Chapter 3

Quantum Chromodynamics and Deep Inelastic Scattering

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This article first describes the parton model which was the precursor of the QCD description of hard scattering processes. After the discovery of QCD and asymptotic freedom, the first successful applications were to Deep Inelastic lepton-hadron scattering. The subsequent application of QCD to processes with two initial state hadrons required the understanding and proof of factorization. To take the fledgling theory and turn it into the robust calculational engine it has become today, required a number of technical and conceptual developments which will be described. Prospects for higher loop calculations are also reviewed.

1. Hard Scattering before QCD

In the preface to his 1972 book, entitled *Photon-hadron interactions*,¹ Richard Feynman indicates how his book was stimulated by his participation in the *International Symposium on Electron and Photon Interactions at High Energies* held at Cornell² the previous year. By the time of that conference the approximate scaling behaviour in Deep Inelastic scattering (DIS) was established and the search for a theoretical framework to interpret it was underway. Scaling in the context of DIS is the observation that the structure functions, F_i , that describe the differential cross section for lepton-hadron scattering, are functions of a single dimensionless variable, x . For a hadron of momentum P interacting with an exchanged virtual photon of momentum q ,

$$x = \frac{-q^2}{2P \cdot q}. \quad (1)$$

The theoretical explanations in vogue were the light cone expansion (that somehow produced free field behaviour on the light cone, but was strongly interacting off the light cone) and various formulations of the parton model (that all required an

ad hoc bound on the transverse momentum of the partons, to explain the observed scaling behaviour). At the Cornell conference, gluons were mentioned in the talk of Bjorken (but they were labelled as speculative and no details of their couplings were given). The fractional charges of quarks were still controversial and were probed by the large x behaviour of the Deep Inelastic scaling functions.

In his parton model, Feynman envisaged that a proton of momentum P was made up of partons of momenta $\xi_i P$ all sharing the longitudinal momentum of the proton in various proportions, ξ_i . By analysing the data on deep inelastic scattering in the parton model, and in particular the scaling behaviour, the relative strengths of the longitudinal and transverse virtual photons, and the fraction of the parton momentum carried by charged constituents, he deduced that the charge carrying partons should have spin $1/2$ and at least some fraction of the momentum of the proton should be carried by electrically neutral partons. As a further development of the theory he considered the possibility that partons should be identified with quarks.

Just one year later the property of asymptotic freedom was discovered^{3,4} and a candidate Lagrangian for the theory of strong interactions, called Quantum Chromodynamics (QCD) was written down. The Lagrangian contained fractionally charged quarks, which were triplets under the $SU(3)$ colour degree of freedom, and eight coloured gluons in the adjoint representation of $SU(3)$. The expression for the classical Lagrangian density of QCD is

$$\mathcal{L}_{\text{classical}} = -\frac{1}{4}F_{\alpha\beta}^A F_A^{\alpha\beta} + \sum_{\text{flavours}} \bar{q}_a (i\not{D} - m)_{ab} q_b. \quad (2)$$

The Lagrangian describes the interaction of spin- $\frac{1}{2}$ quarks with masses specified by the diagonal mass matrix m , and massless spin-1 gluons. $\not{D}$ in Eq. (2) is a symbolic notation for $\gamma_\mu D^\mu$, where $D^\mu = \partial^\mu + ig t \cdot \mathcal{A}^\mu$ is the covariant derivative, and the matrices t are the eight generators of the $SU(3)$ colour group in the fundamental representation. The spinor indices of γ_μ and q_a have been suppressed. $F_{\alpha\beta}^A$ is the field strength tensor derived from the gluon field $\mathcal{A}_\alpha^A$,

$$F_{\alpha\beta}^A = \partial_\alpha \mathcal{A}_\beta^A - \partial_\beta \mathcal{A}_\alpha^A - gf^{ABC} \mathcal{A}_\alpha^B \mathcal{A}_\beta^C. \quad (3)$$

The indices A, B, C run over the eight colour degrees of freedom of the gluon field, whereas the indices a, b run over the three colour degrees of freedom of the quark field. f^{ABC} are the structure constants of the $SU(3)$ colour group.

Since QCD purports to be a complete theory of the strong interactions it should describe both high energy and low energy strong interactions. In the present Chapter we shall be concerned with the high energy manifestations of QCD, which are analysable using the techniques of perturbation theory, because of the property of asymptotic freedom. It can be argued that the discovery of QCD was the best of all possible worlds. On the one hand, the high energy behaviour of the theory can be analysed using the techniques of perturbation theory, that were so successfully used in Quantum Electrodynamics. On the other hand, the theory presented many different features, in particular a confining phase where free quarks and gluons are

not observed, that presented significant intellectual challenges in the application to real world situations. The theoretical physics community has risen to this challenge and provided astonishing advances in calculational technique to elaborate the consequences of QCD. The most significant of these advances for the high energy regime will be described in this review.

