Chapter 7

Hadron Contribution to Vacuum Polarisation

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Precision tests of the Standard Theory require theoretical predictions taking into account higher-order quantum corrections. Among these vacuum polarisation plays a predominant role. Vacuum polarisation originates from creation and annihilation of virtual particle–antiparticle states. Leptonic vacuum polarisation can be computed from quantum electrodynamics. Hadronic vacuum polarisation cannot because of the non-perturbative nature of QCD at low energy. The problem is remedied by establishing dispersion relations involving experimental data on the cross section for e^+e^- annihilation into hadrons. This chapter sets the theoretical and experimental scene and reviews the progress achieved in the last decades thanks to more precise and complete data sets. Among the various applications of hadronic vacuum polarisation calculations, two are emphasised: the contribution to the anomalous magnetic moment of the muon, and the running of the fine structure constant α to the Z mass scale. They are fundamental ingredients to high precision tests of the Standard Theory.

1. Introduction and Historical Perspective

Vacuum polarisation (VP) originates from quantum fluctuations in the exchange of gauge bosons occurring in particle interactions. The simplest case is that of electron–positron pairs in the propagation of a virtual photon: an e^+e^- pair is emitted and re-absorbed by the photon. It is the quantum analogue of the polarisation of molecules in a dielectric material when an external electric field is applied. The field generated by the distorted molecules has the effect of reducing the field in the medium. Similarly at the quantum level the e^+e^- pairs cause a screening effect that reduces the strength of the electromagnetic force carried by the exchanged photon. It is therefore important to take into account VP when evaluating the effective interaction resulting from photon exchange.

Experimental evidence of photon VP dates back to 1947 with the precise measurement of the Lamb shift giving the energy difference between the 2S and 2P levels of the hydrogen atom.¹ Whereas this difference should vanish in the Dirac theory, the measurement reveals a non-zero value mostly from electron self-energy, but also including a small contribution from VP. The explanation, first proposed by Bethe, was refined in the quantum electrodynamic theory (QED) developed by R. P. Feynman, J. Schwinger and S. Tomonaga. The observation of the Lamb shift had therefore a dramatic impact on establishing QED and more generally quantum field theory.

Not only electrons and positrons, but any pair of charged particles needs to be considered when evaluating photon VP effects. In particular pairs of charged leptons heavier than the electron, $\mu^+\mu^-$ and $\tau^+\tau^-$, should occur as well. Their VP effects can be calculated with QED and are found to be suppressed with respect to e^+e^- because of their larger masses. These contributions can be felt only when the experimental observable is very precisely measured or for a large virtuality (large q^2) of the exchanged photon. An example of the first type is the anomalous magnetic moment of the electron, which is experimentally known with a precision of 0.25 ppb² and for which the QED prediction has to include contributions up to the α^5 level to match the experimental accuracy.

The vacuum can also be polarised by fluctuations involving strongly interacting particles, an effect called hadronic vacuum polarisation (HVP). To first order in quantum chromodynamics (QCD) these fluctuations are induced by quark-antiquark pairs. Their effect can be computed at large q^2 in perturbative QCD but not at low scales due to the non-perturbative nature of QCD at large distance. It is possible to overcome this problem by means of a dispersion relation technique involving experimental data measuring the cross section $e^+e^- \rightarrow \text{hadrons}$. This technique was pioneered by C. Bouchiat and L. Michel³ and by N. Cabibbo and R. Gatto,⁴ and provided a first estimate of the HVP contribution to the anomalous magnetic moment of the muon.

