CERN-TH. 6118/91

QED multiphoton corrections to Bhabha scattering at low angles – Monte Carlo solution

S. Jadach[§]

CERN, Theory Division, Geneva 23, Switzerland,

E. Richter-Wąs

*Chair of Computer Science, Jagellonian University,
Kraków, ul. Reymonta 4, Poland*

B.F.L. Ward[†]

*Department of Physics and Astronomy,
The University of Tennessee, Knoxville, Tennessee 37996-1200
and*

SLAC, Stanford University, Stanford, California 94309

and

CERN, Theory Division, Geneva 23, Switzerland,

Z. Wąs

Institute of Nuclear Physics, Kraków, ul. Kawiorów 26a, Poland

Abstract

We calculate the QED-corrected integrated cross-section for small-angle Bhabha scattering with kinematical cuts very close to the real experimental cuts at the LEP/SLC experiments – no distinction between electrons and photons is made. We effectively compare up to four independent Monte Carlo calculations and one semi-analytical calculation. The aim of the exercise is to establish precision of the $\mathcal{O}(\alpha)$ Monte Carlo calculation with the exclusive exponentiation of the Yennie-Frautschi-Suura type. It provides the integrated cross-section for the low-angle Bhabha scattering process ($\vartheta < 10^\circ$), necessary for luminosity measurement, with an overall precision of 0.25%. The corresponding computer program BHLUMI 2.00 is in the form of a stand-alone Monte Carlo event generator. The complete and explicit definition of the multiphoton matrix element is given. Examples of numerical results from Monte Carlo phase space integration are demonstrated and discussed.

To be submitted to Physics Letters

[§] On leave of absence from Institute of Physics, Jagellonian University, Cracow

[†] Work supported in part by the US DOE contracts DE-AS05-76ER03956
and DE-AC03-76SF00515 and by TNRLC grant RCFY9101

CERN-TH. 6118/91

June, 1991

With the advent of higher statistics data the theoretical uncertainty due to radiative QED corrections in the luminosity measurements in LEP/SLC experiments was recently comparable to the pure experimental error. There is no problem of principle in calculating the QED correction to small-angle Bhabha scattering and it can be calculated perturbatively with precision, say $\leq 0.1\%$. The main problem is that this process is always measured for a complicated set of cuts, typically one requires e^+ to be detected in some angular range $(\vartheta_{\min}^{\text{Wide}}, \vartheta_{\max}^{\text{Wide}})$ and e^- in the different range $(\vartheta_{\min}^{\text{Narrow}}, \vartheta_{\max}^{\text{Narrow}})$. Furthermore, the bremsstrahlung photons emitted along the outgoing $e^\pm$ cannot be distinguished from $e^\pm$ themselves. This makes the calculation of the QED corrections non-treatable analytically, and the Monte Carlo (M.C.) method is the only one able to integrate over the allowed (multiphoton) phase space.

Substantial progress in diminishing the theoretical error from about 1% down to 0.3% was achieved in two recent papers [1] and [2]. In the first work [1] the $\mathcal{O}(\alpha)$ calculation for the small-angle Bhabha scattering was revisited. In particular, the analytical calculation for simplified cuts was done. This calculation was shown to agree with a slightly modified version [2] of one of the existing Monte Carlo $\mathcal{O}(\alpha)$ calculations [3] at the unprecedented level of 0.02%. The resulting set of analytical and M.C. calculations in the form of tables, formulas and programs is now available and may be used to check whether *any existing $\mathcal{O}(\alpha)$ calculation is correct down to the technical precision¹ of 0.02%*². In the second paper [2] the method of “upgrading” $\mathcal{O}(\alpha)$ calculations of QED effects in small-angle Bhabha scattering for the arbitrary experimental cuts was given. It consists of adding the leading-logarithmic (LL) $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\alpha^3)$ correction to the existing $\mathcal{O}(\alpha)$ calculation. The corresponding Monte Carlo event generator LUMLOG, necessary for calculation of the $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\alpha^3)$ corrections, see Table 1a, for arbitrary experimental cuts/acceptance was also provided. The main QED uncertainty of the calculation based on the above recipe is related to the neglected next-to-leading $\mathcal{O}(\alpha^2)$ correction and light-pairs production. It was estimated to be 0.2% and 0.1% respectively. The present work completes and improves on the preceding two works on the following two points: First, the “upgrade recipe” of paper [2] requires combining the complete $\mathcal{O}(\alpha)$ correction from one M.C. program [4] and the leading logarithmic $\mathcal{O}(\alpha^2) + \mathcal{O}(\alpha^3)$ obtained for *the same* cuts/acceptance from the *different* Monte Carlo program [2]. The whole procedure was not implemented in the form of a single, stand-alone, M.C. program. Although, from the perturbative calculation point of view, the above “upgrade recipe” is correct, nevertheless, it is inconvenient and dangerous in practical application because, when implemented in the complicated software environment of the experimental data analysis, it is prone to programming and computing errors. Second, the basic

¹As defined in papers [1, 2] *technical precision* includes machine rounding errors, programming bugs, approximations in matrix element and phase space integration, residual dependence on infrared regulators etc., while *physical precision* includes higher orders, missing diagrams etc.

²This check was performed by all LEP Collaborations on their $\mathcal{O}(\alpha)$ Monte Carlo programs [4], see ref. [2].

