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FUSION RULES IN TWO DIMENSIONAL GRAVITY

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ABSTRACT

We analyze the fusion rules for a Minimal Conformal Theory (not necessarily unitary) coupled to gravity in the continuum limit. We show that when the correlators are thought in terms of String Theory, and we include the ghost contributions the standard fusion rules for the ordinary primary fields are recovered, and most of the correlation functions involving null states of the Conformal Field Theory vanish in any Riemann surface.

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1. INTRODUCTION

There are still many puzzles in the understanding of the matrix model formulation of strings in dimensions $d \leq 1$ [1] and the continuum formulation of 2D gravity based on the Liouville field theory coupled to a conformally invariant model (for a review and references to the literature see [2]). Among them one can mention the continuum derivation of the Virasoro constraints satisfied by the partition function [3,4], and the explicit dictionary between the continuum scaling fields and those on the lattice. In particular it was found on the lattice that many operators had a non-vanishing one-point function, and that only for gravitational primaries [5] are the two-point functions diagonal [6]. In [7] a general procedure is introduced which would allow one to pass from the "matrix model frame" to the "conformal frame" for the scaling operators, although the program is carried out in full only for the Ising model. In this case the fusion rules of the order parameters are found to be the same as those in the absence of gravity. In matrix models [1] the number of scaling operators is infinite while the naive count of operators in the continuum description of a conformal model coupled to gravity gives a finite number. This discrepancy was resolved in the one-loop string computation presented in [8] and in the BRST cohomology analysis of these models [9], where it is shown that in the complete system of matter, Liouville and ghosts, the number of scaling operators is infinite, and what is more surprising, the states that in the conformal field theory represented null vectors become dressed with ghosts as well as with Liouville vertex operators and contribute to the partition function. Similar results were obtained in the computation of the three-point functions for Liouville-dressed primary fields [10,11,12]. By using various analytic continuation procedures in the matter central charge of the Virasoro algebra, exact answers were found in the continuum which agree in part with matrix model computations. Here one can also see that the new states mentioned above have in principle non-vanishing three-point functions.

In this paper we want to analyze in more detail the general three point functions of a generic minimal model [13] coupled to gravity. We follow Dotsenko's prescription [12] for the Liouville correlators, which is essentially equivalent to the analytic continuation of Goulian and Li [10] but more practical for explicit computations. We find a prescription to compute three-point functions guaranteeing the vanishing of all the one-point functions and that the two-point functions of primaries are diagonal. Furthermore, we show that the fusion rules computed this way are in general the irrational fusion rules found in [13] (the fusion rules obtained for a conformal theory whose fields are labelled by all the degenerate representation in the Kac table for a generic value of the matter central charge c_M). This is not very satisfactory especially in view of

the results in [7] about the Ising model. The solution comes from the realization that in these computations one systematically ignores the contributions from the ghosts to the three-point functions. One has to remember that we are trying to reproduce in the continuum results obtained for a non-critical string. We know that in string theory a correlation function can be interpreted geometrically as a measure over the moduli space of Riemann surfaces with distinguished points, and here one has to be careful and include the ghost contributions. As we will see, the ghost number assignments obtained in the analysis of Lian and Zuckerman [9] imply that the fusion rules for the dressed fields associated to the primary fields of a minimal conformal field theory are the same as the ones before the theory is coupled to gravity. In the Kac table we can divide the fields $\Psi_{m',m}$ into those which are inside the minimal conformal grid $1 \leq m \leq p-1, 1 \leq m' \leq p'-1$ and those outside which in conformal field theory are identified with the null states, and decouple from physical correlators. In [8,9] it is shown that all these fields contribute to the partition function, although the ghost dressing is different for the fields inside and outside the minimal grid. Ignoring the ghost number assignments, the three-point functions obtained are non-vanishing only when the irrational fusion rules are satisfied. When the ghost number is taken into account, the three-point function involving two fields inside and one outside of the minimal grid is zero as a consequence of ghost number mismatch. This will produce in general a set of non-associative fusion rules. However in the rational limit (for minimal conformal models) there is an inherent ambiguity in how to define the correlators of primary fields. This ambiguity can be used to choose the three-point functions in such a way that associativity is restored and consequently the rational fusion rules are recovered. Furthermore we show that most of the correlation functions involving fields outside the minimal grid vanish. This is in sharp contrast with the result for critical strings that the only non-trivial BRST cohomology classes have ghost numbers equal to $\pm 1/2$ [14,15]. For non-critical strings [9] the ghost number assignment for the BRST cohomology classes associated to the new states are very different from $\pm 1/2$ and the vanishing of many of the three-point functions involving the new states. As we will see this allows us to understand the fusion rules in the continuum.

This paper is organized as follows. In section 2 we review some of the results in matrix models concerning the fusion rules, and we present the general results of [8,9] for later reference. In section 3 we present our computation of the three-point functions, and how one obtains the irrational fusion rules when the ghost contribution is ignored. In section 4 we take into account the ghost contributions and compute the measures in the moduli space of punctured Riemann surfaces associated with the n -point functions of scaling operators. We prove how the ghost number mismatch truncates the fusion

rules and how the standard rational fusion rules are recovered. Section 5 contains the conclusions and speculations. We devote the Appendix to collecting some formulae useful in the derivation of the results in section 3.