The first evidence for HVP came from an experiment at the ACO storage ring at LAL-Orsay⁵ measuring the cross section for the purely leptonic process $e^+e^- \rightarrow \mu^+\mu^-$. A characteristic interference pattern was found around the mass of the $\phi(1020)$ resonance, in agreement with the expected ϕ HVP contribution. A broader range of energies could be explored with the study of the muon magnetic moment exhibiting ever increasing sensitivity. The CERN effort to measure this quantity started in 1958 with an experiment at the synchrocyclotron,⁶ followed by a more and more elaborate programme from 1962 to 1976 using storage rings fed by the proton synchrotron.⁷⁻⁹ In the last phase of the programme the precision reached on the muon magnetic anomaly ($a_\mu = (g_\mu - 2)/2$, where g_μ is the muon gyromagnetic ratio) was 7 ppm, enough to detect the effect of HVP. The final experimental value achieved at CERN for a_μ was $(1\,165\,924 \pm 8.5) \cdot 10^{-9}$, yielding a deviation from pure QED of $(70.4 \pm 8.1) \cdot 10^{-9}$, to be compared to the HVP estimate¹⁰ then

of $(70.2 \pm 8.0) \cdot 10^{-9}$. This measurement established clearly the presence of HVP as a manifestation of the strong interaction in lepton properties. It ranks as one of the highlights achieved in the CERN scientific programme. The pursuit of even higher accuracies was continued at Brookhaven using the CERN technique, and it is still going on now at Fermilab, USA and J-PARC, Japan.¹¹ It is interesting to note that the recent measurements of the electron magnetic moment are now also sensitive¹² to HVP, which must be included in the theoretical prediction in spite of its tiny amount, approximately $m_\mu^2/m_e^2 \sim 40\,000$ time smaller than that of the muon.

Apart from predicting the lepton magnetic moments, the need for accurate calculation of HVP arose with the advent of precision tests in the electroweak sector of the Standard Theory (ST) at LEP, owing to the large statistics accumulated at the Z pole, and at SLC using the electron beam polarisation. The interpretation of these measurements required taking into account VP which affects the effective electromagnetic coupling when evolving from low energies to the Z mass scale, providing a strong impetus to improve HVP calculations and the corresponding e^+e^- cross sections. The variation of the fine structure “constant” amounts to 3.150% from leptonic VP with negligible uncertainty¹³ and to about 2.7% from HVP. Matching the per mil accuracy of the electroweak measurements requires the evaluation of the HVP contribution with percent precision, pushing the state of the art for e^+e^- data.

In this review, after introducing the dispersion approach to HVP, we will discuss the sources of input data on the $e^+e^- \rightarrow$ hadrons cross section which are the necessary ingredient to compute the dispersion integrals. Over the last 50 years the quality and the accuracy of these data has improved dramatically, especially in the last decade, allowing more precise tests of the Standard Theory.

2. Dispersion Relations

Using unitarity and analyticity, spectra of hadron production from e^+e^- annihilation via a spin-one photon propagator are connected to the imaginary part of the two-point correlation (or hadronic vacuum polarisation) functions

$$\Pi_{i,V}^{\mu\nu}(q) \equiv i \int d^4x e^{iqx} \langle 0 | T(V_i^\mu(x) V_i^\nu(0)^\dagger) | 0 \rangle = (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi_{i,V}^{(1)}(q^2) \quad (1)$$

of vector colour-singlet quark currents involving all flavours i , and for time-like momenta-squared $q^2 > 0$. The functions $\Pi_{i,V}^{\mu\nu}(s)$ have a branch cut along the real axis in the complex $s = q^2$ plane. Their imaginary parts reproduce the hadronic annihilation spectrum and provide the basis for comparing short-distance theory with hadronic data. The analytic vacuum polarisation function $\Pi_{i,V}^{(1)}(q^2)$ obeys the dispersion relation

$$\Pi_{i,V}^{(1)}(q^2) = \frac{1}{\pi} \int_0^\infty ds \frac{\text{Im}\Pi_{i,V}^{(1)}(s)}{s - q^2 - i\varepsilon} + \text{subtractions}, \quad (2)$$

where the unknown but in general irrelevant subtraction constants can be removed by taking the derivative of $\Pi_{i,V}(q^2)$. The dispersion relation allows one to connect the experimentally accessible hadron spectrum to the correlation functions $\Pi_{i,V}^{(1)}(q^2)$, which can be derived from theory (QCD).