(a) OLDBIS+LUMLOG				(b) This work, BHLUMI 2.00			
$[1]$	$[\alpha^1]$			$[1]$	$[\alpha^1]$		
$[\alpha^1 L^1]$	$\{\alpha^2 L^1\}$	α^2		$[\alpha^1 L^1]$	$\{\alpha^2 L^1\}^\dagger$	α^2	
$[\alpha^2 L^2]$	$\alpha^3 L^2$	$\alpha^3 L^1$	α^3	$\{\alpha^2 L^2\}^{*\dagger}$	$\alpha^3 L^2^\dagger$	$\alpha^3 L^1$	α^3
$[\alpha^3 L^3]$				$\{\alpha^3 L^3\}^\dagger$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: *List of perturbative QED bremsstrahlung corrections. Order by order expansion is represented by rows starting from the first row which contains the Born approximation. The LL, NLL, etc. corrections are represented by the first, second, etc. columns, $L = \ln(Q^2/m_e^2)$. The two cases (a) and (b) describe works [1, 2] and the present work respectively. Contributions calculated exactly in the corresponding work are marked by [...] and those which are only calculated/estimated quantitatively for the purpose of the theoretical error are marked with {...}. The additional star * means that although this particular contribution is not included in a given calculation its magnitude is nevertheless well known from an independent source and the additional dagger $\dagger$ means that this particular contribution is included partly due to exponentiation and the remaining part (not included) is reduced significantly.*

$\mathcal{O}(\alpha)$ calculation in the recipe does not include exponentiation. Exponentiation is known to improve significantly the convergence of the perturbative expansion, i.e. it is able to diminish higher order effects which have been left out in the calculation [5]. It is able to reduce also, as we shall see in this paper, the next-to-leading logarithmic (NLL) contributions with respect to the leading-logarithmic (LL) ones. In addition, the "upgrade recipe" does not provide an event-by-event picture of the multiphoton effects themselves so that some of the details of the respective multiphoton final states are already integrated.

In this work we present results from the $\mathcal{O}(\alpha)$ exponentiated Monte Carlo calculation which is an improved version of the calculation presented in ref. [6]. It is implemented in a single stand-alone Monte Carlo event generator; therefore the first of the above two problems is solved. The improvement on convergence of perturbative calculation by exponentiation will be demonstrated. We shall show that for a realistic "calorimetric trigger" the ratio of NLL/LL is much less in $\mathcal{O}(\alpha)$ exponentiated calculation than in the ordinary unexponentiated $\mathcal{O}(\alpha)$ calculation.

The essential question discussed in this paper is: *what is the theoretical uncertainty of the presented $\mathcal{O}(\alpha)$ exponentiated calculation?* Let us sketch the answer before we come to details. The presented calculation does not have all of the $\mathcal{O}(\alpha^2)$ LL correction, see also pictorial representation in Table 1. We expect, however, that owing to exponentiation it has most of it. To prove this we use the LL programs of ref. [2] (which have all of $\mathcal{O}(\alpha^2 L^2)$) to calculate, for a few examples of typical experimental cuts, how big the remaining missing part of $\mathcal{O}(\alpha^2 L^2)$ corrections is

in the case of BHLUMI 2.00. As we shall see, it is below 0.2% and for the most interesting case of the asymmetric cut it is even at the level of 0.1%. The next two contributions which are smaller than $\mathcal{O}(\alpha^2 L^2)$, but of similar size, are $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L)$, see Table 1. The first one we can calculate using the LUMLOG Monte Carlo program [2] and we typically find it below 0.02%. The $\mathcal{O}(\alpha^2 L)$ contribution we estimate by looking into NLL/LL $\mathcal{O}(\alpha)$ ratio *for a given set of cuts* and we assume that similar ratio will hold in the $\mathcal{O}(\alpha^2)$. Owing to exponentiation the $\mathcal{O}(\alpha)$ ratio NLL/LL is rather small and even taking a big safety factor of ~ 6 we estimate the neglected $\mathcal{O}(\alpha^2 L)$ contribution to be below 0.1%, less than half that in ref. [2]. The net result is such that, although in our $\mathcal{O}(\alpha)$ exponentiated calculation we miss a (well-controlled) part of the $\mathcal{O}(\alpha^2 L^2)$, nevertheless we reach roughly the same physical precision as in ref. [2] because exponentiation reduces the next-to-leading logarithmic corrections. (In fact there is not much mystery in this reduction: without exponentiation the NLL corrections are in a sense “infrared divergent”, i.e. they grow logarithmically for strong cut-offs while in the exponentiated case they tend in this limit to a constant.)

Let us also note that, since we do not merely estimate but rather *calculate* the missing $\mathcal{O}(\alpha^2 L^2)$ contribution using the well-tested calculation of ref. [2], therefore, the previous calculation of ref. [2] is in a sense *included* as a part of the present one³. In fact, LUMLOG M.C. event generator is now a part of BHLUMI 2.00 and can be activated by appropriate arrangement of the input data.

Master formula: The complete formula for the $\mathcal{O}(\alpha^1)$ exponentiated total cross-section for the process $e^+(p_1) + e^-(q_1) \rightarrow e^+(p_2) + e^-(q_2) + n\gamma(k_i) + n'\gamma(k_{i'})$ as actually implemented in the BHLUMI 2.00 Monte Carlo program reads as follows⁴

$$\begin{aligned} \sigma = & \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{1}{n!} \frac{1}{n'!} \int \frac{d^3 q_2}{q_2^0} \frac{d^3 p_2}{p_2^0} \delta^{(4)}\left(p_1 + q_1 - p_2 - q_2 - \sum_{i=1}^n k_i - \sum_{i'=1}^{n'} k_{i'}\right) \\ & \exp\left(Y(\Omega_1, p_1, p_2) + Y(\Omega_2, q_1, q_2)\right) \\ & \int \prod_{i=1}^n \frac{d^3 k_i}{k_i^0} \tilde{S}(p_1, p_2, k_i) (1 - \Theta(\Omega_1; k_i)) \int \prod_{i'=1}^{n'} \frac{d^3 k_{i'}}{k_{i'}^0} \tilde{S}(q_1, q_2, k_{i'}) (1 - \Theta(\Omega_2; k_{i'})) \\ & \left(\bar{\beta}_0(Q, p_1, p_2, q_1, q_2) + \sum_{i=1}^n \bar{\beta}_1(Q, p_1, p_2, q_1, q_2, k_i) / \tilde{S}(p_1, p_2, k_i) \right. \\ & \quad \left. + \sum_{i'=1}^{n'} \bar{\beta}_1(Q, p_1, p_2, q_1, q_2, k_{i'}) / \tilde{S}(q_1, q_2, k_{i'}) \right) \Xi(p_i, q_i, k_j), \end{aligned} \quad (1)$$

³The calculation of refs. [1, 2] was originally aimed to be a test of BHLUMI [6] but it was published separately and earlier than BHLUMI in an attempt to help the LEP experiments to reduce the theoretical errors in the luminosity measurement in time for LEP data of the 1990 run.