2. SUMMARY OF KNOWN RESULTS

2.1 SCALING DIMENSIONS AND SPECTRUM

There are few exact results available in the continuum description of the coupling of a Conformal Field Theory (CFT) to $2D$ -gravity. Their derivation can be understood through a rather simple argument. In this paper we concentrate on the coupling of a minimal conformal model [13] to Liouville theory. These models are defined by two coprime integers (p', p) ; $p' > p$ and have a central charge

$$c = 1 - 6 \frac{(p' - p)^2}{p'p} = 1 - 3\alpha_0^2 \quad (2.1)$$

The primary fields have dimensions

$$\Delta_{m',m} = \frac{1}{8}[(m\alpha_+ + m'\alpha_-)^2 - \alpha_0^2] = \frac{1}{4p'p}[(mp' - m'p)^2 - (p' - p)^2] \quad (2.2)$$

with

$$\begin{aligned} \alpha_+ + \alpha_- &= \alpha_0 & \alpha_+ \alpha_- &= -2 \\ \alpha_+ &= \sqrt{\frac{2p'}{p}} & \alpha_- &= -\sqrt{\frac{2p}{p'}} \end{aligned} \quad (2.3)$$

The range of m, m' is

$$1 \leq m \leq p-1 \quad 1 \leq m' \leq p'-1 \quad m'p - mp' \geq 0 \quad (2.4)$$

These models are very special because the Verma modules $V_{m',m}$ generated by $\Delta_{m',m}$

have an infinite number of null vectors. Defining

$$\begin{aligned} a_{m',m}(k) &\equiv \frac{1}{4p'p}[(2p'pk + mp' - m'p)^2 - (p' - p)^2] \\ b_{m',m}(k) &\equiv \frac{1}{4p'p}[(2p'pk + mp' + m'p)^2 - (p' - p)^2] \end{aligned} \quad (2.5)$$

the null vector structure of the module $V_{m',m}$ is described by the diagram [16,17]

$$\begin{array}{ccccccc} & & a(0) \rightarrow b(0) & \rightarrow a(1) & \rightarrow b(1) & \rightarrow a(2) & \rightarrow \dots \\ E_{m',m}(p',p) & & \searrow & \times & \times & \times & \times \\ & & b(-1) \rightarrow a(-1) & \rightarrow b(-2) & \rightarrow a(-2) & \rightarrow \dots \end{array} \quad (2.6)$$

The meaning of this graph is as follows. Each node represents a Verma module with $a(k)$ or $b(k)$ as the dimension of its highest weight vector, and an arrow connecting two spaces $E \rightarrow F$ means that the module F is contained in the module E . This diagram allows us to compute the Virasoro character of the irreducible representation $L_{m',m}(p',p)$ obtained by eliminating the null vectors; the result is [17]

$$\begin{aligned} \chi_{m',m} &= \text{tr} q^{L_0 - c/24} = \\ &= \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} \left(q^{\frac{1}{4p'p}(2p'pk + mp' - m'p)^2} - q^{\frac{1}{4p'p}(2p'pk + mp' + m'p)^2} \right) \end{aligned} \quad (2.7)$$

and $\eta(q)$ is the Dedekind η -function. All the properties of the minimal models we have described can be represented explicitly in terms of a Coulomb gas formulation [18] which will be used in the next section to compute general three-point functions.

In the evaluation of n -point functions of gravitationally dressed operators we can split the computation into the matter and the Liouville parts as long as we assume that there is no mixing between both sets of fields in the action. The Liouville part for a genus h surface (we follow the notation of [2]) is

$$\langle \prod_i e^{\beta_i \phi(P_i)} \rangle = \int d\mu(\tau) \mathcal{D}\phi e^{-\frac{1}{8\pi} \int (\nabla_i \phi \nabla^i \phi + Q R \phi + \mu \Psi_{\min} e^{\beta_{\min} \phi})} \prod_i e^{\beta_i \phi(P_i)} \quad (2.8)$$

Here $d\mu(\tau)$ represents the measure over the moduli space of Riemann surfaces of genus h , and we lump there as well the contribution due to the ghost determinant. $\Psi_{\min}$ is the primary field in the CFT with lowest dimension, and $\beta_{\min}$ is its gravitational

dressings [19]. For a unitary theory this field is the identity. To evaluate (2.8) we insert an "area constraint" δ -function and integrate over all areas. Changing the order of integration we obtain

$$\langle \prod_i e^{\beta_i \phi(P_i)} \rangle = \int d\mu(\tau) \mathcal{D}\phi \int_0^\infty dA e^{-\mu A/8\pi} \delta\left(\int \Psi_{\min} e^{\beta_{\min} \phi} - A\right) e^{-\frac{1}{8\pi} \int (\nabla_i \phi \nabla^i \phi + Q R \phi)} \prod_i e^{\beta_i \phi(P_i)} \quad (2.9)$$

Next we eliminate the δ -function by splitting the integration over ϕ into the zero mode and the non-zero mode pieces $\phi = \phi_0 + \tilde{\phi}$. Using the Gauss-Bonnet theorem and standard properties of δ -functions yields

$$\langle \prod_i e^{\beta_i \phi(P_i)} \rangle = \frac{1}{\beta_{\min}} \int_0^\infty dA e^{-\mu A/8\pi} A^{\frac{1}{\beta_{\min}} (\sum \beta_i - Q(1-h)) - 1} \int d\mu(\tau) \mathcal{D}\tilde{\phi} \tilde{A}^{-\frac{1}{\beta_{\min}} (\sum \beta_i - Q(1-h))} e^{-\frac{1}{8\pi} \int (\nabla_i \tilde{\phi} \nabla^i \tilde{\phi} + Q R \tilde{\phi})} \prod_i e^{\beta_i \tilde{\phi}(P_i)} \quad (2.10)$$