2. The Discovery of Asymptotic Freedom

Politzer,³ Gross and Wilczek^{4,5} discovered that non-Abelian gauge theories are asymptotically free, viz. that the coupling of the theory decreases at high energy, according to the β function equation.^a The running of the coupling constant α_S is determined by the renormalization group equation,

$$Q^2 \frac{\partial \alpha_S}{\partial Q^2} = \beta(\alpha_S), \quad (4)$$

where Q denotes the energy scale. In QCD, the β function has the perturbative expansion⁶⁻⁹

$$\beta(\alpha_S) = -b\alpha_S^2(1 + b'\alpha_S + b''\alpha_S^2 + b'''\alpha_S^3 + O(\alpha_S^4)), \quad (5)$$

where the one-loop coefficient of the β -function is,

$$b = \frac{(11C_A - 2n_{\text{lf}})}{12\pi} = \frac{(33 - 2n_{\text{lf}})}{12\pi}, \quad (6)$$

and n_{lf} is the number of active light flavours. The coefficients b', b'', b''' are also known. The form of the beta function utilizing all the known coefficients is shown in Fig. 1. Since the beta function is negative the coupling constant decreases as the energy increases. We may solve Eq. (4) for the dependence of the coupling on the scale Q^2 . In the lowest approximation the result is that the running coupling vanishes as an inverse power of a logarithm of the scale,

$$\alpha_S(Q^2) = \frac{1}{b \ln(Q^2/\Lambda^2)}. \quad (7)$$

Λ is a constant of integration that determines the scale at which the coupling constant becomes large. Because the coupling constant vanishes at large scale we may expand perturbative quantities as a series in α_S . For a physical quantity f , the relationship between the perturbative series evaluated at two different scales, Q_0 and Q_1 is,

$$\begin{aligned} f &= c_1 \alpha_S(Q_0^2) + c_2 \alpha_S^2(Q_0^2) + O(\alpha_S^3) \\ &\equiv c_1 \alpha_S(Q_1^2) + (c_2 - bc_1 \ln(Q_0^2/Q_1^2)) \alpha_S^2(Q_1^2) + O(\alpha_S^3). \end{aligned} \quad (8)$$

^aIt is often wrongly stated that the coupling decreases at small distances, but since the theory only introduces a logarithmic modification of the Coulomb potential, $1/r$, this cannot be the case.

Fig. 1. Perturbative expansion of the QCD beta function as a function of α_s in the one-loop approximation,^{3,4} two-loop approximation,⁶ three-loop approximation,^{7,8} and four-loop approximation.⁹ The one- and two-loop approximations are scheme independent. The three- and four-loop approximations are in the Minimal Subtraction (MS) scheme.

This illustrates the special character of QCD perturbation series, where the difference between two choices of scales, is itself a non-leading effect. Therefore a meaningful discussion about the choice of scale requires at least calculation of next-to-leading (NLO) effects.

3. Deep Inelastic Scattering

The first application of QCD was to the Deeply inelastic scattering of a charged lepton off a proton target by virtual photon exchange, $lp \rightarrow lX$. With four quark flavours, the result for the *QCD improved parton model* is

$$F_2^{\text{em}}(x, Q^2) = x \left[\frac{4}{9} (u(x, Q^2) + \bar{u}(x, Q^2) + c(x, Q^2) + \bar{c}(x, Q^2)) + \frac{1}{9} (d(x, Q^2) + \bar{d}(x, Q^2) + s(x, Q^2) + \bar{s}(x, Q^2)) \right]. \quad (9)$$

This is in contradistinction with the *naive parton model* in which the parton distribution functions are independent of the scale.^b For compactness of notation we shall use the variable t to represent Q^2 .

^bFeynman would bristle at the use of the word *naive* to qualify his parton model. Several times I heard him remonstrate, saying ‘Was it so naive?’