⁴Note that taking only $n + n' = 0, 1$ and expanding the form-factor $\exp(Y(\Omega_1) + Y(\Omega_2))$ one recovers the ordinary non-exponentiated $\mathcal{O}(\alpha^1)$ expression for the differential cross-sections. For instance defining $\Omega_{1,2}$ by $k^0 < \varepsilon\sqrt{s}/2$ in the laboratory frame one recovers exactly the eq. (1) of ref. [1].

where $\tilde{S}(p_1, p_2, k) = -(\alpha/4\pi^2)((p_1/kp_1) - (p_2/kp_2))^2$ is the real photon infrared factor and

$$\begin{aligned} Y(\Omega, p_1, p_2) &= 2\alpha\tilde{B}(\Omega, p_1, p_2) + 2\alpha\Re B(p_1, p_2) \\ &= -2\alpha \frac{1}{8\pi^2} \int \frac{d^3k}{k^0} \Theta(\Omega; k) \left(\frac{p_1}{kp_1} - \frac{p_2}{kp_2} \right)^2 \\ &\quad + 2\alpha\Re \int \frac{d^4k}{k^2} \frac{i}{(2\pi)^3} \left(\frac{2p_1 - k}{2kp_1 - k^2} - \frac{2p_2 - k}{2kp_2 - k^2} \right)^2 \end{aligned} \quad (2)$$

is the standard Yennie-Frautschi-Suura form-factor [7]. It is infrared finite and $\Theta(\Omega; k) = 1$ for $k \in \Omega$ and $\Theta(\Omega; k) = 0$ for $k \notin \Omega$. The infrared Ω region includes the $k = 0$ infrared point and its definition may implicitly involve the dependence on fermion four-momenta p_i and q_i [7]. The Ω domain is typically defined through the $k^0 < E_{\min}$ condition in a certain reference frame. We shall define $\Omega_{1,2}$ and give the corresponding explicit formula for the form-factors later in the paper while describing the M.C. algorithm. (No physically sensible result depends on the choice of Ω .) The perturbative $\mathcal{O}(\alpha)$ QED matrix element is located in the $\bar{\beta}$'s which are

$$\bar{\beta}_0^{(1)}(Q, p_1, p_2, q_1, q_2) = \bar{\beta}_0^{(0)}(Q, p_1, p_2, q_1, q_2)(1 + \delta_0 + \delta_\gamma + \delta_Z), \quad (3)$$

$$\delta_0 = 2\Re F_1(Q^2) - 2\Re B(Q^2) = \frac{1}{2}\beta_t, \quad \beta_t = 2\frac{\alpha}{\pi} \left(\ln \frac{|Q^2|}{m_e^2} - 1 \right), \quad (4)$$

$$\bar{\beta}_0^{(0)}(Q, p_1, p_2, q_1, q_2) = \frac{2\alpha_r(t)^2}{s} \frac{(s^2 + u^2 + s_1^2 + u_1^2)}{4t_p t_q}, \quad (5)$$

$$\begin{aligned} \bar{\beta}_1^{(1)}(Q, p_1, p_2, q_1, q_2, k_i) &= \frac{\alpha_r(t)^2}{2s} \frac{\alpha}{4\pi^2} D_1^{(1)}(Q, p_1, p_2, q_1, q_2, k_i) \\ &\quad - (\tilde{S}(p_1, p_2, k) + \tilde{S}(q_1, q_2, k)) \bar{\beta}_0^{(0)}(Q, p_1, p_2, q_1, q_2), \end{aligned} \quad (6)$$

$$\begin{aligned} D_1^{(1)}(Q, p_1, p_2, q_1, q_2, k_i) &= \\ &\frac{1}{(kp_1)(kp_2)} \left\{ \frac{s^2 + u_1^2}{|t_q|} \left(1 - \frac{2m_e^2}{|t_q|} \frac{kp_1}{kp_2} \right) + \frac{s_1^2 + u^2}{|t_q|} \left(1 - \frac{2m_e^2}{|t_q|} \frac{kp_2}{kp_1} \right) \right\} \\ &+ \frac{1}{(kq_1)(kq_2)} \left\{ \frac{s^2 + u_1^2}{|t_p|} \left(1 - \frac{2m_e^2}{|t_p|} \frac{kq_1}{kq_2} \right) + \frac{s_1^2 + u^2}{|t_p|} \left(1 - \frac{2m_e^2}{|t_p|} \frac{kq_2}{kq_1} \right) \right\}, \end{aligned} \quad (7)$$

$$t = Q^2 = (p_2 + \sum_{i=1}^n k_i - p_1)^2, \quad t_p = -2p_1 p_2, \quad t_q = -2q_1 q_2,$$

$$s = 2p_1 q_1, \quad s_1 = 2p_2 q_2, \quad u = -2p_1 q_2, \quad u_1 = -2q_1 p_2.$$

We implement vacuum polarization through the QED running coupling constant $\alpha_r(t) = \alpha/[1 + \Pi(t)]$ at the proper $Q^2 = t$ scale. This takes into account the vacuum polarization correction in the $\mathcal{O}(\alpha^2 L^2)$, as was pointed out in ref. [8]. The correction $\delta_\gamma = t/s$ is due to s -channel γ exchange and the correction δ_Z represents here the interference of the t -channel photon with the s -channel Z

$$\delta_Z = \frac{\alpha_r(s)}{\alpha_r(t)} \left(\frac{t}{s} \right) \frac{2s^2}{s^2 + (t+s)^2} \left(1 + \frac{t}{s} \right)^3 (v^2 + a^2) \Re \left(\frac{s}{s - M^2 + is\Gamma/M} \right) \quad (8)$$

where $a = -1/(4 \sin \theta_W \cos \theta_W)$, $v = a(1 - 4 \sin^2 \theta_W)$, M and Γ are the usual coupling constants, mass and width of Z . We use $\sin^2 \theta_W = 0.2306$, $M = 91.161 \text{ GeV}$ and $\Gamma = 2.534 \text{ GeV}$ and these values are already precise enough for the purpose of luminosity measurement. In the above two corrections we keep terms, which are necessary for the precision $< 10^{-4}$ for angles $\vartheta < 10^\circ$.

