \nu (E - \nu Y(\bar{\phi}))^3}{30} \right)^{1/4} \\ & - \frac{1}{2} \ln \left\{ 2 \left(\frac{\pi^2 \nu (E - \nu Y(\bar{\phi}))^3}{30} \right)^{-1/4} \left(\frac{E - \nu Y(\bar{\phi})}{\delta E} \right)^2 \right\} \\ & - \frac{2\nu}{48} \left(m^2 + \frac{\lambda}{2}\bar{\phi}^2 \right) \left(\frac{30}{\pi^2 \nu} (E - \nu Y(\bar{\phi})) \right)^{1/4}. \end{aligned} \quad (6.10)$$

For example, the inverse temperature in the presence of the classical field defined in eq. (4.16) is given by

$$\bar{\beta} = \frac{\partial \Sigma_E}{\partial E} \sim \left(\frac{\pi^2 \nu}{30(E - \nu Y(\bar{\phi}))} \right)^{1/4} - \frac{5}{8} \frac{1}{E - \nu Y(\bar{\phi})}, \quad (6.11)$$

where terms proportional to $(m^2 + \lambda\bar{\phi}^2/2)$ are neglected. It is useful to compare eq. (6.11) with eq. (5.7) in section 5. The difference is the replacement: $E \rightarrow E - \nu Y(\bar{\phi})$.

It is interesting to see the symmetry behaviour at high energy density. For this purpose, let us examine the case $m^2 < 0$, where spontaneous symmetry breaking occurs at zero energy. Since we are considering the classical field which is independent of the coordinate, we define the effective potential by

$$V_{eff}^E[\bar{\phi}] = -\frac{1}{\nu} \Sigma_E[\bar{\phi}].$$

Similarly to the discussion of Dolan and Jackiw, the condition that the symmetry is restored is^[4]

$$\frac{d^2 V_{eff}^E[\bar{\phi}]}{d\bar{\phi}^2} \Big|_{\bar{\phi}=0} > 0.$$

This condition gives in our approximation

$$\frac{\lambda}{24} \left(\frac{30E}{\pi^2 \nu} \right)^{1/2} > |m^2|.$$

Therefore, the critical energy density ε_c above which the symmetry is restored is

$$\varepsilon_c \sim \frac{\pi^2}{30} \left(\frac{24|m^2|}{\lambda} \right)^2.$$

Using $\bar{\beta}$ in eq. (6.11), $\bar{\beta}$ at the critical energy density satisfies

$$\bar{\beta}^{-2} |E/\nu = \varepsilon_c| \cong \frac{24|m^2|}{\lambda}. \quad (6.12)$$

Thus we recover the result of Dolan and Jackiw. In eq. (6.12), we have neglected finite-energy corrections, which give non-zero contributions in general.

In this section, we have calculated the one-loop effective action at finite energy in ϕ^4 theory. In the case with spontaneous symmetry breaking at zero energy, we showed that symmetry restoration occurs at high energy, which is consistent with the result of finite-temperature QFT in the thermodynamical limit. Before closing this section, we would like to make several comments. We have performed only a one-loop analysis. If one wants to go beyond the one-loop approximation, one might be required some ideas to calculate the quantities effectively. Probably, the tadpole method of Weinberg would be useful.^[5, 9] Another point is that in the above procedure to obtain the effective action, the role of φ^a is not clear. This is mainly because φ^3 is related to $\bar{\phi}^3$ in a non-trivial way even in the case where $\bar{\phi}^3$ is independent of α .

7 Conclusion and Discussion

Let us first summarize the results we have obtained so far. In this paper, quantum field theory at finite energy is formulated in the functional integral approach using the microcanonical distribution and the general formalism of the effective action is developed in the context of the grand microcanonical ensemble. We showed some sample computations using the real scalar field theory with ϕ^4 interaction. From these results, we deduce the general properties of our formalism. Firstly, as a consequence of the convolution property, the Green functions have the structure of the bootstrap. Secondly, as in finite-temperature field theory, the system prefers higher symmetry at high energy. The third point is that in the thermodynamical limit the microcanonical formulation gives equivalent results to the canonical one.

The merit of the microcanonical formulation is that we can analyse isolated finite systems. In the previous sections, we used two methods to evaluate the physical quantities. The saddle point method is available when the thermodynamical limit is a good approximation. Thus it is not appropriate to investigate a finite system. The other one, which we call the Taylor expansion method, is valid even in the finite system, although the procedure would be complicated.