As an example this formalism is applied to the running electromagnetic coupling. In quantum field theory all contributions from self-energy and vertex correction graphs to the photon vacuum polarisation function $\Pi_\gamma(q^2)$ cancel and only vacuum polarisation modifies the charge of an elementary particle. One can therefore write for the running electron charge (charge screening) at energy scale s

$$\alpha(s) = \frac{\alpha(0)}{1 - \Delta\alpha(s)}, \quad (3)$$

where $\alpha(0)$ is the fine structure constant in the long-wavelength Thomson limit and $\Delta\alpha(s) = \Delta\alpha_{\text{lep}}(s) + \Delta\alpha_{\text{had}}(s) = -4\pi\alpha \cdot \text{Re}[\Pi_\gamma(s) - \Pi_\gamma(0)]$. The leptonic part $\Delta\alpha_{\text{lep}}(s)$ is calculable within QED and known to high accuracy. Quark loops, however, are modified by long-distance hadronic physics that cannot be calculated within QCD. Instead, the optical theorem

$$12\pi\text{Im}\Pi_\gamma(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \equiv R(s), \quad (4)$$

and the dispersion relation

$$\Pi_\gamma(s) - \Pi_\gamma(0) = \frac{s}{\pi} \int_0^\infty ds' \frac{\text{Im}\Pi_\gamma(s')}{s'(s' - s) - i\varepsilon}, \quad (5)$$

lead to

$$\Delta\alpha_{\text{had}}(s) = -\frac{\alpha(0)s}{3\pi} \text{Re} \int_0^\infty ds' \frac{R(s')}{s'(s' - s) - i\varepsilon}. \quad (6)$$

The hadronic vacuum polarisation contribution to the anomalous magnetic moment of the muon is derived in a similar way. The dominant leading order part is given by the dispersion integral^{3,14}

$$a_\mu^{\text{had,LO}} = \frac{1}{3} \left(\frac{\alpha}{\pi} \right)^2 \int_{m_\pi^2}^\infty ds \frac{K(s)}{s} R^{(0)}(s). \quad (7)$$

The integration kernel $K(s)/s \sim s^{-2}$ strongly emphasises the low-energy part of $R(s)$ with the main contribution and uncertainty stemming from the $\rho(770) \rightarrow \pi\pi$ resonance.

The computation of the dispersion integrals in Eqs. (6) and (7) requires the knowledge of $R(s)$ at any scale s , where in practice a mix of information is exploited. At low scales and just above the heavy quark thresholds e^+e^- annihilation data to hadrons (or, in some cases data from hadronic τ decays owing to isospin symmetry) are used. Narrow resonances can be described using analytical formulas with experimentally determined Breit–Wigner parameters, and the continuum region can be obtained with the use of perturbative QCD.

3. e^+e^- Data

3.1. *Experimental progress toward precision*

Exclusive low-energy $e^+e^- \rightarrow$ hadrons cross sections have been measured by experiments running at e^+e^- colliders. Fixed-energy measurements have been performed in Orsay and Novosibirsk. The most recent measurements of the $\pi^+\pi^-$ channel by the CMD2^{15–18} and SND¹⁹ experiments at the VEPP-2M collider at Novosibirsk have achieved comparable statistical uncertainties and energy-dependent systematic uncertainties down to 0.8% and 1.3%, respectively. These measurements involve running the collider at a series of different centre-of-mass energies. However, doing so, the detector conditions can also evolve between the different data-taking periods, hence the problem of evaluating precisely the systematic uncertainties and their correlations between measurements at different $\sqrt{s}$ values.