Phase space integration: For a given acceptance Ξ , the master formula (1) and the following definitions determine the integrated cross-section and any inclusive distribution *completely and uniquely*. The only practical problem is to integrate properly and exactly over the multi-photon phase space.⁵ The integral is difficult to evaluate because of sharp bremsstrahlung and t -channel γ exchange peaks combined with the total four-momentum conservation constraint. Furthermore, any experimental acceptance/cut-off function $\Xi(p_i, q_i, k_j)$ is, in the typical luminosity measurement, complicated enough to render the integral unfeasible analytically. In practice, the phase space integral is calculable exactly only by the M.C. method. The problem of the M.C. integration/simulation of multiple photons in the presence of t -channel γ exchange was solved in ref. [6] and we employ this solution here. It is always highly desirable to have a possibility to turn the M.C. integration into an event generator with non-variable weights. This is easy to do in the present M.C. because its weight distribution is very regular, see ref. [9].

Trigger: The trigger function $\Xi = \Xi_{NW}$ used in the numerical calculations in this work closely resembles the typical asymmetric trigger of LEP experiments and the MINISAM trigger of MARK-II. The calorimetric detector is assumed to detect the electron and γ energy in the narrow angular range $\vartheta_{\min}^N < \vartheta_1 < \vartheta_{\max}^N$ on the e^+ beam side, ϑ_1 counted from e^+ beam direction, and in the wide angular range $\vartheta_{\min}^W < \vartheta_2 < \vartheta_{\max}^W$ on the e^- beam side, ϑ_2 counted from e^- beam direction. Furthermore, the detector on each side is divided into $n_\phi = 6$ azimuthal sectors. Let us define the total energy from all $e^\pm$ and γ 's deposited in each narrow i -th ϕ -sector as E_i^N and in the wide j -th sector as E_j^W . Let us number ϕ -sectors in such a way that $i = j$ for the sectors opposite with respect to the interaction point. For a triggered event, $\Xi = 1$, we require that in a pair of opposite sectors the deposited energy to be above a certain minimum $z_{\min} E_{\text{beam}}$, i.e. $E_i^N > z_{\min} E_{\text{beam}}$ and $E_i^W > z_{\min} E_{\text{beam}}$ simultaneously, otherwise $\Xi = 0$. The equivalent definition is $z > z_{\min}$ where $z = \max_{i=1, n} \min_{S=N, W} E_i^S / E_{\text{beam}}$. Note that taking $z_{\min} \rightarrow 1$ forces photons emitted along beams to be soft, but not photons collinear to outgoing $e^\pm$. In the numerical calculations we shall examine the dependence on the energy-cut parameter $z_{\min}$. The numerical values of $\vartheta_{\min, \max}^{N, W}$ are the same as in ref. [2]. In order to gain insight into the dependence on the limits of the angular range we introduce in addition to the narrow-wide asymmetric trigger Ξ_{NW} the two symmetric triggers Ξ_{NN} and Ξ_{WW} . For them the same angular range, either $(\vartheta_{\min}^N, \vartheta_{\max}^N)$ or $(\vartheta_{\min}^W, \vartheta_{\max}^W)$,

⁵It is often the case in the QCD calculations that part of the definition of differential/total cross-section is hard-wired into M.C. phase space integration algorithm such that the phase space and the matrix element are impossible to separate. In our case we follow the standard method "phase space times matrix element" as in the usual order by order QED calculations.

is used simultaneously on both sides, as in ref. [2].

More on the Monte Carlo: The basic difference of the present version 2.00 of BHLUMI with respect to older versions 1.xx is not in the M.C. algorithm but rather in the matrix element (functionally these things are well separated in the program). The main modification in the matrix element is the neglect of up-down interference. This contribution was found in ref. [1] to be very small in small-angle Bhabha, for the cut of this paper it is only 0.016%. This approximation allows us to consider bremsstrahlung from upper e^+ and lower e^- fermion lines independently, and to simplify the multiphoton bremsstrahlung matrix element considerably. In the process of writing $\mathcal{O}(\alpha)$ multiphoton matrix element in the YFS exponentiation it is necessary to extend/interpolate the single bremsstrahlung matrix element beyond the three-body phase space⁶. Instead of doing it by means of manipulating four-momenta arguments in the corresponding expressions, as in refs. [11, 6], (so-called reduction procedure) we rather extrapolate the single bremsstrahlung matrix element expressed in terms of Mandelstam variables, see eq. (1). This method gives the same numerical result while it leads to more compact and explicit expressions which are faster and numerically more stable in the computer evaluation. Since photon emission is considered independently for e^+ and e^- lines we may therefore define the infrared cuts $\Omega_{1,2}$ on soft photons independently for these two kinds of emissions, see eq. (1). This saves computing time with respect to older solutions in BHLUMI 1.xx where the single soft-photon cut was implemented by rejection in the laboratory frame (necessary if up-down interference is kept). In fact we define the soft-photon region Ω_i in terms of Sudakov variables y and z (of ref. [6]) by the simple condition $\max(y, z) < \delta$. The corresponding YFS formfactor for Ω_1 then reads

$$Y(\Omega_1, p_1, p_2) = \frac{\alpha}{\pi} \left\{ 2 \left(\ln \frac{2p_1 p_2}{m_e^2} - 1 \right) \ln \delta + \frac{1}{2} \ln \frac{2p_1 p_2}{m_e^2} - 1 \right\}. \quad (9)$$

In addition to the above changes we have removed certain annoying numerical instabilities in the differential cross-section evaluation⁷. Finally, we have replaced the primary three-parameter M.C. trigger of BHLUMI 1.xx defined in terms of angles and energies of “bare” final state electrons by a simpler two-parameter primary trigger in terms of $|t| = Q^2$, see the following discussion. This brings us closer to the typical LEP/SLC experimental trigger. For more details on the Monte Carlo algorithm in BHLUMI 2.00 and the usage of the program see ref. [9].