where

$$\tilde{A} = \int \Psi_{\min} e^{\beta_{\min} \tilde{\phi}} \quad (2.11)$$

and

$$Q = \sqrt{\frac{25 - c_M}{3}}, \quad Q = \frac{2}{\gamma} + \gamma, \quad \gamma = \frac{Q - \sqrt{Q^2 - 8}}{2} \quad (2.12)$$

$$s = \frac{1}{\beta_{\min}} (Q(1-h) - \sum \beta_i)$$

We obtain finally

$$\langle \prod_i e^{\beta_i \phi(P_i)} \rangle = \frac{1}{\beta_{\min}} \left(\frac{\mu}{8\pi}\right)^s \Gamma(-s) \int d\mu(\tau) \mathcal{D}\tilde{\phi} \tilde{A}^s e^{-\frac{1}{8\pi} \int (\nabla_i \tilde{\phi} \nabla^i \tilde{\phi} + Q R \tilde{\phi})} \prod_i e^{\beta_i \tilde{\phi}(P_i)} \quad (2.13)$$

When the theory is unitary, $\Psi_{\min} = 1$, the action in the integrand of (2.8) is the standard Liouville action, and μ is the cosmological constant. In order to facilitate the comparison with matrix model results it is convenient to use (2.8) instead of the ordinary Liouville action. However, in the computation of three-point functions we will

work with the standard action. This is equivalent to fine tuning the coefficients of the most relevant operators (those of negative dimensions) in the non-unitary case. Using (2.13) we can read off the gravitational dimensions of fields on any genus surface and obtain the results of [19,20,21]. Furthermore, as noted in [8] for genus 1 and $\beta_i = 0$, the remaining functional integral is a free field functional integral and it can be carried out easily. So far we have ignored the ghost contribution, but it should now be included. The result of ref. [8] is that the partition function is proportional to

$$\int_{\mathcal{F}} d^2\tau \tau_2^{-3/2} |\eta(q)|^2 Z_M(q, \bar{q})$$

$$Z_M(q, \bar{q}) = \frac{1}{|\eta(q)|^2} \sum_{\substack{1 \leq m \leq p-1 \\ 1 \leq m' \leq p'-1}} |\chi_{m',m}|^2 \quad (2.14)$$

where the matter characters appear in (2.7). Since the η factors cancel, there are minus signs in the expansion of the integrand which are interpreted as ghost number counting contributions $(-1)^{\mathcal{G}}(-1)^{\bar{\mathcal{G}}}$. Hence, it seems that the would-be null states in CFT resuscitate although with ghost dressing. We can therefore distinguish the dressing of the standard primary fields which only get Liouville contributions from those of the new states which also get ghost dressing. The ghost structure of these new states is best understood in terms of the BRST analysis of Lian and Zuckerman [9].

2.2 BRST ANALYSIS

In the full theory of Liouville + matter and ghosts, the total value of the central charge is $c = 0$, and we can define a BRST charge Q

$$Q = \oint \frac{dz}{2\pi i} J(z)$$

$$J(z) =: (T_L(z) + T_M(z) + \frac{1}{2} T_{\text{gh}}(z)) c(z) : \quad (2.15)$$

The energy-momentum tensor for Liouville is

$$T_L(z) = -\frac{1}{2}(\partial\phi)^2 + \frac{Q}{2}\partial^2\phi \quad c_L = 1 + 3Q^2 \quad (2.16)$$

and the conformal dimension of exponential operators is

$$\Delta(e^{\beta\phi}) = -\frac{1}{2}\beta(\beta - Q) \quad Q = \sqrt{\frac{25 - c_M}{3}} \quad (2.17)$$

We will label the Fock spaces generated by the Liouville oscillators out of the state

created by the vertex operator $e^{\beta\phi}$ by $\mathcal{F}_{\beta,Q}$. The general analysis of the cohomology of the BRST charge can be carried out in terms of an arbitrary representation of the Virasoro algebra with energy-momentum tensor $T(z)$ and central charge $26 - c_M$. Let Λ^{bc} be the ghost Fock space, which can be thought of as a space of semi-infinite forms. If $L_{m',m}(p',p)$ is the irreducible Virasoro module associated to the primary field with dimension (2.2) and central charge (2.1), and V is a Virasoro module with central charge $26 - c_M$, the first result of Lian and Zuckerman [9] implies that the cohomology groups of Q (the BRST closed modulo exact states) acting on the space $\Lambda^{bc} \otimes V \otimes L_{m',m}(p',p)$ which will be denoted by $H_*^{\text{abs}}(V)$, are non-vanishing only when V is a vector space in the following sequence of embeddings

$$\begin{array}{ccccccc}
 & & 1 - a(0) \leftarrow 1 - b(0) & \leftarrow 1 - a(1) & \leftarrow 1 - b(1) & \leftarrow 1 - a(2) & \leftarrow \dots \\
 \tilde{E}_{m',m}(p',p) & & \searrow & \times & \times & \times & \times \\
 & & 1 - b(-1) \leftarrow 1 - a(-1) & \leftarrow 1 - b(-2) & \leftarrow 1 - a(-2) & \leftarrow \dots &
 \end{array} \tag{2.18}$$

where again every node in the graph represents a Verma module whose highest weight conformal dimension is given, and the arrows have exactly the same meaning as in (2.6). This exhibits a very interesting duality noticed in [16] between representations of Virasoro characterized by c, Δ and those characterized by $26 - c, 1 - \Delta$. The fact that the arrows in (2.18) are reversed with respect to those in (2.6) implies that the module $1 - a(0)$ has no null vectors, the first two joined with one arrow to $1 - a(0)$ have this module as a common null submodule, etc.