The scale dependent parton distribution functions satisfy sum rules that count the valence quarks in the proton, and the total momentum of the partons

$$\begin{aligned} \int_0^1 dx (u(x, t) - \bar{u}(x, t)) &= 2, \\ \int_0^1 dx (d(x, t) - \bar{d}(x, t)) &= 1, \\ \int_0^1 dx x \left[g(x, t) + \sum_{i=d,u,s,c} (f_i(x, t) + \bar{f}_i(x, t)) \right] &= 1. \end{aligned} \quad (10)$$

The parton distributions themselves must be determined by experiment, but their t -dependence is determined in QCD. The t -dependence is simplest for (flavour) non-singlet combinations of quark distributions, $V = q_i - q_j$. A first formulation of the evolution equation was in terms of the moments of the parton distributions:

$$f(j, t) = \int_0^1 dx x^{j-1} f(x, t), \quad f = q_i, g. \quad (11)$$

In terms of moments, the evolution equation is

$$t \frac{\partial}{\partial t} V(j, t) = \frac{\alpha_S(t)}{2\pi} \gamma_{qq}^{(0)}(j) V(j, t), \quad (12)$$

where $\gamma_{qq}^{(0)}(j)$ are anomalous dimensions given in Eq. (13). In general the evolution equation is a matrix equation in the space of quarks and gluons. The complete set of leading-order anomalous dimensions in the matrix equation is,

$$\begin{aligned} \gamma_{qq}^{(0)}(j) &= C_F \left[-\frac{1}{2} + \frac{1}{j(j+1)} - 2 \sum_{k=2}^j \frac{1}{k} \right], \\ \gamma_{qg}^{(0)}(j) &= T_R \left[\frac{(2+j+j^2)}{j(j+1)(j+2)} \right], \\ \gamma_{gq}^{(0)}(j) &= C_F \left[\frac{(2+j+j^2)}{j(j^2-1)} \right], \\ \gamma_{gg}^{(0)}(j) &= 2C_A \left[-\frac{1}{12} + \frac{1}{j(j-1)} + \frac{1}{(j+1)(j+2)} - \sum_{k=2}^j \frac{1}{k} \right] - \frac{2}{3} n_f T_R, \end{aligned} \quad (13)$$

where $C_F = 4/3$, $C_A = 3$ and $T_R = 1/2$. A graphical illustration of the j -dependence of these functions in the LO and NLO approximations is shown in Fig. 2. Note that

$$\begin{aligned} \gamma_{qq}^{(0)}(1) &= 0, \\ \gamma_{qq}^{(0)}(2) + \gamma_{gq}^{(0)}(2) &= 0, \\ 2n_f \gamma_{qg}^{(0)}(2) + \gamma_{gg}^{(0)}(2) &= 0. \end{aligned} \quad (14)$$

Inserting the lowest-order form Eq. (7) for the running coupling, we obtain the solution of Eq. (12) for the moments of the non-singlet distribution,

$$V(j, t) = V(j, t_0) \left(\frac{\alpha_S(t_0)}{\alpha_S(t)} \right)^{d_{qq}(j)}, \quad d_{qq}(j) = \frac{\gamma_{qq}^{(0)}(j)}{2\pi b}, \quad (15)$$

where b is given in Eq. (6). Finally, the distribution in x space can be obtained using the inverse Mellin transform integral,

$$V(x, t) = \frac{1}{2\pi i} \int_C dj \, x^{-j} V(j, t), \quad (16)$$

where the integration contour in the complex j plane is parallel to the imaginary axis and to the right of all singularities of the integrand. Except in very special cases, the inverse transformation can only be performed by numerical integration.

It is straightforward to show that $d_{qq}(1) = 0$ and that $d_{qq}(j) < 0$ for $j > 1$, cf. Fig. 2. This in turn implies that as t increases the non-singlet distribution function *decreases* at large x and *increases* at small x in accord with experimental observation.

The description of Deep Inelastic scattering beyond the leading order requires higher-order calculation of the DIS anomalous dimensions¹⁰⁻¹⁵ as well as the calculation of higher order corrections to the coefficient functions.^{16,17}

Fig. 2. Anomalous dimensions in moment space.

4. Factorization and the QCD Improved Parton Model

The treatment of Deep Inelastic scattering outlined above was done in the context of operator product expansion, building on the work of Wilson.¹⁸ At the time it was not widely known which class of diagrams was resummed by including the effects of the anomalous dimensions. This became clear after the papers of Altarelli and Parisi¹⁹ and Dokshitzer.²⁰ Earlier papers by Gribov and Lipatov^{21,22} were also important. These papers made clear that the diagrams summed by the leading order anomalous dimension were ladder diagrams, in any physical gauge in which the vanishing of the helicity amplitudes in the forward direction was manifest. These papers also effectively separated the ladder sum from any particular physical process, opening the door to generalisations to processes other than deep inelastic scattering.