Numerical results: In Fig. 1 we present the total QED correction $\delta = \sigma/\sigma_{\text{Born}} - 1$ to small-angle Bhabha scattering for the asymmetric calorimetric trigger/efficiency

⁶This extrapolation is inherent to any kind of exponentiation and it is due to the fact that infrared singularities were subtracted and summed up to infinite order, see refs. [7, 10] for more comments.

⁷Most of them we traced to the simple fact that even in 64-bit floating point arithmetic the dot product of two very collinear four-vectors (photon and electron momenta) may easily change by a factor of two or more after these four-vectors are transformed from one Lorentz frame (frame of M.C. generation of photons) to another frame (laboratory frame). This can be cured by keeping track of the original values of the dot product before the Lorentz boost.

function Ξ_{NW} with six azimuthal sectors, as a function of the energy cut $z_{\min}$. The results for two symmetric calorimetric triggers are also included. The corrected $\mathcal{O}(\alpha^1)_{exp}$ cross-section, as defined in eq. (1) is calculated using BHLUMI 2.00. The vacuum polarization and the residual Z and s -channel γ exchange are included. The typical experimental energy cut is close to $z_{\min} = 0.5$. In Table 2 we present numerical values of the QED-corrected integrated cross-section for the same calorimetric asymmetric trigger as in Fig. 1, for one value of $z_{\min} = 0.5$. The principal components of the total cross-section are also listed. In Fig. 2 we calculate the second and higher-order corrections beyond $\mathcal{O}(\alpha)$ in the exponentiated $\mathcal{O}(\alpha)$ calculation of BHLUMI 2.00 by taking the difference of the BHLUMI correction and the non-exponentiated $\mathcal{O}(\alpha)$ correction from the OLDBIS M.C. [1] which is a slightly modified version⁸ of the OLDBAB M.C. [3], the standard M.C. program of PETRA experiments. In this case the Z and s -channel γ are switched off. The above difference is compared with the similar difference $\mathcal{O}(\alpha^3)_{exp}^{LL} - \mathcal{O}(\alpha^1)^{LL}$ from the LL program LUMLOG of ref. [2]. In ref. [2] experimental physicists were advised to take the combination $\mathcal{O}(\alpha^3)_{exp}^{LL} - \mathcal{O}(\alpha^1)^{LL} + \mathcal{O}(\alpha^1)$ as the best available QED calculation for luminosity measurement. In Fig. 2 we effectively compare this combination with the BHLUMI correction. If BHLUMI 2.00 had all $\mathcal{O}(\alpha^2 L^2)$ corrections included then the two series of curves in Fig. 2a would be almost identical. They may differ because the present version of BHLUMI does not include all of the $\mathcal{O}(\alpha^2 L^2)$ corrections or owing to lack of technical precision in BHLUMI. (Note that the technical precision of LUMLOG is known to be 0.02% [2] and is negligible.) In fact the two families of curves in Fig. 2a resemble each other very closely. To see it more clearly we calculate the difference between them and plot it in Fig. 2b. The plotted difference $\mathcal{O}(\alpha^1)_{exp} - \mathcal{O}(\alpha) - \mathcal{O}(\alpha^3)_{exp}^{LL} + \mathcal{O}(\alpha^1)^{LL}$ represents the missing $\mathcal{O}(\alpha^2 L^2)$ correction in BHLUMI (plus its technical error). As we see from Fig. 2b the missing $\mathcal{O}(\alpha^2 L^2)$ in BHLUMI is below 0.15% for $0.25 < z_{\min} < 0.80$. This estimate includes the technical precision of BHLUMI unless a special test is done to calculate it separately. Fig. 2b is *the main result of this paper* and we would like to stress that it combines high statistics M.C. with more than 10^8 events from three completely independent Monte Carlo programs. The fact that the plotted combination of the cross-sections is so close to zero for a wide range of the energy-cut parameter $z_{\min}$ and for three different types of angular cut (N-W, N-N and W-W) is highly non-trivial!

Subleading corrections: In Fig. 3 we examine the magnitude of the subleading correction in the exponentiated and non-exponentiated $\mathcal{O}(\alpha)$ calculations for the same calorimetric trigger. This exercise was done in the previous work [2] for a less realistic trigger on “bare” electrons and only for symmetric triggers in the non-exponentiated $\mathcal{O}(\alpha)$ case. As we see, in the present case of our calorimetric trigger the subleading correction is relatively large, up to 1.5% for $z_{\min} = 0.5$, higher than in ref. [2] where it was about 0.7% (for the trigger on “bare” $e^\pm$). On the contrary,

⁸In OLDBIS program we have implemented option of weighted events, reorganized matrix element and replace random number generator. The generation algorithm and most of the code is identical to that of OLDBAB.