As we have mentioned in the introduction, there are lots of places one can apply the present formulation. One of the interesting problems is thermodynamics in a black-hole background.[24] The relation to the thermo-field theory is discussed by Israel.[25] It is pointed out by York et al. that one should not use the thermodynamical limit in order to make the problem a well-posed one.[26] Therefore, the microcanonical formulation of quantum field theory will be useful to investigate this system.

Another interesting problem would be the finite-energy effect on the phase transition in the Landau-Ginzburg model with ϕ^3 and ϕ^4 potentials.[27] Apart from the relation to nucleation theory, this problem is interesting because it is related to the bag model if one couples the Landau-Ginzburg model to fermions.[16, 28]

In a similar way as we have done for ϕ^4 theory in this paper, one can formulate the quantum field theory at finite energy for gauge theories such as QCD. In this case, the formulation would be useful in describing the dynamical evolution of a strongly-interacting system, e.g. ion-ion collisions. One could also study the behaviour of QCD running coupling constant as a function of energy in analogy with the temperature behaviour.

Although we discussed the equilibrium properties in this paper, one can gen-

eralize it to the non-equilibrium situation and see how the system approaches the equilibrium, which is appropriate for discussions of quark-gluon plasma and the early universe.

Acknowledgements

We would like to thank R. Hagedorn, A. Niemi and H. Satz for several useful discussions. One of us (I.S.) is grateful to *ICTP*, *IAEA* and *UNESCO* for support.

Appendix A. Propagator of the Scalar Field

Let us consider a real scalar field. Assuming that the volume of the system is sufficiently large, the Fourier components of the free field and their commutation relations are given by

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left(a_k e^{i(kx - \omega_k x_0)} + a_k^\dagger e^{-i(kx - \omega_k x_0)} \right), \quad (A.1)$$

$$[a_k, a_{k'}^\dagger] = \delta^3(k - k'), \quad \text{others} = 0.$$

The two-point Green function in the media of finite-energy E in the time-path formalism is defined as

$$G_E(x - y) = \int \frac{d\alpha}{2\pi} e^{i\alpha E} < T_c(\phi(x)\phi(y)) >_\alpha,$$

where $< >_\alpha$ on the right-hand side means

$$< O' >_\alpha \equiv \text{Tr} \left(e^{-i\alpha H} O' \right) / \text{Tr} \left(e^{-i\alpha H} \right).$$

Since the parameter α has an infinitesimal negative part, which is discussed in section 3, the trace on the right-hand side of the above expression is well-defined. Because of the contour ordering, we introduce the contour step function $\theta_c(x_0 - y_0)$, which is 1 if x_0 is closer to the end of the contour than y_0 , 1/2 when $x_0 = y_0$ and otherwise 0. Then

$$< T_c(\phi(x)\phi(y)) >_\alpha = \theta_c(x_0 - y_0) < \phi(x)\phi(y) >_\alpha + \theta_c(-x_0 + y_0) < \phi(y)\phi(x) >_\alpha. \quad (A.2)$$

Through the standard procedure, one obtains

$$\begin{aligned} < \phi(x)\phi(y) >_\alpha &= \int \frac{d^4k}{(2\pi)^3} e^{-ik(x-y)} \epsilon(k_0) \delta(k^2 - m^2) (1 + f_\alpha^{(-)}(k_0)), \\ < \phi(y)\phi(x) >_\alpha &= \int \frac{d^4k}{(2\pi)^3} e^{-ik(x-y)} \epsilon(k_0) \delta(k^2 - m^2) f_\alpha^{(-)}(k_0), \end{aligned}$$

where we are using the metric $(+, -, -, -)$ and $\epsilon(k)$ equals $k/|k|$ for $k \neq 0$ and 0 for $k = 0$. The quantity $f_\alpha^{(-)}$ is defined in eq. (3.5). Let us define $D^{>}$ and $D^{<}$ by

$$\begin{aligned} iD_\alpha^{>}(k) &= 2\pi\epsilon(k_0)\delta(k^2 - m^2)f_\alpha^{(+)}(k_0), \\ iD_\alpha^{<}(k) &= 2\pi\epsilon(k_0)\delta(k^2 - m^2)f_\alpha^{(-)}(k_0), \end{aligned} \quad (A.3)$$

and their Fourier transformation by $D_\alpha(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} D_\alpha(k)$. Since the two-point Green function and the propagator are related by

$$G_E(x - y) = \int \frac{d\alpha}{2\pi} e^{-i\alpha E} iD_\alpha(x - y),$$

the propagator is obtained as

$$D_\alpha(x - y) = \theta_c(x_0 - y_0) D_\alpha^{>}(x - y) + \theta_c(-x_0 + y_0) D_\alpha^{<}(x - y). \quad (A.4)$$