For example, for the flavour non-singlet combination, the DGLAP equation at lowest order reads,

$$t \frac{d}{dt} V(x, t) = \frac{\alpha_S(t)}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z, \alpha_S(t)) V\left(\frac{x}{z}, t\right). \quad (17)$$

The leading-order splitting functions are

$$\begin{aligned} P_{qq}^{(0)}(z) &= C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right], \\ P_{qg}^{(0)}(z) &= T_R \left[z^2 + (1-z)^2 \right], \quad T_R = \frac{1}{2}, \\ P_{gq}^{(0)}(z) &= C_F \left[\frac{1+(1-z)^2}{z} \right], \\ P_{gg}^{(0)}(z) &= 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] \\ &\quad + \delta(1-z) \frac{(11C_A - 4n_f T_R)}{6}, \end{aligned} \quad (18)$$

where the ‘plus prescription’ on the singular parts of the functions is defined in Eq. (19):

$$\int_0^1 dz f(z)[g(z)]_+ = \int_0^1 dz (f(z) - f(1))g(z). \quad (19)$$

We shall refer to the splitting functions without the plus prescription and the end-point contributions $\delta(1-z)$, as unregularised splitting functions, $\hat{P}$.

It took some time for the whole community to accept that the DGLAP equation was a systematically improvable approximation, rather than just a leading log statement. This became clear with the work of Refs. 23, 24 and the work of Curci, Furmanski and Petronzio.^{12,13} The latter authors used the method of Ellis *et al.*²³ and calculated the previously known NLO anomalous dimensions in a compact form, without reference to a particular hard process.

The first NLO application of QCD to hadron-hadron processes²⁵ was the calculation of higher-order corrections to muon pair production in hadronic collisions, the so-called Drell-Yan process. At fixed target energies it was found that the corrections were large, so large that the applicability of perturbation theory was called into question. The ratio of the NLO to LO predictions is often referred to as the K-factor, a term introduced by Altarelli.²⁶ In the parton model with a colour degree of freedom, there is a factor of $1/N_c$ expressing the fact that only a red and anti-red quark, etc. can annihilate to give a colour singlet final state. Since the experimental results in 1979 were larger than the expected in the coloured quark model without NLO corrections, the introduction of the colour degree of freedom was not universally accepted by experimentalists. Nowadays the validity of QCD in this process is beautifully confirmed by the data on W and Z production²⁷ which is compared with NNLO QCD.²⁸

5. Parton Shower Monte Carlo

The sequential emission of partons in a high energy collision will give rise to a shower of emitted partons, just an energetic hadron hitting the atmosphere will give rise to an extended air shower. The formulation of the DGLAP equation given previously is not the most useful when we want to study the detailed features of a parton shower. We therefore consider an equivalent formulation. In the simplest case, in which we have only one species of parton, the DGLAP equation may be rewritten by introducing the Sudakov form factor,

$$\Delta(t) \equiv \exp \left[- \int_{t_0}^t \frac{dt'}{t'} \int dz \frac{\alpha_S}{2\pi} \hat{P}(z) \right]. \quad (20)$$

We shall see shortly that this function has a simple physical interpretation. Now we can write

$$t \frac{\partial}{\partial t} f(x, t) = \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) f(x/z, t) + \frac{f(x, t)}{\Delta(t)} t \frac{\partial}{\partial t} \Delta(t) \quad (21)$$

and hence

$$t \frac{\partial}{\partial t} \left(\frac{f}{\Delta} \right) = \frac{1}{\Delta} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) f(x/z, t). \quad (22)$$

Notice that this is similar in form to the Altarelli-Parisi equation (17), except that f is replaced by f/Δ and the regularised splitting function P is replaced by the unregularised one $\hat{P}$. This equation can be integrated to give an integral equation for $f(x, t)$ in terms of the initial parton distribution $f(x, t_0)$:

$$f(x, t) = \Delta(t) f(x, t_0) + \int_{t_0}^t \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) f(x/z, t'). \quad (23)$$

The first term on the right-hand side is the contribution from paths that do not branch between scales t_0 and t . Thus the Sudakov form factor $\Delta(t)$ is simply the probability of evolving from t_0 to t without branching. The second term is the contribution from all paths which have their last branching at scale t' . The factor of $\Delta(t)/\Delta(t')$ represents the probability of evolving from t' to t without branching. This is consistent with the above interpretation of $\Delta(t)$ because we see from Eq. (20) that $\Delta(t_0) = 1$.