If we particularize this result to the case when V is the Fock module $\mathcal{F}_{\beta,Q}$, the result of [9] needed here is that the dimensions and ghost numbers of the absolute BRST cohomology groups are given by

$$\dim H_{n+1/2}^{\text{abs}}(\mathcal{F}_{\beta,Q}) = \delta_{n \pm d_{\beta}, 0} + \delta_{n \pm d_{\beta}, -1} \tag{2.19}$$

In this formula d_{β} is the distance along the arrows in (2.18) from the selected node in the graph to the tip $1 - a(0)$. The $+$ (resp. $-$) sign occurs when $\beta \leq Q/2$ (resp. $\beta > Q/2$).

As a final remark we should mention that in the comparison between the gravitational scaling dimensions that one finds in the matrix models and those obtained from (2.13) one is still missing in the continuum a detailed understanding of a set of operators whose gravitational scaling dimensions coincide with those one would obtain if we included the boundaries of the Kac table, in other words, states with dimensions $\Delta_{m',0}, m' \neq 0 \pmod{p}$. These states are also peculiar from the point of view of the

Virasoro modules analyzed in [16]. They only have one generating null vector and the embedding diagram for null vectors contains a single line of arrows. It should be interesting to find out whether one can construct modular invariants with their associated character, and if so, why such characters do not appear in the one-loop results of ref. [8].

3. IRRATIONAL CONFORMAL THEORIES COUPLED TO 2D GRAVITY

3.1 GENERALITIES

Recently, various groups [10], [11], [12], [22], have calculated the one-, two- and three-point functions in interacting Liouville theory coupled to minimal models. All these calculations rely on some (not yet fully understood) analytic continuation techniques, in the framework of the Coulomb gas representation of minimal models.

In the Goulian and Li prescription (see also [23]), after integrating the zero mode in the Liouville sector, one is left with an effective Coulomb gas amplitude but with a non-integer number of screenings. Then, analytic continuation in the central charge of the matter theory [10] or the tachyon momenta [22] is performed from the general case of integer number of screenings. On the other hand, Dotsenko's proposal uses an effective action with both macro- and microscopic Liouville screenings. Correlators are evaluated in a formal perturbative expansion giving us a Coulomb gas procedure allowing a negative number of Liouville screenings. Then analytic continuation of negative products gives results that are consistent with the ones obtained by means of the "dimensional regularization" techniques whenever they have been compared.

As a common feature, all these calculations in the interacting theory give a factorized three-body S-matrix, just as in the case of the free field (bulk) correlators in $c=1$ [24]. Thus, the so called normalization invariant ratio:

$$\frac{Z\langle A_1 A_2 A_3 \rangle^2}{\langle A_1 A_1 \rangle \langle A_2 A_2 \rangle \langle A_3 A_3 \rangle} \quad (3.1)$$

only depends on the scaling exponents, and agreement with some matrix model results [6] is quite trivial at this point. However, no descendant structure (in the sense of topological gravity) is seen, and apparently almost no fusion rules survive, while it is well known [6] that at least the gravitational primaries in unitary matrix models exhibit exact BPZ [13] fusion rules. The observation in [7] consisted of stressing the basic

ambiguity in any generally covariant theory, where the correlators are integrated over the surface and they are bound to pick up contact terms in the boundaries of moduli space. This freedom of adding distributions supported on the boundaries of moduli can be traced back to analytic redefinitions of the coupling constants in the matrix model potential. The authors of [7] carry out a technical analysis of the Ising model, where they find the precise analytic redefinition of couplings leading to BPZ fusion rules on the sphere. They also claim a complete solution of the operator dictionary between all critical points in the hermitian one-matrix model and $(2, 2k-1)$ minimal models. The transformation is completely determined by the requirement of orthogonality of the two-point functions, and is nicely supported by heuristic arguments involving the two-dimensional Wheeler-DeWitt equation. However, for theories other than the Ising model their transformation is probably only the first term in a more complicated transformation, because it fails to reproduce BPZ fusion rules.

We investigate here which are the fusion rules in the Liouville construction of the theory, by means of the usual tricks of analytic continuation. We would like to remark that these tricks imply an implicit convention for contact terms, and even the explicit forms of the Coulomb gas integrals, as given in [18] assume certain analytic continuations in regulating contact divergences. As usual in the Coulomb gas construction of minimal models, we work out first the irrational case, and investigate possible decouplings in the rational limit. Unless otherwise stated, the central charge is not rational in this section.

Our matter fields are $\Psi_{n',n} = e^{i\alpha_{n',n}\psi}$, ψ being the Feigin-Fuks free field. The central charge depends on the background charge as $c_M = 1 - 3\alpha_0^2$. The so called degenerate representations are given by the Kac's formula (see (2.3)):

$$\alpha_{n',n} = \frac{1-n}{2}\alpha_+ + \frac{1-n'}{2}\alpha_- \quad (3.2)$$

and the conformal weights are given by $\Delta_M(e^{i\alpha\psi}) = \frac{1}{2}(\alpha^2 - \alpha\alpha_0)$.