With the enhanced luminosity at colliders like the DAPHNE ϕ -factory in Frascati and PEP-II B-factory at SLAC, the use of the innovative technique of radiative return became possible. For this technique, events with a hard photon emitted from the initial state (ISR) are considered. This allows to cover, for a fixed value of the centre-of-mass energy of the e^+e^- collider and with the same detector conditions, a wide range of the spectrum of the hadronic final state. This type of measurement was first performed by the KLOE experiment,²⁰ where events with the ISR photon emitted along the beam-pipe were considered, using theory for the evaluation of the ISR effective luminosity. The BABAR experiment pioneered a luminosity-independent technique which consists in measuring the ratio of the cross sections $e^+e^- \rightarrow \pi^+\pi^-\gamma(\gamma)$ to $e^+e^- \rightarrow \mu^+\mu^-\gamma(\gamma)$,^{21,22} where large cancellations of systematic effects occur. Considering events with a hard ISR photon reconstructed in the detector, the BABAR analysis covers the full region of interest for $g-2$, from the $\pi\pi$ threshold up to 3 GeV. In order to achieve the desired sub-percent systematic precision, additional photons emitted either from the initial or from the final state have been taken into account in this analysis. The experimental systematic uncertainties are kept to 0.5% in the ρ peak region (0.6–0.9 GeV), increasing to 1% outside. More recently, the KLOE experiment has also performed measurements with the hard photon reconstructed in the detector,²³ as well as considering the ratio between the $\pi\pi$ and $\mu\mu$ spectra.²⁴ Figure 1 shows a summary of the cross section measurements for $e^+e^- \rightarrow \pi^+\pi^-$, in the dominant ρ resonance region.

In addition to the $\pi\pi$ channel, which brings about 75% of the hadronic contribution to the $g-2$ of the muon, the channels with kaons and those with higher multiplicities have a non-negligible contribution. The BABAR experiment has implemented a systematic programme for a precise measurement of these cross sections, exploiting the ISR method. This allowed to complete and often to significantly improve the precision of past measurements of these channels, performed at Orsay and Novosibirsk.

Fig. 1. Cross section for the process $e^+e^- \rightarrow \pi^+\pi^-$ measured by different experiments (top) and their relative difference with respect to the combined cross section (bottom). The uncertainty bars contain both the statistical and systematic components added in quadrature.

Several five- and six-pion modes involving π^0 's, as well as $K\bar{K}[n\pi]$ final states are still unmeasured. The isospin invariance is exploited to relate their contributions to those of known channels using dynamics information obtained in the BABAR study of the final states.²⁵

3.2. Progress in combining data

Using the input described in the previous section to compute the dispersion integral in Eq. (7) requires some special care. The requirements for averaging and integrating

cross section data are: (i) properly propagate all the uncertainties in the data to the final integral uncertainty, (ii) minimise biases, i.e. reproduce the true integral as closely as possible in average and measure the remaining systematic uncertainty, and (iii) optimise the integral uncertainty after averaging while respecting the two previous requirements. Different techniques have been used in past studies. For example, a “clustering” method (see e.g. Refs. 26–28) consisted in combining weighted measurements of different experiments within a prescribed energy interval. We describe here the approach used in our recent studies.

In order to better address the three requirements above, a new methodology has been developed.²⁹ The first item practically requires the use of pseudo-Monte Carlo (MC) simulation, which needs to be a faithful representation of the measurement ensemble and to contain the full data treatment chain (interpolation, averaging, integration). The second item requires a flexible data interpolation method (the trapezoidal rule is not sufficient as explained below) and a realistic truth model used to test the accuracy of the integral computation with pseudo-MC experiments. Finally, the third item requires optimal data averaging taking into account all known correlations to minimise the spread in the integral measured from the pseudo-MC sample.

The combination procedure transforms the bare cross section data and associated statistical and systematic covariance matrices into fine-grained energy bins, taking into account to our best knowledge the correlations within each experiment as well as between the experiments (such as uncertainties in radiative corrections). A detailed splitting of the systematic uncertainties in sub-components allows for an improved treatment of these correlations. The covariance matrices are obtained by assuming common systematic uncertainty sources to be fully correlated. To these matrices are added statistical covariances, present for example in binned measurements as provided by KLOE, BABAR or the τ data, which are subject to bin-to-bin migration that has been unfolded by the experiments, thus introducing correlations.