for the exponentiated calculation from BHLUMI the subleading correction, see also Fig. 3, is generally rather small, below 0.25%, and $NLL/LL \sim 1/10$. This is a strong indication that exponentiation reduces not only the higher order corrections but also the leading versus next-to-leading correction⁹. The subleading corrections for BHLUMI we define as the difference between the calculation of eq. (1) and its LL version with $\tilde{\beta}_0$ and $\tilde{\beta}_1$ taken in the LL approximation. The truncation to LL is done as follows. In $\tilde{\beta}_0$ we should drop the non-logarithmic pure α/π term but it is (similarly as in s -channel [11]) accidentally equal to zero so $\tilde{\beta}_0$ remains unmodified. In $\tilde{\beta}_1$ we drop all terms explicitly proportional to m_e^2 and replace $(s_i^2 + u_j^2)/2s^2$ factors by the corresponding Altarelli-Parisi factors $\chi(x) = (1 + (1 - x)^2)/2$ where the definition of x depends on whether radiation is in the initial or final state. In practice we take $\chi(z)$ for $z > y$ and $\chi(y)$ for $y > z$ where y and z are Sudakov variables defined exactly as in ref. [6]. Note that the same definition and calculation procedure for the LL matrix element was already used to construct the $\mathcal{O}(\alpha^2 L^2)$ expressions for $\tilde{\beta}_{0,1,2}$ in the YFS2 M.C. program [11].

Overall precision: In Table 2b we summarize our evaluation of the total physical and technical uncertainty for BHLUMI 2.00. Error {1} due to hadronic uncertainty in the photon vacuum polarization is taken to be 0.8% according to ref. [12]¹⁰. The $\mathcal{O}(\alpha^2 L^2)$ missing correction {2} is calculated using LUMLOG, see also Fig. 2b. The second order subleading correction {3} is estimated to be 0.6 of {2} (averaged over $0.2 < z_{\min} < 0.8$). This is a rather conservative estimate in view of the fact that in the $\mathcal{O}(\alpha)_{exp}$ correction (without vacuum polarization) the ratio NLL/LL is only 1/10, see Fig. 3. This estimate is to be compared with the previous 0.2% estimate of ref. [2] (no exponentiation). The $\mathcal{O}(\alpha^3 L^3)$ correction {4} is calculated using LUMLOG, while the up-down QED interference correction {5} comes from OLDBIS. The dominant $e^\pm$ pair correction is calculated in the LL approximation using the non-singlet LL structure functions in LUMLOG, as in ref. [2]. Looking into singlet LL structure functions we estimate that for $z_{\min} \leq 0.5$ this contribution is always much below the non-singlet one {6}. We therefore set the upper limit for {7} as being half of {6}. By inspecting the s -channel examples of pair-production corrections [13, 14, 15] we estimate other contributions from the μ and quark pairs and any kind of NLL contribution {8} to be less than the sum of {6} and {7}. The $\sqrt{s} = 92.5$ GeV was chosen to maximize the Z contribution which in our case is $\delta_Z = -0.9\%$, see Table 2a. The Z contribution is included only in $\tilde{\beta}_0$ and not in $\tilde{\beta}_1$. This is sufficient since $\tilde{\beta}_1$ contributes less than 0.2% to the total cross-section¹¹. The only sizeable uncertainty in δ_Z may arise from improper treatment of the bremsstrahlung to this interference type correction. We therefore estimate its theoretical uncertainty {9} simply as $\delta_Z \cdot \delta_{brems} = 3 \cdot 10^{-4}$. The uncertainty due to s -channel γ exchange {10} is small because the whole contribution is very small. The sign of the corrections in Table 2b is indicated wherever it is known. The technical

⁹This result will help us to lower the limit on the subleading $\mathcal{O}(\alpha^2 L)$ correction, see Table 1.

¹⁰We thank Drs. H. Burkhardt and B. Pietrzyk for useful discussion on this point.

¹¹BHLUMI provides numerical results for $\tilde{\beta}_{0,1}$ separately.

(a) BHLUMI 2.00 cross-sections and corrections			
Cross-sections [nb]		Corrections	
$\sigma_0 = \sigma_{\text{Born}}$	26.606 ± 0.0	—	—
σ_1 ; brems. only	25.780 ± 0.011	$\delta_{\text{brems}} = (\sigma_1 - \sigma_0)/\sigma_0$	-0.0310 ± 0.0004
σ_2 ; vac. pol. on	27.178 ± 0.011	$\delta_{VP} = (\sigma_2 - \sigma_1)/\sigma_1$	0.0526 ± 0.0000
σ_3 ; δ_Z on	26.933 ± 0.011	$\delta_Z = (\sigma_3 - \sigma_2)/\sigma_2$	-0.0093 ± 0.0000
σ_4 ; δ_s on	26.897 ± 0.011	$\delta_s = (\sigma_4 - \sigma_2)/\sigma_2$	-0.0013 ± 0.0000
$\sigma_{\text{tot}} = \sigma_4$	26.897 ± 0.011	$\delta_{\text{tot}} = (\sigma_{\text{tot}} - \sigma_0)/\sigma_0$	0.0109 ± 0.0004

(b) BHLUMI 2.00 precision		
Type of contribution	Physical	Technical
{1} Vacuum polarization hadronic	0.08%	
{2} Missing part of $(\frac{\alpha}{\pi})^2 L^2$ brems.	0.15%	
{3} Missing part of $(\frac{\alpha}{\pi})^2 L$ brems.	0.09%	
{4} Missing part of $(\frac{\alpha}{\pi})^3 L^3$ brems.	0.000%	0.008%
{5} Up-down interference	+0.016%	0.001%
{6} e^+e^- pairs LL non-singlet	-0.015%	0.002%
{7} e^+e^- pairs LL singlet	< 0.007%	
{8} pairs NLL, quarks, μ	< 0.020%	
{9} Z exchange	0.030%	
{10} γ exchange s-channel	< 0.001%	
Total	0.25%	

Table 2: The integrated cross section at $\sqrt{s} = 92.5$ GeV and its precision. The cross sections and corrections in (a) are from BHLUMI 2.00. The program calculates the $\mathcal{O}(\alpha^1)$ exponentiated cross section as defined in eq. (1). The phase space is restricted by the asymmetric calorimetric trigger Ξ_{NW} with $z_{\min} = 0.5$. In (b) we list theoretical uncertainties of the corrections which are included in BHLUMI 2.00, e.g. {1,9,10}, or corrections which are left out in, e.g. {2-8}, but are calculated {2,4-6} or estimated {3,7,8}. Total theoretical error is indicated.