For the propagators of fields on the path $C_1 + C_2$, the result in eq. (3.5) is found by Fourier-transforming the above expression and using the definition of $\tilde{D}$ in eq. (3.3). The propagator of fields on C_3 in eq. (3.9) is obtained directly from eq. (A.4).

As we have discussed in section 2, the propagator has to satisfy the periodic boundary condition in eq. (2.3). In order to check this periodicity, let us discuss keeping T finite in the definition of the path C . For $x_1 = (-T, \mathbf{x}_1) \in C_1$ and $\forall x_2 \in C$, one finds

$$D_\alpha(x_1 - x_2) = D_\alpha^{<}(-T - x_{2,0}, \mathbf{x}_1 - \mathbf{x}_2), \quad (A.5)$$

where we have used the fact that x_1 is placed at the starting point of the path C . For $x_1 = (\alpha - T, \mathbf{x}_1) \in C_3$, at the endpoint of the path C , and $\forall x_2 \in C$, the propagator becomes

$$D_\alpha(x_1 - x_2) = D_\alpha^{>}(-T + \alpha - x_{2,0}, \mathbf{x}_1 - \mathbf{x}_2). \quad (A.6)$$

Using the expressions of $D^{>}$ and $D^{<}$ in eq. (A.3) and the property $f_\alpha^{(-)}(-u) = -f_\alpha^{(+)}(u)$ for $u > 0$, one finds

$$D_\alpha^{>}(-T + \alpha - x_{2,0}, \mathbf{x}_1 - \mathbf{x}_2) = D_\alpha^{<}(-T - x_{2,0}, \mathbf{x}_1 - \mathbf{x}_2).$$

Therefore, the right-hand side of eq. (A.5) is equal to the right-hand side of eq. (A.6). Thus we have found the desired periodicity.

Appendix B. Decoupling of the C_3 Path

The decoupling of the C_3 part from $C_1 + C_2$ is equivalent to the vanishing of the propagator connecting C_3 , $a = 1, 2$ and C_3 . After integrating out the field, the part connecting sources on C_3 and C_3 is given by

$$\begin{aligned} & i \int_{C_3} d^4x \int_{C_3} d^4x' j_a(x) D_\alpha(x - x') j_3(x') \\ &= i \int \frac{d^4k}{(2\pi)^4} e^{-ik_0 T} \int_{C_3} d^4x \int_0^\infty dt' \int d^3x' e^{-ik(x-x')} j_a(x) D_\alpha^<(k) j_3(-T + t', x'), \end{aligned} \quad (\text{B.1})$$

where we have changed the integration variable over time on C_3 . Since j_3 is chosen to be a smooth function of $x' \in (C_3, \mathbf{R}^3)$, eq. (B.1) has a form in which we can use the Riemann-Lebesgue theorem with respect to k_0 integration. Thus, implemented with a standard adiabatic switching-on and -off mechanism for the external source, the right-hand side of eq. (B.1) vanishes in the limit $T \rightarrow \infty$.

Appendix C. Evaluation of the Entropy

This appendix contains the calculation of the one-loop and two-loop order- λ^1 contributions to the entropy.

Let us start with the one-loop contribution, namely the determinant $\ln \text{Det}(\bar{D}^{-1})$. Because of decoupling between $C_1 + C_2$ and C_3 , the determinant is separated as

$$\ln \text{Det}(\bar{D}^{-1}) = \ln \text{Det}(\bar{D}_{(C_1+C_2)}^{-1}) + \ln \text{Det}(\bar{D}_{(C_3)}^{-1}),$$

where " Det " contains the determinant with respect to coordinate space. In order to calculate the $C_1 + C_2$ part, the form of propagator in eq. (3.7) is not convenient. We rewrite it in the form:

$$\begin{aligned} & i \bar{D}_\alpha(k) \\ &= \theta(k_0) \begin{pmatrix} 1 & f^{(-)} \\ -1 & -f^{(+)} \end{pmatrix} \begin{pmatrix} i/(k^2 - m^2 + i\epsilon) & 0 \\ 0 & i/(k^2 - m^2 - i\epsilon) \end{pmatrix} \begin{pmatrix} f^{(+)} & -f^{(-)} \\ -1 & 1 \end{pmatrix} \\ &+ \theta(-k_0) \begin{pmatrix} f^{(+)} & -1 \\ -f^{(-)} & 1 \end{pmatrix} \begin{pmatrix} i/(k^2 - m^2 + i\epsilon) & 0 \\ 0 & i/(k^2 - m^2 - i\epsilon) \end{pmatrix} \begin{pmatrix} 1 & -1 \\ f^{(-)} & -f^{(+)} \end{pmatrix}, \end{aligned} \quad (\text{C.1})$$

where $f^{(\pm)}$ are those defined in eq. (3.5) with their arguments being ω_k . The inverse of (C.1) is

$$\begin{aligned} & \bar{D}_\alpha^{-1}(k) \\ &= i\theta(k_0) \begin{pmatrix} -i(k^2 - m^2) + \epsilon f^{(+)} + f^{(-)} & 2\epsilon f^{(-)} \\ 2\epsilon f^{(+)} & i(k^2 - m^2) + \epsilon(f^{(+)} + f^{(-)}) \end{pmatrix} \\ &+ i\theta(-k_0) \begin{pmatrix} f^{(-)} \leftrightarrow f^{(+)} \end{pmatrix}. \end{aligned}$$

Therefore, the determinant becomes

$$\begin{aligned} \ln \text{Det}(\bar{D}^{-1}) &= \int \frac{d^4k}{(2\pi)^4} \ln \text{Det}(\bar{D}^{-1}(k)) \\ &= \int \frac{d^4k}{(2\pi)^4} \ln(k^2 - m^2 + i\epsilon)(-k^2 + m^2 + i\epsilon), \end{aligned} \quad (\text{C.2})$$

where det is performed with respect to the 2×2 matrix. Noticing the position of poles, the right-hand side is easily shown to vanish.^[9]

For the purpose of obtaining the determinant of the inverse propagator on the C_3 path, let us define $F[\mu^2]$ by:

$$F[\mu^2] \equiv -\frac{1}{2} \text{Tr} \left[L_n \left(\bar{D}_\sigma^{-1}(\mu^2) \right) \right], \quad (\text{C.3})$$

where $\bar{D}_\sigma^{-1}(\mu^2)$ is a shorthand notation of $\bar{D}_\sigma^{33-1}|_{m^2=\mu^2}$ and Tr is also done over the coordinate space. The difference of $F[\mu^2]$ between two masses m_1 and m_2 is given by

$$F[m_1^2] - F[m_2^2] = \frac{1}{2} \text{Tr} \int_{m_2^2}^{m_1^2} d\mu^2 \bar{D}_\sigma(\mu^2).$$

To go to the right-hand side in the above equation, we have used the fact that $\bar{D}_\sigma^{-1}(\mu^2)$ is linear in μ^2 . The μ^2 -integral is carried out and the contributions of $\mu^2 = m_1^2$ and $\mu^2 = m_2^2$ are separated in the result. Thus one finds up to a trivial additive constant

$$F[m^2] = -\frac{1}{2} \left\{ -i \frac{\Gamma(-n/2)}{(4\pi)^{n/2}} (m^2)^{n/2} + i \int \frac{d^3k}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{2i}{\alpha n} e^{-i\alpha\omega_k} \right\} \cdot \alpha \mathcal{V}|_{n \rightarrow 4}, \quad (\text{C.4})$$

where the factor $\alpha \mathcal{V}$ comes from $\int_{C_3} d^4x$.

Next we will consider the two-loop contribution coming from the diagram in Fig. 5(b). As we have mentioned in the text, the $C_1 + C_2$ part cancels. The contribution of the C_3 path is given by

$$\frac{i}{8} \lambda \alpha \mathcal{V} \left(\bar{D}_\sigma^{33}(0) \right)^2. \quad (\text{C.5})$$

Using the dimensional regularization, eq. (C.5) is evaluated:

$$\frac{i}{8} \lambda \alpha \mathcal{V} \left[i \frac{\pi^{n/2} \Gamma(1 - n/2)}{16\pi^4 (m^2)^{1-n/2}} + i \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} f_\alpha^{(-)}(\omega_k) \right]^2.$$

The divergences in one loop and two loops are removed by mass renormalization and the subtraction of zero-point energy by defining the normalization factor properly. Finally one finds eq. (5.1).