The Liouville vertices are $\Phi_{m',m} = e^{\beta_{m',m}\phi}$ and analogously:

$$c_L = 1 + 3Q^2, \quad \Delta_L(e^{\beta\phi}) = -\frac{1}{2}(\beta^2 - \beta Q) \quad (3.3)$$

The dressing interplay, $c_M + c_L = 26$, $\Delta_M + \Delta_L = 1$ implies the relations:

$$Q = \sqrt{8 + \alpha_0^2}, \quad \beta_+ = \alpha_+, \quad \beta_- = -\alpha_- \quad (3.4)$$

and the possible dressing combinations are:

$$e^{i\alpha_{n',n}\psi} e^{\beta_{-n',n}\phi} \quad \text{or} \quad e^{i\alpha_{n',n}\psi} e^{\beta_{n',-n}\phi} \quad (3.5)$$

where

$$\beta_{n',n} = \frac{1-n}{2}\beta_+ + \frac{1-n'}{2}\beta_- \quad (3.6)$$

It is useful to define the irrational number:

$$\rho = \frac{1}{\rho'} = \frac{\beta_+}{\beta_-} = -\frac{\alpha_+}{\alpha_-} \quad (3.7)$$

In Dotsenko's formulation [12] one calculates using a Coulomb gas technique based on both Liouville screenings; $e^{\beta_-\phi} = e^{\gamma\phi}$, $e^{\beta_+\phi} = e^{\frac{2\phi}{\gamma}}$. The resulting perturbative expansion is only formal, in the sense that negative numbers of screenings occur in the Liouville sector. If N and N' are the screening numbers for the matter correlator, one finds N and $-N' - 1$ for the Liouville part.

These negative products are defined as:

$$\prod_{i=1}^{-N'-1} f(i) = \prod_{i=0}^{N'} \frac{1}{f(-i)} \quad (3.8)$$

which curiously agrees with the more conventional Goulian-Li dimensional regularization, at least for diagonal fields [11]. Of course, the issue of the equivalence between these different tricks and even their precise physical meaning remains to be clarified. Under the analytic continuation (3.8), Liouville essentially cancels the bulk part of the matter structure constants and we end up with a leg-factorized result. However, one should be careful in two points:

First, we would like to obtain orthogonal two-point functions at least for the would-be gravitational primaries, and eventually (in the spirit of [7]) for all conformal primaries.

Second, since we keep a fully interacting Liouville sector we should ensure $\beta \leq Q/2$, the microscopic branch [2], for all insertions. This amounts to distinguishing matter operators in the irrational Kac table, according to their chirality. That is, we define the $(-)$ chirality operators as those $\Psi_{n',n}$ such that $n' < n\rho$, and $(+)$ operators as those such that $n' > n\rho$. This chiral distinction introduces selection rules in the dressing conventions (see also [24]): $\Psi_{n',n}^{(-)} \Phi_{-n',n}$ for $(-)$ fields, and $\Psi_{n',n}^{(+)} \Phi_{n',-n}$ for $(+)$ fields.

Now, defining $\alpha = (a', a) \in \mathbf{Z}_+^2$, we have four possible chirality classes of 3-point functions: $G_{\alpha\beta\delta}$, $G_{\beta\delta}^\alpha$, $G_\alpha^{\beta\delta}$, and $G^{\alpha\beta\delta}$. Here a superindex denotes a (+) field, and a subindex a (-) field.

Before considering the full correlators, we present some properties of the matter structure constants. Denote $\bar{\alpha}_{a',a} = \alpha_{-a',-a} = \alpha_0 - \alpha_{a',a}$ as usual, and define also the Coulomb gas given correlators:

$$C_{\alpha\beta\delta} = \langle \Psi_\alpha \Psi_\beta \Psi_\delta \rangle \quad (3.9)$$

and

$$C_{\beta\delta}^\alpha = C_{-\alpha\beta\delta} = \langle \Psi_{\bar{\alpha}} \Psi_\beta \Psi_\delta \rangle \quad (3.10)$$

Explicit formulae are found in the appendix.

The screening condition for $C_{\beta\delta}^\alpha$ ensures the truncation from above in the usual irrational fusion rules:

$$\alpha \leq \beta + \delta - 1, \quad \alpha + \beta + \delta = \text{odd} \quad (3.11)$$

For thermal algebras (fields of the form $\Psi_{1,m}$ or $\Psi_{m',1}$) the gamma functions appearing in $C_{\beta\delta}^\alpha$ develop zeroes which truncate the full irrational fusion rule [18] given by:

$$1 + |\beta - \delta| \leq \alpha \leq \beta + \delta - 1, \quad \alpha + \beta + \delta = \text{odd} \quad (3.12)$$

But this is no longer true for the general (n', n) theories. In fact, it is easy to check that in irrational theories, $C_{\beta\delta}^\alpha$ doesn't contain any zero along the diagonal of the Kac table. This is solved in [18] by going to the so-called symmetric structure constants, constructed as follows:

$$(D_{\beta\delta}^\alpha)^2 = \frac{C_{\beta\delta}^\alpha C_{\beta\alpha}^\delta}{C_{\beta\beta}^1} = \frac{\langle \Psi_{\bar{\alpha}} \Psi_\beta \Psi_\delta \rangle \langle \Psi_\alpha \Psi_\beta \Psi_{\bar{\delta}} \rangle}{\langle \Psi_1 \Psi_\beta \Psi_\beta \rangle} \quad (3.13)$$

normalized so that $D_{\alpha\alpha}^1 = 1$.