component of the error is in almost all cases below 25% of the correction itself and is usually not indicated. In the three cases when it is indicated it represents simply the M.C. statistical error. In {2} the M.C. statistical error of 0.05% is included and the technical precision content may be higher than 25%. In fact we have other independent evidence that the technical precision of BHLUMI is $\leq 0.2\%$. In ref. [16] we have found that results of BHLUMI for cuts on “bare” final electrons agree with the semi-analytical exponentiated calculation to better than 0.2% ¹². We have also checked that numerical results from BHLUMI 2.00 are stable to within a statistical error of 0.04% with respect to change of the type of random number generator¹³. The results from BHLUMI in Table 2 are stable with respect to lowering the primary cut $|t| > |t_{\min}|$ in the MC generation to within 0.03%, see Fig. 3 and the corresponding discussion in the following. The total precision 0.25% of BHLUMI 2.00 is obtained by taking the direct sum of entries {2} and {3} in Table 2b (they are probably correlated) and combining this in the quadrature with all other components in the table. This is a definite progress with respect to precision 1% of BHLUMI 1.00 quoted originally in ref. [6]. Our total precision is dominated by the vacuum polarization and missing second order corrections. We should remember that the calorimetric trigger also accepts 2γ and 3γ events. These contributions are small ($\sim 10^{-3}$) and are routinely subtracted from the luminosity data as a background. The estimate of theoretical error in Table 2 should not be treated as the ultimate solution but rather as a summary of our actual knowledge which may, and should, be gradually improved as a result of more precise calculations of the listed correction. In particular more precise information on the fermion-pair production, and the second order subleading correction better knowledge/separation of technical precision is of prime importance.

What do we recommend at the moment for the calculation of the QED correction in small-angle Bhabha scattering? If the pure experimental error is above 0.5% then BHLUMI alone is enough. Its precision tag of 0.25% for cuts similar to those in the present paper is well established and sufficient. For the experimental error is $\leq 0.5\%$ we recommend more detailed error analysis, as was done in this paper. The full detector simulation (so called digitalization) should be based on events from multiphoton BHLUMI. In addition, the OLDBIS+LUMLOG combination should be employed for cross-checks and theoretical error calculations, as in Table 2. For this purpose it is enough to run them with semi-realistic cuts (no digitalization). It is rather trivial to do it in practice because LUMLOG and OLDBIS are included as a part of BHLUMI 2.00 with the same format of the output Monte Carlo events.

Let us comment finally on the important question of uncertainty due to possible incompleteness of the phase space. In all present LEP/SLC experiments the experimental cut in the luminosity measurement eliminates the Compton peak at $Q^2 = 0$

¹²The intermediate version 1.12 was used for this exercise and it should be repeated for the actual BHLUMI 2.00 of this paper in a systematic way.

¹³The user of BHLUMI 2.00 may choose between two good quality random number generators RANMAR [17] and RANECU [18].

rather efficiently. In the actual experiments this is done in a complicated way – there is no chance of controlling this analytically. In BHLUMI 2.00 we generate $Q^2 = |t|$ transfer¹⁴ in the range $|t_{\min}| < |t| < |t_{\max}|$ and the condition $|t| > |t_{\min}|$ does the job of eliminating the Compton peak from the phase space. The primary M.C. cut $|t_{\min}|$ should be set low enough such that all phase space allowed by the experimental trigger is inside the $|t| > |t_{\min}|$ region and, therefore, none of the final numerical results (cross-sections etc.) for events passing through the experimental trigger depends on the actual value of $|t_{\min}|$. It cannot be pushed too low, however, because implementation of the experimental trigger will lead to excessive rejection. In Fig. 4 we show how to reach a reasonable compromise. We plot the $d\sigma/d|t|$ distribution for M.C. events before and after the experimental trigger defined for us as Ξ_{NW} with $x_{\min} = 0.5$. The content of the tail in the “triggered” distribution close to $|t_{\min}|$ should be reasonably small, say 0.03% and stable with respect to further lowering of $|t_{\min}|$. In Fig. 4 we see that the content of the tail is really below 0.03% and we have checked independently that the tail looks the same after lowering $|t_{\min}|$ by a factor of two. In Fig. 4 and for all other presented results we have chosen $|t_{\min}| = 0.8 z_{\min} s (1 - \cos \vartheta_{\min}^{\text{Wide}})/2$. We conclude that the uncertainty due to incompleteness of the phase space (an important component of technical precision of BHLUMI) can easily be reduced below 0.03%. An independent new answer to the question of the phase space completeness may be found in the recent dedicated study of ref. [19], where it was found that contribution to total cross section from events $e^+e^- \{e^-\gamma\}$ and $e^+e^- \{\gamma\gamma\}$, where $\{\dots\}$ are seen in the trigger, are very small; for angular range similar to ours they are below 0.02%.

Acknowledgments. Useful discussions with drs. M. Dallavalle, M. Dam, B. Pietrzyk, and W.D. Schlatter are acknowledged. One of us (B.F.L.W) acknowledges the support and kind hospitality of dr. J. Ellis of the CERN Theory Division.

References

- [1] S. Jadach, E. Richter-Wąs, B.F.L. Ward and Z. Wąs, Phys. Lett. **B253** (1991) 469.
- [2] S. Jadach, E. Richter-Wąs, B.F.L. Ward and Z. Wąs, Phys. Lett. **B260** (1991) 438.
- [3] F.A. Berends and R. Kleiss, Nucl. Phys. **B228** (1983) 537.
- [4] M. Böhm, A. Denner and W. Hollik, Nucl. Phys. **B304** (1988) 687; F. A. Berends, R. Kleiss and W. Hollik, Nucl. Phys. **B304** (1988) 712.