Appendix D. Evaluation of the One-loop Effective Action

In this appendix, we will explain the evaluation of the one-loop effective action and the procedure to renormalize it briefly. At the end, a comment on the evaluation of the effective action using the Taylor-expansion method is given.

The evaluation of the determinant in eq. (6.2) is the same as we explained in appendix C, except replacing m^2 by $m^2 + \lambda\bar{\phi}^2/2$. Adding the tree-level classical action, one obtains eq. (6.3).

The renormalization condition in eq. (6.5) is the one discussed in the appendix of Ref.[22]. This renormalization condition is suitable for investigating the relativistic limit of the system. The counterterms determined by the condition in eq. (6.5) are

$$\begin{aligned}\delta m_1^2 &= \lambda \frac{m^2}{(4\pi)^2} \left(\frac{1}{2\epsilon} + \frac{1}{2} - \frac{\gamma}{2} - \frac{1}{2} \ln \frac{m^2}{4\pi} \right), \\ \delta \lambda_1 &= \frac{3\lambda^2}{16\pi^2} \left(\frac{1}{2\epsilon} + \frac{3}{4} - \frac{\gamma}{2} \right) - \frac{1}{4} \frac{6\lambda^2}{16\pi^2} \ln \frac{m^2 + \lambda M^2/2}{4\pi} + \lambda',\end{aligned}$$

where λ' is a function of m , M and λ , mentioned in the text, γ is the Euler constant and $\epsilon = 2 - n/2|_{n \rightarrow 4}$.

In section 6, we showed two ways to estimate the effective action in eq. (6.7). Here, we would like to comment on the more precise method, that is, expansion of the exponent of eq. (6.7). As we have mentioned in the text, the evaluation in the general case is difficult. In the case where $(m^2 + \lambda\bar{\phi}^2/2)$ is small, one can expand with respect to $(m^2 + \lambda\bar{\phi}^2/2)$ and approximate $e^{\mathcal{L}_S}$ by low-order terms. For example, up to first order in $(m^2 + \lambda\bar{\phi}^2/2)$, one obtains

$$\begin{aligned}e^{\mathcal{L}_S} &\cong \delta E \left\{ \frac{1}{E - \mathcal{V}Y(\bar{\phi})} \sum_{N=1}^{\infty} \frac{1}{N!(3N-1)!} y^{3N} \right. \\ &\quad \left. - \frac{\mathcal{V}}{24} \left(\frac{m^2}{2} + \frac{\lambda}{2} \bar{\phi}^2 \right) \sum_{N=1}^{\infty} \frac{y^{3(N-1)}}{(N-1)!(3N-3)!} \right\}, \quad (D.1) \\ y^3 &= \frac{\pi^2}{90} \mathcal{V} (E - \mathcal{V}Y(\bar{\phi}))^3.\end{aligned}$$

The difference of this formula from the one obtained by the saddle point method is that one needs not assume large energy, E , in eq. (D.1). The physical solution to eq. (4.15) is the one which maximizes the right-hand side of (D.1). If the system does not have spontaneous symmetry breaking at zero energy, $\bar{\phi} = 0$ is the physical solution. However, in the case where $Y(\bar{\phi})$ can become negative, like the case with spontaneous symmetry breaking at $E = 0$, the situation is subtle. If the energy is not large, from eq. (D.1) one can see that the system prefers the

configuration $\bar{\phi} \neq 0$. In order to make a definite statement, numerical calculations will be required.

Appendix E. The Different Choice of the Time Path

This appendix is used to discuss the choice of the time path different from C in Fig. 1. The merit of the path C is that the real time part $C_1 + C_2$ and the equilibrium part C_3 decouple and it makes the theory simple. However, there is always a possibility to choose the other path. If one wants to do the real-time formulation, the alternative to Fig. 1 would be the one in Fig. 6, $C' := C'_1 : -T \rightarrow T$ and $C'_2 : T \rightarrow -T + \alpha - i\epsilon$.¹¹ The procedure for formulating the quantum field theory by this choice of path is the same as we discussed in the text by choosing C . For the same reason as discussed in the text, which is the completeness of the states inserted in the path integral at each time, the choice of path does not change the results. In order to convince ourselves, we show the calculation of the one-loop effective action using C' .