The interpolation between adjacent measurements of a given experiment uses second order polynomials. This is an improvement with respect to the previously applied trapezoidal rule, corresponding to a linear interpolation, which leads to systematic biases in the integral.²⁹ The averaging of the interpolated measurements from different experiments contributing to a given energy bin is the most delicate step in the analysis chain. It takes into account correlations between measurements and experiments, as well as the different measurement densities or bin widths that the experiments have within a given energy interval.²⁹ If the χ^2 value of a bin-wise average exceeds the number of degrees of freedom (n_{dof}), the uncertainty in this averaged bin is rescaled by $\sqrt{\chi^2/n_{\text{dof}}}$ to account for inconsistencies. This rescaling is applied locally, correctly taking into account the energy dependence of the potential tension between the measurements. Such inconsistencies frequently occur because most experiments are dominated by systematic uncertainties, which are difficult to estimate. For example, in the $\pi\pi$ channel, the existing tension between the BABAR

and KLOE measurements (see Fig. 1) prevents the expected precision improvement in the combination.²⁹

The consistent propagation of all uncertainties into the evaluation of $a_\mu^{\text{had,LO}}$ is ensured by generating large samples of pseudo experiments, representing the full list of available measurements and taking into account all known correlations. For each generated set of pseudo measurements, the identical interpolation and averaging treatment leading to the computation of Eq. (7) as for real data is performed, hence resulting in a probability density distribution for $a_\mu^{\text{had,LO}}$, the mean and RMS of which define the 1σ allowed interval. Common sources of systematic uncertainties also occur between measurements of different final state channels and are taken into account when summing up the exclusive contributions. Such correlations mostly arise from luminosity uncertainties, if the data stem from the same experimental facility, and from radiative corrections. These correlations have a non-negligible impact on the evaluated uncertainty of $a_\mu^{\text{had,LO}}$.

Figure 2 shows the cross section for the process $e^+e^- \rightarrow \text{hadrons}$ versus centre-of-mass energy $\sqrt{s}$. The result of the combination of the experimental measurements with its uncertainty, as well as the QCD prediction are shown.

Fig. 2. Cross section for the process $e^+e^- \rightarrow \text{hadrons}$ versus centre-of-mass energy $\sqrt{s}$. The band represents the combined experimental measurements within their uncertainty.²⁹ The red line shows the perturbative QCD prediction, the data points show the inclusive measurements from the BES experiment.^{30–33}

4. Use of tau Data

The use of tau data of semi-leptonic τ decays in the evaluation of a_μ^{had} and $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ was originally proposed in Ref. 26 based on the fact that hadronic spectral functions (or normalised invariant mass-squared distributions) from τ decays are directly related to the isovector vacuum polarisation currents when isospin invariance, or the conserved vector current (CVC), and unitarity hold. The CVC hypothesis relates the isovector, vector matrix element of the decay $\pi^- \rightarrow \pi^0 e^- \bar{\nu}_e$ to the electromagnetic form factor of the pion. At that time, the spectral functions, in particular of the $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ decay mode, and the τ -based evaluation of a_μ^{had} , were more precise than the corresponding e^+e^- results by a factor of two, as it is shown in Fig. 3. The precision of the τ -based evaluation for $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$, however, was comparable with the e^+e^- -based one. This is due to different QED kernels in the two evaluations with the former one giving a much larger weight to the low energy data.