Here we have two superimposed screening conditions:

$$\frac{\alpha + \beta - \delta - 1}{2} \in \mathbf{Z}_+^2, \quad \frac{\beta + \delta - \alpha - 1}{2} \in \mathbf{Z}_+^2 \quad (3.14)$$

Moreover, the equivalence of D's and C's is shown in [18]

$$D_{\beta\delta}^\alpha = C_{\beta\delta}^\alpha a_\beta^{1/2} a_\delta^{1/2} a_\alpha^{-1/2}, \quad a_\alpha^{-1} = C_{\alpha\alpha}^1 \quad (3.15)$$

This means that $D_{\beta\delta}^\alpha$ are fully symmetric and the screening conditions above (3.14) define the domain of D as the usual fusion rules for the irrational case (3.12). So

the symmetric and non-symmetric structure constants are equivalent up to leg factors *within* the domain defined in (3.12). Now by dressing the product $\langle \Psi_{\bar{\alpha}} \Psi_{\beta} \Psi_{\delta} \rangle \langle \Psi_{\alpha} \Psi_{\beta} \Psi_{\bar{\delta}} \rangle$ with Liouville, we get at least the irrational fusion rules, which ensure vanishing one point functions (but for the identity), and orthogonal two-point functions.

There is however a stronger combinatorial property, whose proof we sketch in the appendix, stating that $C_{\alpha\alpha}^1$ is a suitable metric to raise indices:

$$C_{\alpha\beta\delta} = \Delta(\rho)\Delta(\rho' - 1)C_{\alpha\alpha}^1 C_{\beta\delta}^{\alpha} \quad (3.16)$$

again restricted to the intersection of the domains of both $C_{\alpha\beta\delta}$ and $C_{\beta\delta}^{\alpha}$. This relation is fundamental in the next section.

3.2 STRUCTURE CONSTANTS OF MATTER COUPLED TO GRAVITY

We can now apply Dotsenko's tricks to correlators in any chiral class. There are four possibilities; $G_{\alpha\beta\delta}$, $G_{\beta\delta}^{\alpha}$, $G_{\alpha}^{\beta\delta}$, and $G^{\alpha\beta\delta}$. In each case the Liouville part is obtained up to the scaling exponent, just by transforming the matter correlator under $\rho \rightarrow -\rho$ induced by $c \rightarrow 26-c$, and $(n', n) \rightarrow (-n', n)$ or $(n', -n)$ according to the corresponding dressing rule. Our computation differs from the one presented in reference [12] in that we use the physical structure constants of the operator algebra for the matter part in the four cases discussed below. In [12] the non-symmetric structure constants were used instead.

The cancellation quoted in [12] works perfectly for the mixed chirality cases: $G_{\beta\delta}^{\alpha}$ and $G_{\alpha}^{\beta\delta}$. Let us illustrate this for the first one ($\langle + - - \rangle$):

$$G_{\beta\delta}^{\alpha} = D_{\beta\delta}^{\alpha} L_{\beta\delta}^{\alpha} = a_{\beta}^{1/2} (C_{\beta\delta}^{\alpha})^{1/2} (C_{\beta\alpha}^{\delta})^{1/2} L_{\beta\delta}^{\alpha} \quad (3.17)$$

The vector of screenings for $C_{\beta\delta}^{\alpha}$ is:

$$(N', N) = \frac{\beta + \delta - \alpha - 1}{2} \quad (3.18)$$

and for $C_{\beta\alpha}^{\delta}$ is:

$$(\tilde{N}', \tilde{N}) = \frac{\alpha + \beta - \delta - 1}{2} \quad (3.19)$$

On the other hand, the Liouville screenings are:

$$N_L = \frac{b + d - a - 1}{2}, \quad N'_L = \frac{-b' - d' + a' - 1}{2} = -N' - 1 \quad (3.20)$$

Hence Dotsenko's cancellation works with $C_{\beta\delta}^{\alpha}$, not with $C_{\beta\alpha}^{\delta}$. However, as we know

from (3.15), they are related within the irrational constraint (3.12) by:

$$C_{\beta\delta}^\alpha = C_{\beta\alpha}^\delta a_\alpha a_\delta^{-1} \quad (3.21)$$

Now cancelling $C_{\beta\delta}^\alpha$ and $L_{\beta\delta}^\alpha$ as in [12] we find:

$$G_{\beta\delta}^\alpha = \mu^s K_{\beta\delta}^\alpha a_\alpha^{-1/2} a_\beta^{1/2} a_\delta^{1/2} \Delta_{-\alpha} \Delta_\beta \Delta_\delta \quad (3.22)$$

with $s = \frac{1}{\gamma}(Q - \sum \beta) = N'_L + \rho N_L$ and

$$\Delta_\alpha = \Delta(1 + a'\rho' - a), \quad K_{\beta\delta}^\alpha = K_{-\alpha\beta\delta} \quad (3.23)$$

$$K_{\alpha\beta\delta} = \pi^{a+b+d-2} \rho^{a'+b'+d'-a-b-d+2} \Delta(1 - \rho')^{a'+b'+d'} \quad (3.24)$$

There is an overall singular factor $\frac{1}{\Gamma(0\rho')}$ which has been absorbed in the partition function. It is remarkable that this term does not appear in Goulian-Li's dimensional regularization. This concludes the calculation for the $\langle + - - \rangle$ case.