Equation (23) can be used to develop a parton shower in a Monte Carlo program by a sequential series of branchings. Each individual branching is characterised by a lower scale t_2 at which the branching occurs, and a value of z which characterises the fractional longitudinal momenta carried away in the branching. Both of these quantities can be generated with the appropriate probabilities specified by Eq. (23) by solving the following equations for t_2 and x_2/x_1 ,

$$\frac{\Delta(t_2)}{\Delta(t_1)} = \mathcal{R},$$

$$\int_{\epsilon}^{x_2/x_1} dz \frac{\alpha_S}{2\pi} P(z) = \mathcal{R}' \int_{\epsilon}^{1-\epsilon} dz \frac{\alpha_S}{2\pi} P(z), \quad (24)$$

where $\mathcal{R}$ and $\mathcal{R}'$ are random numbers. The effects of coherence can be approximately introduced in the parton shower by imposing angular ordering, so that successive branchings occur at ever smaller angles.²⁹

6. Jet Cross Sections

At collider energies, the event displays of a small fraction of events, show conglomerations of tracks or energy deposits, which one is tempted to interpret as the fragmentation products of individual partons. At the high energy the existence of jets in these events is evident to the naked eye. To put this simple idea on a more rigorous footing one must introduce a jet algorithm. For both experimenters and theorists a jet algorithm defines a procedure for including a set of particles into a jet and for defining the momentum of the resulting jet. A good algorithm can be applied to both real experimental data and simulated Monte Carlo data. In the latter case it should be applicable at both the hadronic and partonic level. The necessity for this consensus between experimenters and theorists has the result that, according to Tannenbaum,³⁰ “large sections of the publications read more like legal contracts between experimentalists and theorists than like scientific papers”. The paper of Sterman and Weinberg³¹ established that any infrared finite quantity would be calculable in perturbation theory. Therefore the primary legal requirement for a theorist is that the jet algorithm should be infrared finite, i.e. it should be insensitive to the emission of soft or collinear parton radiation. Within this overarching requirement, many algorithms are legal and no one algorithm or jet size will be suitable in all circumstances.

Jet algorithms come in two main types: cone algorithms and sequential recombination jet algorithms. Historically, many proposed cone algorithms have had issues of infrared safety. A good modern example of an infrared finite cone algorithm is the Seedless, infrared safe cone (Siscone) algorithm.³² Sequential recombination jet algorithms proceed by defining a distance measure between particles or clusters of energy. If, according to the distance measure, two particles are closer than some jet resolution size, they are combined by some well-defined prescription into a pseudo-particle. The whole procedure is re-iterated until no further recombinations can occur. Two famous examples are the longitudinally invariant k_T clustering algorithm³³ which has a clustering pattern, approximating the QCD branching structure, and the anti- k_T jet algorithm,³⁴ a sequential recombination jet algorithm giving rise to approximately conical jets. For a full review, see the article by Salam.³⁵

7. Technical Advances

The method of dimensional regularisation³⁶ was introduced in order to have a regularisation for non-Abelian theories that would respect the Ward identities of the theory. Ultraviolet loop divergences are regulated by reducing the number of space-time dimensions, d , to $d < 4$. Thus the integration measure becomes,

$$\frac{d^4 k}{(2\pi)^4} \longrightarrow (\mu)^{2\epsilon} \frac{d^{4-2\epsilon} k}{(2\pi)^{4-2\epsilon}}, \quad (25)$$

where $\epsilon = 2 - \frac{d}{2}$. For a detailed discussion of renormalisation and dimensional regularisation we refer the reader to the book by Collins.³⁷ It was quickly realised that dimensional regularization could also be used to regulate both infrared singularities coming from soft emission and mass singularities present in the region of collinear emission.^{38,39} Essentially all higher loop QCD calculations are now done using dimensional regularisation.

Because of the gauge structure of QCD, the final results of the calculation of an amplitude are often simpler than the intermediate results, (e.g. the result from an individual Feynman diagram). In this circumstance symbolic manipulation programs, such as Veltman's *Schoonschip*⁴⁰ and its successor Vermaseren's *Form*^{41,42} are invaluable. Commercial products such as *Maple* and *Mathematica* are also widely used. This software (and corresponding improvements in computer hardware) have greatly extended the complexity of the calculations that can be successfully undertaken.