¹⁴This transfer has physical meaning only if up-down interference is neglected and/or in the LL approximation. Otherwise it is an intermediate parameter in the M.C. generation being a complicated function of the photon momenta dependent on details of the M.C. generation algorithm.

- [5] S. Jadach, M. Skrzypek and B.F.L. Ward, Phys. Lett. **B257** (1991) 173.
- [6] S. Jadach and B. F. L. Ward, Phys. Rev. **D40** (1989) 3582.
- [7] D. R. Yennie, S. Frautschi and H. Suura, Annal. Phys. (NY) **13** (1961) 379.
- [8] W. Beenakker, F. A. Berends and S. C. van der Marck, Nucl. Phys. **B355** (1991) 281.
- [9] S. Jadach, E. Richter-Wąs, B.F.L. Ward and Z. Wąs, Long-Write-up of BH-LUMI 2.00, in preparation.
- [10] S. Jadach and B.F.L. Ward, "Exclusive exponentiation in the Monte Carlo Yennie-Frautschi-Suura approach", in Proc. of Sussex University Conference "Electroweak Physics", eds. N. Dombey and F. Boudjema (Plenum Publ. Co., London, 1989),
- [11] S. Jadach and B.F.L. Ward, Comp. Phys. Commun. **52** (1990) 351.
- [12] H. Burkhardt et.al., Z. Phys. **C43** (1989) 497.
- [13] E. A. Kuraev and V. S. Fadin, Sov. J. Nucl. Phys. **41** (1985) 466.
- [14] F.A Berends, W.L. Van Neerven and G.J.H. Burgers, Nucl. Phys. **B297** (1988) 429.
- [15] B.A. Kniehl, M.Krawczyk, J.H.Hühn and R.G. Stuart, Phys. Lett. **B209** (1988) 337.
- [16] S. Jadach, E. Richter-Wąs, B.F.L. Ward and Z. Wąs, "High precision improved analytical exponentiation..." Univ. of Tennessee, Knoxville, preprint, UTHEP-91-0101, (1991).
- [17] F. James, Comp. Phys. Commun. **60** (1990) 329; G. Marsaglia, A. Zaman and W.-W. Tsang, Stat. Prob. Lett. **9** (1990) 35.
- [18] P. L'Ecuyer, Commun. ACM **31** (1988) 742.
- [19] R. Kleiss, "Magnitude of luminosity cross section with lost fermions at LEP", CERN-TH-6141/91 preprint, (1991).

Figure 1: The dependence of the total QED correction $\delta = \sigma/\sigma_{\text{Born}} - 1$ on the energy cut z_{min} where σ is defined in eq. (1) and calculated with BHLUMI version 2.00 Monte Carlo. (The vacuum polarization, Z and γ s -channel contribution are included.) In addition to δ for the most interesting asymmetric acceptance function Ξ_{NW} we also include results for symmetric triggers Ξ_{NN} and Ξ_{WW} . Statistical M.C. errors are indicated (they are usually below the size of dots). Results come from the M.C. sample of $2.5 \cdot 10^7$ weighted events.

Figure 2: Second and higher order correction for asymmetric calorimetric trigger Ξ_{NW} in BHLUMI and LUMLOG for varying energy cut $z_{\min}$. Plotted are $\delta^{B,L} = (\sigma_1^{B,L} - \sigma_2^{B,L})/\sigma_{\text{Born}}$, where σ_{Born} is calculated for the Narrow-Narrow angular range. $\sigma_1^B = \sigma\{\mathcal{O}(\alpha^1)_{\text{exp}}\}$, is from BHLUMI and unexponentiated $\sigma_2^B = \sigma\{\mathcal{O}(\alpha^1)\}$ is from OLDBIS [3, 1]. $\sigma_1^L = \sigma\{\mathcal{O}(\alpha^3)_{\text{exp}}\}$ and $\sigma_2^L = \sigma\{\mathcal{O}(\alpha^1)\}$ are both from LUMLOG [2]. In (a) we plot $\delta^{B,L}$ separately and in (b) their difference which represents the missing $\mathcal{O}(\alpha^2 L^2)$ bremsstrahlung correction in BHLUMI (together with its technical precision). Dotted lines in (b) mark the 0.15% limit quoted in Table 2b. Vacuum polarization, Z and s channel γ are switched off. MC statistical errors correspond to $2.5 \cdot 10^7$ of weighted events from BHLUMI and 10^8 events from OLDBIS and LUMLOG.

Figure 3: Subleading $\mathcal{O}(\alpha)$ corrections $\delta^{subl} = (\sigma - \sigma^{LL})/\sigma_{\text{Born}}$ as a function of the energy cut $z_{\min}$ for the asymmetric trigger Ξ_{NW} and the symmetric trigger Ξ_{NN} . In the non-exponentiated case the σ and σ^{LL} cross-sections are calculated using OLDBIS [3, 1] and $\mathcal{O}(\alpha^1)^{LL}$ comes from LUMLOG [2]. For the first order exponentiated calculation both $\mathcal{O}(\alpha^1)_{exp}$ and $\mathcal{O}(\alpha^1)^{LL}_{exp}$ cross-sections are from BHLUMI 2.00. The curves for the Ξ_{WW} trigger are not included but they are very similar to those which are presented.

Figure 4: The distribution $d\sigma/dQ^2$ for all generated M.C. events (solid thick line) for two triggers, symmetric Ξ_{WW} and asymmetric Ξ_{NW} , both with energy cut $z_{\min} = 0.5$ (the curve for Ξ_{NN} is very similar to that of Ξ_{NW} type and is not included). The transfer $t = Q^2$ is the primary transfer generated in the BHLUMI M.C. within limits $(t_{\min}, t_{\max})$ wider than the acceptance of the above two triggers. Vacuum polarization, Z and s channel γ are switched off. Figure (b) is the same as (a) but the vertical axis is expanded. The 0.1% of the Narrow-Wide cross-section with $z_{\min} = 0.5$ is visualized as a rectangle box.