The uniform chirality cases are more subtle. Consider for example the combination $\langle - - - \rangle$:

$$G_{\alpha\beta\delta} = D_{\beta\delta}^\alpha L_{\alpha\beta\delta} \quad (3.25)$$

In this case $(D_{\beta\delta}^\alpha)^2 \sim \langle \bar{\alpha}\beta\delta \rangle \langle \alpha\beta\bar{\delta} \rangle \sim \langle + - - \rangle \langle - - + \rangle$. Then the screening content of the Liouville term is:

$$N_L = \frac{a+b+d-1}{2}, \quad N'_L = \frac{-a'-b'-d'-1}{2} \quad (3.26)$$

while that of $\langle \bar{\alpha}\beta\delta \rangle$ is :

$$(N', N) = \frac{-\alpha + \beta + \delta - 1}{2} \quad (3.27)$$

and that of $\langle \alpha\beta\bar{\delta} \rangle$ is:

$$(\tilde{N}', \tilde{N}) = \frac{\alpha + \beta - \delta - 1}{2} \quad (3.28)$$

Here none of the screening numbers match as $N_{Liouville} = -N_{Matter} - 1$, and Dotsenko's cancellation is not obvious.

The calculation goes through nevertheless by noticing that $N_L = M$ and $N'_L = -M' - 1$ for M, M' the screening numbers of the correlator $C_{\alpha\beta\delta}$. Using (3.16) we can write:

$$G_{\alpha\beta\delta} = D_{\beta\delta}^\alpha L_{\alpha\beta\delta} = \Delta(1 - \rho)\Delta(2 - \rho')(a_\alpha a_\beta a_\delta)^{1/2} C_{\alpha\beta\delta} L_{\alpha\beta\delta} \quad (3.29)$$

Cancelling $L_{\alpha\beta\delta}$ against $C_{\alpha\beta\delta}$ yields

$$G_{\alpha\beta\delta} = \mu^s K_{\alpha\beta\delta} (a_\alpha a_\beta a_\delta)^{1/2} \Delta(2 - \rho')\Delta(1 - \rho)\Delta_\alpha \Delta_\beta \Delta_\delta \quad (3.30)$$

Analogous considerations for the $G_\alpha^{\beta\delta}$ and $G^{\alpha\beta\delta}$ cases lead to similar results. Here, however, cancellations occur in the reflected sector. This is because now one has to analytically continue in the unprimed screenings. The net result amounts to transforming the Kac indices $n \leftrightarrow n'$ and $\rho \leftrightarrow \rho'$. The final structure constants for all chiralities are:

$$\begin{aligned} G_{\alpha\beta\delta} &= \mu^s K_{\alpha\beta\delta} (a_\alpha a_\beta a_\delta)^{1/2} \Delta(2 - \rho')\Delta(1 - \rho)\Delta_\alpha \Delta_\beta \Delta_\delta \\ G_\alpha^{\beta\delta} &= \mu^s K_{\beta\delta}^\alpha (a_\alpha)^{-1/2} (a_\beta a_\delta)^{1/2} \Delta_{-\alpha} \Delta_\beta \Delta_\delta \\ G^{\alpha\beta\delta} &= \mu^s K^{\alpha\beta\delta} (a_\alpha a_\beta a_\delta)^{1/2} \Delta(2 - \rho')\Delta(1 - \rho)\Delta^\alpha \Delta^\beta \Delta^\delta \\ G_\alpha^{\beta\delta} &= \mu^s K^{-\alpha\beta\delta} (a_\alpha)^{-1/2} (a_\beta a_\delta)^{1/2} \Delta^{-\alpha} \Delta^\beta \Delta^\delta \end{aligned} \quad (3.31)$$

where now:

$$\Delta^\alpha = \Delta(1 - a' + \rho a) \quad (3.32)$$

$$K^{\alpha\beta\delta} = \pi^{a'+b'+d'-2} \rho^{a'+b'+d'-a-b-d-2} \Delta(1 - \rho)^{a+b+d} \quad (3.33)$$

We note that Kitazawa's [11] calculation is fully equivalent to the case $\langle - - - \rangle$ sketched here, since $\rho > 1$ and all diagonal fields are of type $(-)$. The pattern of cancellations is the same, although the exact structure constants differ from what we call $K_{\alpha\beta\delta}$.

In view of (3.31), the important point to be stressed is that, although we find quite simple structure constants, there is some residual interaction left, embodied in the selection rules for different chiralities. In particular, the leg factors that a particular operator picks up depend on whether it is on majority or minority in the corresponding chiral region.

We conclude this section with some remarks on the rational limit of (3.31). Since the Liouville correlators cancel the bulk part of the matter structure constants, essentially no new decouplings occur in the rational limit $\rho \rightarrow p/p'$, and the fusion rules appear to be the irrational ones. In particular, the familiar symmetry $(n', n) \rightarrow (p' - n', p - n)$ is lost as a consequence of our chirality rules. So, the usual mapping between conformal weights (or scaling dimensions) and operators is ambiguous, reflecting the different compatible screening conventions we have in the rational case. Naively one has irrational fusion rules, restricted to a certain fundamental region obtained by modding the Kac table by the symmetries $(n', n) \rightarrow (-n', -n)$ and $(n', n) \rightarrow (p' + n', p + n)$, for example, $0 \leq n' \leq p' - 1$, $n > pn'/p'$, and then the fusion rules depend on the choice for this fundamental region or equivalently, on the choice made for the screening rules. In the next section however we will show how the ghosts decouple all microscopic operators sitting outside the minimal grid, and how the BPZ fusion rules are obtained.