The spinor-helicity method has also led to great advances in understanding the structure of QCD amplitudes. This method starts from the observation that the polarisation vector of an external gluon field can be represented as a bispinor. Thus a polarisation vector of a gluon with momentum k (and gauge vector b) can be written as,

$$\varepsilon_{\pm}^{\mu}(k, b) = \pm \frac{\langle k \pm | \gamma^{\mu} | b \pm \rangle}{\sqrt{2} \langle b \mp | k \pm \rangle}. \quad (26)$$

Physical results for gauge invariant (sub-)amplitudes will be independent of the choice for b . The function $u_{\pm}(k_i) \equiv |k_i \pm\rangle$ is a massless Weyl spinor of momentum k_i and positive or negative chirality. In terms of these solutions of the free Dirac equation, the spinor products are defined by,

$$\langle i j \rangle = \langle i^- | j^+ \rangle = \bar{u}_-(k_i) u_+(k_j), \quad (27)$$

$$[i j] = \langle i^+ | j^- \rangle = \bar{u}_+(k_i) u_-(k_j). \quad (28)$$

The spinor products are given, up to a phase, by the square root of a dot product, $\langle i j \rangle [j i] = 2k_i \cdot k_j$. Spinor products are the natural language to represent gauge theory amplitudes, because the amplitudes contain square root singularities. The spinor-helicity formalism,^{43,44} has been used to great effect for tree graphs, and for one-loop calculations.^{45,46} The result for one four-gluon helicity amplitude expressed in terms of spinor products is

$$m(1_g^-, 2_g^-, 3_g^+, 4_g^+) = -i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}. \quad (29)$$

This formula, which generalises to an arbitrary number of gluons,⁴⁷ is an example of a maximum helicity violating amplitude (MHV). The amplitudes with less helicity violation $m(1_g^+, 2_g^+, 3_g^+, 4_g^+)$ and $m(1_g^-, 2_g^+, 3_g^+, 4_g^+)$ are equal to zero at tree graph level.

7.1. One-loop calculations

Scalar integrals are loop integrals with no powers of the loop momentum in the numerator; they are classified by the number of external lines, $N = 1, 2, 3, 4$, corresponding to scalar tadpoles, bubbles, triangles and boxes. Any one-loop amplitude can be expressed as a sum of scalar tadpoles, bubbles, triangles and box integrals. The existence of this decomposition is one of the most important results for one-loop calculations; the result relies on simple Lorentz invariance, which allows a decomposition of tensor integrals to invariant form factors, and on the four-dimensional nature of space-time, which allows scalar higher-point integrals to be reduced to sums of boxes. Using the property of Lorentz invariance, Passarino and Veltman⁴⁸ demonstrated how to reduce the tensor integrals (with $N \leq 4$ and with k powers of the loop momentum in the numerator) to scalar integrals. In a renormalisable theory, tensor integrals occur with a limited number of powers of the loop momentum in the numerator, $k \leq N$. Higher point scalar integrals can also be reduced to lower point functions. Thus a scalar pentagon in d dimensions can be written as a sum of the five box diagrams obtainable by removing one propagator, if we neglect terms of order ϵ .^{49–51} The general one-loop N -point integral in $d = 4 - 2\epsilon$ dimensions for $N \geq 6$ can be recursively obtained as a linear combination of pentagon integrals,^{49,50} provided that the external momenta are restricted to four dimensions.

Thus any integral I_N can be written as a linear combination of one-loop scalar integrals, that include four-, three-, two- and one-point functions and a remnant of the dimensional regularisation procedure that is called *the rational part* $\mathcal{R}$,

$$I_N = c_{4;j}I_{4;j} + c_{3;j}I_{3;j} + c_{2;j}I_{2;j} + c_{1;j}I_{1;j} + \mathcal{R} + \mathcal{O}(D-4). \quad (30)$$

The index j labels the combination of momenta that enter into each particular scalar integral $I_{n;j}$. The significance of Eq. (30) is that once the scalar one-loop integrals are tabulated for $n \leq 4$, *any* one-loop calculation is reduced to the determination of both the coefficients $c_{n;j}$ and the rational part $\mathcal{R}$.

Complete results for finite scalar one-loop integrals were given by 't Hooft and Veltman.⁵² A corresponding set of complete results for integrals containing soft or collinear divergences, regulated using dimensional regularisation, were provided by the program QCDLoop.⁵³ As far as one-loop diagrams are concerned, the scalar integrals are completely known. Therefore the calculation of one-loop amplitudes is reduced to the determination of the coefficients of these known one-loop integrals.

The reduction coefficients and the rational part, can be obtained by efficient numerical methods based on an algebraic understanding of the structure of the integrand, as first noted by Ossola, Papadopoulos and Pittau.⁵⁴ This procedure exploits the general parameterisation of the structure of the one-loop *integrand* involving a mix of terms that give rise to the wanted coefficients, $c_{n;j}$, and evanescent terms that vanish upon integration. By repeated evaluation of the integrand at differing values of the loop momentum (typically values at which some subset of the denominators vanish), we can determine the coefficients of both the physical and the evanescent terms. A detailed description of this method, including its extension to d dimensions to determine the rational terms, can be found in the *Physics Report* of Ellis *et al.*⁵⁵