Similarly to the discussion in Appendix B, the paths C'_1 and C'_2 decouple. The C'_1 path gives the one-loop contribution:

$$\int_{C'_1} d^4x F[m^2 + \frac{\lambda}{2}\bar{\phi}^2] = 2T \mathcal{V} F[m^2 + \frac{\lambda}{2}\bar{\phi}^2],$$

where $F[\mu^2]$ is defined in eq. (C.3). C'_2 gives the contribution:

$$\int_{C'_2} d^4x F[m^2 + \frac{\lambda}{2}\bar{\phi}^2] = (-2T + \alpha) \mathcal{V} F[m^2 + \frac{\lambda}{2}\bar{\phi}^2].$$

Adding these two contributions, one recovers a result identical to eq. (6.3).

¹¹We would like to thank A. Niemi for drawing our attention to this choice of path.

REFERENCES

- [1] H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States.*, North-Holland, Amsterdam, 1982.
- [2] R. Mills, *Propagators for Many-particle Systems.* New York, 1969.
- [3] J. Rammer and H. Smith, *Rev. of Mod. Phys.* 58 (1986) 323.
- [4] L. Dolan and R. Jackiw, *Phys. Rev. D9* (1974) 3320.
- [5] S. Weinberg, *Phys. Rev. D9* (1974) 3357.
- [6] N. P. Landsman and Ch. G. Van Weert, *Phys. Rep.* 145 (1987) 141.
- [7] T. Matsubara, *Prog. Theor. Phys.* 14 (1955) 351.
- [8] Y. Takahashi and H. Umezawa, *Collective Phenomena* 2 (1975) 55.
- [9] A. J. Niemi and G. W. Semenoff, *Ann. of Phys.* 152 (1984) 105, *Nucl. Phys. B230* [FS10] (1984) 181.
- [10] H. A. Weldon, *Ann. of Phys.* 193 (1989) 166 and 177.
- [11] A. Strominger, *Ann. of Phys.* 146 (1983) 419.
- [12] R. P. Feynman and F. L. Vernon, *Ann. of Phys.* 24 (1963) 118.
- [13] A. J. Niemi, *Phys. Lett. B203* (1988) 425.
- [14] J. D. Bjorken, *Phys. Rev. D27* (1983) 140.
- [15] U. Heinz, *Phys. Rev. Lett.* 51 (1983) 351.
- [16] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, *Phys. Rev. D9* (1974) 3471.
- [17] R. Hagedorn, *Suppl. Nuovo Cimento* 3, 147 (1965) and 1971 *Cargèse Lectures in Physics*, Vol.6 (1973), ed. E. Schatzmann.
- [18] C. J. Hammer and S. C. Frautschi, *Phys. Rev. D4* (1971) 1130.
- [19] H. Matsumoto, Y. Nakano and H. Umezawa, *J. Math. Phys.* 25 (1984) 3076.
- [20] R. Hagedorn, *Hadron Matter under Extreme Conditions*, Kiev 1985.
- [21] J. Kapusta, *Nucl. Phys. B196* (1982) 1.
- [22] S. Coleman and E. Weinberg, *Phys. Rev. D7* (1973) 1888.
- [23] R. Jackiw, *Phys. Rev. D9* (1974) 1686.
- [24] S. W. Hawking, *Commun. Math. Phys.* 43 (1975) 199.
- [25] W. Israel, *Phys. Lett.* 57A (1976) 107.
- [26] B. F. Whiting and J. W. York, *Phys. Rev. Lett.* 61 (1988) 1336.
- [27] J.S. Langer, *Ann. of Phys.* 41 (1967) 108; *Ann. of Phys.* 54 (1969) 258.
- [28] M. Creutz, *Phys. Rev. D10* (1974) 1749; M. Creutz and K.S. Soh, *Phys. Rev. D12* (1975) 443.

Figure Captions

1. The time path C used to formulate finite-energy QFT.
2. One-loop contribution to the real scalar boson self-energy.
3. Summation to obtain full propagator. The left-hand side is the full propagator and the shaded circles on the right-hand side are self-energy insertions.
4. Graphical explanation of the generating functional of connected Green functions.
5. (a) one-loop and (b) two-loop diagrams which contribute to the entropy.
6. The path C' .

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6