With more precise data from both the e^+e^- annihilation and the τ decays, significant deviations were observed between the two sets of the evaluation.^{27,59} A few isospin-breaking sources are known. These include the short-distance electroweak radiative correction S_{EW} , final state radiation corrections in the $\pi^+\pi^-$ channel $\text{FSR}(s)$, long-distance radiative corrections $G_{\text{EM}}(s)$ of order α to the photon inclusive $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ spectrum (including both virtual and real photonic corrections), the $\pi^\pm - \pi^0$ mass splitting effects $\beta_0^3(s)/\beta_-^3(s)$ (important close to the threshold), and the ratio of electromagnetic to weak form factors $|F_0(s)/F_-(s)|$ corresponding to effects of the π mass splitting $\delta m_\pi = m_{\pi^\pm} - m_{\pi^0}$, the ρ mass splitting $\delta m_\rho = m_{\rho^\pm} - m_{\rho^0}$ expected from $\gamma - \rho$ mixing, and the difference $\delta\Gamma_\rho$ in the ρ meson widths. These isospin breaking effects were carefully studied in Refs. 65–68, which improved the agreement between the two calculations, though not resolving all discrepancies.^{25,65}

In Ref. 70, an additional correction originating from the $\gamma - \rho$ mixing was considered for the τ data, with a claimed improvement in the compatibility of the e^+e^- - and τ -based evaluations. However, unlike for the analogous $\gamma - Z$ mixing, the correction here is model-dependent because of the ρ hadronic structure.

5. Use of Theory

Sufficiently far from the quark thresholds, the inclusive hadronic cross section can be computed reliably using perturbative QCD. In a recent evaluation²⁵ we use four-loop perturbative QCD,⁵² including $\mathcal{O}(\alpha_s^2)$ quark mass corrections, between 1.8 and 3.7 GeV, as well as above 5 GeV. Non-perturbative contributions at 1.8 GeV were determined from data⁵³ and found to be small. The uncertainties of the R_{QCD} contributions account for the uncertainty in α_s , the truncation of the perturbative series, the full difference between fixed-order perturbation theory (FOPT) and, so-called,

Fig. 3. Summary of the evolution as function of time for a_μ^{had} and $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$. The corresponding references are Refs. 25–28,34–51,53–65,69–73. The error bars represent the total uncertainty by adding the experimental and theoretical uncertainties in quadrature.

contour-improved perturbation theory (CIPT), as well as quark mass uncertainties. The former three uncertainties are taken to be fully correlated between the various energy regions, whereas the (smaller) quark-mass uncertainties are taken to be uncorrelated.

Agreement within uncertainties is observed between the BES data and the QCD prediction below the $D\bar{D}$ threshold. Comparing the $a_\mu^{\text{had,LO}}$ predictions in the energy interval 2–3.7 GeV, we find $26.5 \pm 0.2 \pm 1.7$ for BES data, and 25.2 ± 0.2 for perturbative QCD. To study the transition region between the sum of exclusive

measurements and QCD, we have computed $a_{\mu}^{\text{had,LO}}$ in two narrow energy intervals around 1.8 GeV. For the energy interval 1.75–1.8 GeV we find (in units of 10^{-10}) $2.74 \pm 0.06 \pm 0.21$ (statistical and systematic uncertainties) for the sum of the exclusive data, and 2.53 ± 0.03 for perturbative QCD. For the interval 1.8–2.0 GeV we find $8.28 \pm 0.11 \pm 0.74$ and 8.31 ± 0.09 for data and QCD, respectively. The excellent agreement represents another support for the use of QCD beyond 1.8 GeV centre-of-mass energy.

6. Applications

We recall the two historically most important applications of dispersion-relation based hadronic vacuum polarisation evaluations: the hadronic contributions to the anomalous magnetic moment of the muon and to the running of the electromagnetic coupling constant.