8. The Age of the Automation

8.1. Tree graphs

Although completely specified by the Feynman rules, the calculation of tree graphs can lead to considerable algebraic complexity. Here we document examples of attempts to deal with this complexity. Helas⁵⁶ is a set of Fortran subroutines for helicity amplitudes which enable the calculation of an arbitrary tree diagram using building blocks supplied by a number of subroutine calls. Two years later it was realized by Stelzer and Long,⁵⁷ working in Madison, that a script, dubbed Madgraph, could be written to assemble the needed subroutine calls for an arbitrary process. This has subsequently been further developed into a sophisticated tool that can generate both tree diagrams and one-loop diagrams, for both the standard model and BSM models, and that can sum over parton distributions and generate events for further processing by parton showers.⁵⁸

Subsequently there have been many further developments in the automatic generation of Feynman graphs. As an example of these programs we quote *qgraf* by Nogueira,⁵⁹ which can handle both tree diagrams and loop diagrams. Automatic graph generation is an essential feature for higher-order calculations. In addition, much effort has been put into automatic Feynman rule generation (for models different than QCD). A standard interface for the Feynman rule output that can serve as the input for other amplitude generators, called Universal Feynman-rule Output (UFO)⁶⁰ has been proposed.

Also of great importance is the generation of Feynman amplitudes recursively. The earliest of these methods, due to Berends and Giele,⁶¹ generates colour-ordered off-shell currents which are further used as the input for higher-point currents by attaching all possible branching processes to the off-shell line. The computer evaluation of amplitudes using off-shell recursion has been compared with analytic formula for QCD amplitudes and the conclusion is that a hybrid approach may be the most efficient.⁶² Thus for a large number of legs, and for helicity choices far from the MHV, Berends–Giele recursion is computationally superior to analytic formulas.

Berends–Giele recursion relations perform recursion with off-shell currents. By contrast BCFW recursion⁶³ performs recursion with on-shell colour-ordered amplitudes. The proof of BCFW recursion relies heavily on the analyticity properties of the colour-ordered amplitudes. Thus for example, using BCFW recursion we can show that a four-point MHV gluon amplitude can be calculated as a product of two three-point amplitudes with shifted external momenta, connected by a scalar propagator. For an elementary introduction to BCFW recursion, both at tree graph level and at one-loop level I refer the reader to the lectures of Lance Dixon.⁶⁴

8.2. NLO calculations

With the factorisation theorem in place, and advances in the theory of jet algorithms, the conceptual framework needed to perform NLO one-loop calculations of physical quantities was in place. If the Born diagram for a given process is a tree graph, the NLO calculation will require the calculation of the Born diagram with one additional parton, and interference of the one-loop virtual contribution with the Born diagram. We shall refer to these as the real and virtual contributions respectively. In QCD both of these contributions are individually singular because of the emission of a soft gluon or because of the collinear emission of effectively massless partons. The form of the singular terms is completely understood.^{65,66} For suitable inclusive quantities there is a cancellation between real and virtual contributions.

The subtraction method, first used for 3-jet structure in e^+e^- annihilation,⁶⁷ introduces a subtraction term that cancels the behaviour of the real diagrams in

the singular limit,

$$d\sigma_{ab,\text{NLO}} = \int_{n+1} [d\sigma_{ab,\text{NLO}}^R - d\sigma_{ab,\text{NLO}}^S] + \int_n [d\sigma_{ab,\text{NLO}}^V + d\sigma_{ab,\text{NLO}}^S]. \quad (31)$$

The subtraction terms are chosen to be sufficiently simple that they can be analytically integrated (in d dimensions) over the kinematic variables of the additional parton. The resultant integrated counterterm contains poles in ϵ that, when added to the virtual term, yield a finite result. The subtraction procedure was turned into a precise algorithm by Catani and Seymour⁶⁸ who also introduced a procedure for mapping the n -parton phase space onto the $(n+1)$ -parton phase space, both for initial and final radiation. Also worthy of note is the subtraction procedure of Frixione *et al.*⁶⁹ which uses energy and angle variables to define the subtraction term.

An alternative to the subtraction procedure is the slicing method which separates the $(n+1)$ -parton phase space into two regions. In the first region, $(n+1)$ -partons are resolved, whereas in the second region n -partons are resolved. If the invariant mass of two partons, s_{ij} , is smaller than some resolution parameter $s_{\min}$ only n partons are resolved. All of the divergences from the $(n+1)$ -parton final state occur in regions where only n -partons are resolved. The resolution parameter $s_{\min}$ must be judiciously chosen, small enough that the sub-leading effects can be ignored in the n -parton phase space, but not so small that the divergences in the $(n+1)$ -parton phase space are numerically unstable. A review of the slicing method, with references to the original literature is given in Ref. 70.