6.1. The anomalous magnetic moment of the muon

The Dirac equation predicts a muon magnetic moment, $\vec{M} = g_{\mu} \frac{e}{2m_{\mu}} \vec{S}$, with gyromagnetic ratio $g_{\mu} = 2$. Quantum loop effects lead to a small calculable deviation from $g_{\mu} = 2$, parameterised by the anomalous magnetic moment a_{μ} . That quantity has been accurately measured by the E821 experiment at Brookhaven National Lab (BNL) which studied the precession of μ^{+} and μ^{-} in a constant external magnetic field as they circulated in a confining storage ring. Assuming CPT invariance, it found the combined value^{74,75} $a_{\mu}^{\text{exp}} = (11\,659\,209.1 \pm 5.4 \pm 3.3) \cdot 10^{-10}$, where the first error is statistical and the second systematic.

Comparison of experiment and theory tests the Standard Theory at its quantum loop level. A deviation in a_{μ}^{exp} from the ST expectation would signal effects of new physics, with current sensitivity reaching up to mass scales of $\mathcal{O}(\text{TeV})$ ^{76,77} (see Ref. 78 for a thorough review).

The ST prediction for a_{μ}^{ST} is generally divided into three parts,

$$a_{\mu}^{\text{ST}} = a_{\mu}^{\text{QED}} + a_{\mu}^{\text{EW}} + a_{\mu}^{\text{had}}. \quad (8)$$

The dominant QED and much smaller electroweak (EW) contributions have been calculated to five and two-loop levels, respectively, with negligible uncertainty.

The main uncertainty stems from the hadronic contribution a_{μ}^{had} , which is further split into lowest order and higher order parts. The dominant contributions originate from two-vertex functions and are obtained via a dispersion relation approach such as Eq. (7) for the lowest order term. Accurate measurements and limited use of perturbative QCD lead to the results $a_{\mu}^{\text{had,LO}} = (692.3 \pm 4.2) \cdot 10^{-10}$ and $(701.5 \pm 4.6) \cdot 10^{-10}$ for the $e^{+}e^{-}$ based and $e^{+}e^{-} + \tau$ based evaluations, respectively. The higher order hadronic contribution,⁷⁹ $a_{\mu}^{\text{had,NLO}} = (7 \pm 26) \cdot 10^{-11}$, is dominated

by the uncertainty in a term involving four photon vertices that is theoretically estimated and model dependent.

Adding all terms together gives the representative e^+e^- data based ST prediction⁷⁵

$$a_\mu^{\text{ST}} = (11\,659\,180.3 \pm 0.1 \pm 4.2 \pm 2.6) \cdot 10^{-10}, \quad (9)$$

where the errors are due to the electroweak, lowest-order hadronic, and higher-order hadronic contributions, respectively. The difference to experiment $a_\mu^{\text{exp}} - a_\mu^{\text{ST}} = (28.8 \pm 6.3 \pm 4.9) \cdot 10^{-11}$ (with all errors combined in quadrature) represents an interesting but not conclusive discrepancy of 3.6 times the estimated uncertainty. Using τ data in the relevant channels reduces that number to 2.4. The left column of Fig. 3 gives a historical view of the evaluations of $a_\mu^{\text{had,LO}}$. The individual results differ in the experimental datasets used, the averaging procedures, and the amount of theory injected into the calculation.

6.2. Running electromagnetic fine structure constant at M_Z^2

The running electromagnetic fine structure constant, $\alpha(s) = \alpha(0)/(1 - \Delta\alpha_{\text{lep}}(s) - \Delta\alpha_{\text{had}}(s))$, at the scale of the Z mass-squared, $s = M_Z^2$, is an important ingredient of the ST fit to electroweak precision data at the Z pole. Similar to a_μ , the leptonic contribution has a negligible uncertainty so that $\alpha(M_Z^2)$ is limited by the hadronic vacuum polarisation part.

Summing all the contributions to the integral (6) gives for the e^+e^- based hadronic term⁸⁰

$$\Delta\alpha_{\text{had}}(M_Z^2) = (275.0 \pm 1.0) \cdot 10^{-4}, \quad (10)$$

which is, contrary to the evaluation of $a_\mu^{\text{had,LO}}$, not dominated by the uncertainty in the low-energy data, but by contributions from all energy regions, where both experimental and theoretical errors are of similar magnitude. The corresponding τ -based result is $\Delta\alpha_{\text{had}}(M_Z^2) = (276.1 \pm 1.1) \cdot 10^{-4}$. The use of perturbative QCD instead of experimental data in the energy interval between 1.8 and 3.7 GeV (cf. Fig. 2) significantly improves the accuracy in the evaluation of $\Delta\alpha_{\text{had}}(M_Z^2)$. Excluding the analytical top-quark contribution from the result (10) gives $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2) = (275.7 \pm 1.0) \cdot 10^{-4}$. A historical compilation of $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$ evaluations is given on the right column of Fig. 3.

Adding the three-loop leptonic contribution,⁸¹ $\Delta\alpha_{\text{lep}}(M_Z^2) = 314.97686 \cdot 10^{-4}$, with negligible uncertainty, one finds

$$\alpha^{-1}(M_Z^2) = 128.962 \pm 0.014. \quad (11)$$

The running electromagnetic coupling at M_Z enters at various levels the global ST fit to electroweak precision data. It contributes to the radiator functions that modify the vector and axial-vector couplings in the partial Z boson widths to

fermions, and also to the ST prediction of the W mass and the effective weak mixing angle. Overall, the fit exhibits a -39% correlation between the Higgs boson mass and $\Delta\alpha_{\text{had}}(M_Z^2)$.⁸² The current precision of $\Delta\alpha_{\text{had}}(M_Z^2)$ is sufficient to not deteriorate the constraints obtained by the present electroweak fit.

7. Perspectives

Recent measurements of the cross section for $e^+e^- \rightarrow \text{hadrons}$ at energies less than 5–10 GeV have played a major role in particle physics. The increase in precision and completeness of these data have been essential for progressing in important areas of the electroweak and strong sectors. However even better measurements are necessary to match future needs. Fortunately, the BABAR ISR programme is near completion with results on the few last significant processes to be published soon. Also new experiments are underway: CMD-3 and SND-2 at VEPP-2000, BES3 at BEPC2, and in the future Belle-2 at SuperKEKB. Experimental progress requires an even better control of systematic uncertainties and also resolving inconsistencies in the present data which limit the accuracy of the combined results. It is not unreasonable to gain a factor of two in the course of the next few years over the energy range up to 5 GeV.

A short-term goal is to improve the theoretical prediction for the muon magnetic anomaly as a new generation of experiments are aiming at an increase in precision by a factor of four.^{83,84} Assuming the expected improvement in the leading order hadronic vacuum polarisation part, and even if there is no improvement in the current uncertainty assigned to the light-by-light contribution, the discrepancy would reach about 8 standard deviations if measured at the same central value. This would represent a compelling signal for new physics beyond the Standard Model.

An independent approach being explored by several groups is a first-principle calculation of HVP using lattice QCD.⁸⁵ This method, which allows one to cover the low-energy region in a purely theoretical way, involves the computation of the two-point quark-connected correlator for Euclidean q^2 . The major problem at present is to get precise enough lattice data at small q^2 and extrapolation at $q^2 = 0$ introduces systematic uncertainties which are difficult to estimate.⁸⁶ Currently the precision claimed is ranging from a few times to an order of magnitude worse than for the dispersive approach based on experimental data, but further progress is expected.

Improving the determination of $\alpha(M_Z^2)$ would also be required for future precision electroweak tests for example with the very large statistics available at a Z factory, as foreseen in the GigaZ option of the International Linear Collider (ILC).^{87,88}

Again with the same improvement of a factor of two in the accuracy of $\Delta\alpha_{\text{had}}(M_Z^2)$ and the precision expected for the electroweak observables at ILC with GigaZ, an indirect determination of M_H with an uncertainty of 7 GeV could be obtained⁸⁸ and compared to the known value of M_H as a precise test of the Standard Theory.

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