9. Outlook for NNLO

Since at high energy the coupling constant $\alpha_S \approx 0.1$, it might be expected that NLO calculations would give about 10% precision. This is not the case because of the special nature of asymptotically-free perturbation theory where the coupling decreases only logarithmically at large energy. Leading order calculations give little idea of the correct choice of scale, cf. Eq. (8). Thus the NLO calculation is the first serious approximation. With this viewpoint it is clear that NNLO has an important role to play. NNLO gives the first information about the behaviour of the asymptotic series.

The pioneers of these calculations were van Neerven and collaborators^{28,71} who performed NNLO calculations of the $2 \rightarrow 1$ processes such as muon pair production in hadron-hadron collisions and Higgs boson production by gluon-gluon fusion. There are two major challenges in these calculations, the calculation of the two-loop matrix elements and the treatment and isolation of the singular regions in the real radiation contributions.

The NNLO mass factorised partonic cross section has three contributions: the double real, real-virtual and double virtual corrections,

$$d\sigma_{ab,\text{NNLO}} = \int_{n+2} d\sigma_{ab,\text{NNLO}}^{RR} + \int_{n+1} \left[d\sigma_{ab,\text{NNLO}}^{RV} + d\sigma_{ab,\text{NNLO}}^{MF,1} \right] + \int_n \left[d\sigma_{ab,\text{NNLO}}^{VV} + d\sigma_{ab,\text{NNLO}}^{MF,2} \right]. \quad (32)$$

Each of these terms is separately divergent, either containing singularities in regions of single or double unresolved phase space or explicit IR poles in ϵ . The sum of all three contributions can be rearranged, so that all singularities (poles in ϵ) cancel,

$$d\sigma_{ab,\text{NNLO}} = \int_{n+2} \left[d\sigma_{ab,\text{NNLO}}^{RR} - d\sigma_{ab,\text{NNLO}}^S \right] + \int_{n+1} \left[d\sigma_{ab,\text{NNLO}}^{RV} - d\sigma_{ab,\text{NNLO}}^T \right] + \int_n \left[d\sigma_{ab,\text{NNLO}}^{VV} - d\sigma_{ab,\text{NNLO}}^U \right]. \quad (33)$$

We mention three methods for choosing the subtractions. First, the antenna subtraction method⁷² was developed for e^+e^- annihilation, but has subsequently been adapted and successfully used for hadron-hadron collisions. Second, the sector improved subtraction technique of Czakon⁷³ has been applied to several physical processes. Lastly, the NNLO corrections for processes involving the production of colourless particles, such as W, Z and Higgs bosons are also known.^{74,75}

Two loop (and higher) calculations generate families of integrals with differing denominators and numerators. Although it would in principle be possible to calculate all of these integrals individually, it is more efficient to note that there are relations between the integrals that allow reduction to a set of master integrals. The most significant of these relationships are the integration by parts (IBP) identities.^{76,77} Here I provide a simple example, which illustrates the power of the method. Consider an integral that is a total derivative and that vanishes in dimensional regularisation,

$$0 = \int \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{\partial}{\partial p^\mu} \frac{1}{p^2(p-k)^2 q^2(q-k)^2} \frac{(p-q)^\mu}{(p-q)^2}. \quad (34)$$

Performing the differentiation it follows that in $d = 4 - 2\epsilon$ dimensions,

$$\begin{aligned} & \epsilon \int \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{1}{p^2(p-k)^2 q^2(q-k)^2(p-q)^2} \\ &= \int \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{1}{p^2(p-k)^4 q^2(p-q)^2} \\ & \quad - \int \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{1}{p^2(p-k)^4 q^2(q-k)^2}. \end{aligned} \quad (35)$$

Even with IBP identities in hand, it is quite onerous to find the correct reduction to a set of master integrals. The Laporta Algorithm⁷⁸ provides a systematic lexicographic ordering of the integrals, that allows a decision on the most complicated integral that should be eliminated using the IBP and other identities. Publicly available programs implement this algorithm,^{79,80} however it remains a compute-intensive problem. The calculation of phenomenologically relevant two-loop amplitudes is going through a period of rapid development at the time of writing. The current state of the art can be assessed by looking at recent work.^{81,82}

10. Epilogue

There can be no doubt that the enormous human effort invested into high-energy QCD, has had a great impact on the interpretation of the experiments at e^+e^- and hadron-hadron colliders throughout the world. Although this support role is important, the continuing challenge of understanding the workings of QCD, an interacting quantum field theory, is no less important.

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