The appearance of the factors Δ_α and Δ^α , similar to the "leg-poles" found in $c=1$ [25], is remarkable. They pick zeroes or poles at the boundaries of the minimal grid, and those states obtained by the action of the symmetries mentioned in the previous paragraph. In particular, the term Δ_α develops zeroes for $\alpha = (kp', n)$, $n \leq kp$ and poles for $n > kp$. Conversely Δ^α gives zeroes for $\alpha = (n', kp)$, $n \leq kp'$ and poles for $n > kp'$. The products $\Delta_\alpha \Delta_{-\alpha}$ and $\Delta^\alpha \Delta^{-\alpha}$ are regular. In addition, the matter normalizations a_α are regular except for these points. In particular, those of the form (kp', n) for $n \neq 0(\text{mod } p)$, $n < kp$ are poles and those with $n > kp$ are zeroes; analogously, all points (n', lp) with $n' \neq 0(\text{mod } p')$ give poles for $n' < lp'$ and zeroes for $n' > lp'$. Whether this structure is reminiscent of the similar phenomenon in $c=1$ is not yet clear.

4. AMPLITUDES, GHOSTS AND THE TRUNCATION OF THE FUSION ALGEBRA

Amplitudes in string theory define measures on the moduli space of Riemann surfaces with punctures. The operator representation of these measures for the bosonic and fermionic string was presented in [26,27]. In this section we generalize some aspects of these measures to the case of non-critical strings.

We recall the main properties of the operator formalism in the case of the bosonic string. The operator formalism is based on interpreting a conformal field theory as a way to associate quantum states to Riemann surfaces. The Schrödinger representation of these states can be obtained using the functional integral representation and imposing boundary conditions at the punctures (see [26,27] for details and references). For

the standard 26-dimensional bosonic string the two conformal field theories involved are the ghost system and 26 free bosonic fields. For non-critical strings the relevant conformal field theories contain the Liouville field together with the matter sector. We assume that at least formally it is possible to use the Liouville theory to associate states to Riemann surfaces using the functional integrals. The explicit form of these states will not be relevant for our general discussion.

Denote by $|\Sigma_{g,N}\rangle^{m,gh}$ the state associated to a Riemann surface of genus g and n punctures, where the superscript m denotes collectively the matter and Liouville fields and gh is the ghost contribution. The scattering measure for N external states $|\chi_1\rangle, \dots, |\chi_N\rangle$ is given by

$$\langle\chi_1|\langle\chi_2|\dots\langle\chi_N|\prod_{i=1}^{3g-3+N}\bar{b}(v_i)b(v_i)|\Sigma_{g,n}\rangle^{(m)}|\Sigma_{g,n}\rangle^{(gh)} \quad (4.1)$$

where the $3g - 3 + N$ vector fields v_i represent a basis of the tangent space to the moduli space at the point given by the surface $\Sigma_{g,N}$.

The states $|\Sigma_{g,N}\rangle$ depend on what local coordinates we have chosen close to the punctures. A necessary condition for the amplitude (4.1) to define a good moduli measure is invariance under reparametrizations of the punctures. The transformation law of $|\Sigma_{g,N}\rangle$ under a change of coordinates $z \mapsto z + \epsilon v(z)$ is generated by the energy-momentum tensor $\oint T(z)v(z)dz$, with the integral taken around the puncture. Using this transformation law for the states $|\Sigma_{g,N}\rangle$ and taking into account that local changes of coordinates are generated by vector fields which are holomorphic in a neighborhood of the puncture and vanishing at its location, we obtain the following condition for the external states

$$L_n^{\text{tot}}|\chi_i\rangle = 0 \quad n \geq 0 \quad (4.2)$$

which are necessary for the reparametrization invariance of (4.1).

Defining the ghost number for the ghost fields $b(z)$ (resp. $c(z)$) as -1 (resp. $+1$), the ghost number of a generic state $|\Sigma_{g,N}\rangle$ is $3g - 3 + 3N/2$. In particular the state $|0\rangle$ associated to the Riemann sphere with one puncture has ghost number $-3/2$. This state is defined by conditions

$$b_n|0\rangle = 0 \quad n \geq -1 \quad c_n|0\rangle = 0 \quad n \geq 2 \quad (4.3)$$

These ghost number assignments imply for the external states appearing in (4.1)

$$\mathcal{G}|\chi_1\rangle \otimes |\chi_2\rangle \otimes \dots \otimes |\chi_N\rangle = -\frac{N}{2}|\chi_1\rangle \otimes |\chi_2\rangle \otimes \dots \otimes |\chi_N\rangle \quad (4.4)$$

From the point of view of the BRST cohomology, physical states are required to be in non-trivial cohomology classes. In the case of the critical bosonic string in flat Minkowski space (with signature $(25, 1)$) and non-zero momentum for the center of mass, the only non-vanishing cohomology groups are H_m with $m = \pm 1/2$ [15]. The only way to saturate the constraint (4.4) is therefore with the ghost number of each of the external states in (4.1) taken as $-1/2$. To write explicitly the form of the states χ_i , we introduce two states associated to the ghost Hilbert space on the sphere : $|\pm\rangle$, with ghost numbers $\pm 1/2$ and satisfying the conditions

$$b_n|-\rangle = 0 \quad n \geq 0 \quad c_n|+\rangle = 0 \quad n \geq 0 \quad (4.5)$$

These states satisfy $L_0|\pm\rangle = -1|\pm\rangle$; hence the state χ_i becomes $|\chi_i\rangle = |\chi_i\rangle^{(m)} \otimes |-\rangle$. For these states the BRST condition $Q|\chi\rangle = 0$ implies the condition (4.2) .

Now we are ready to extend the previous result to the case of non-critical strings. The main change comes from the BRST cohomology structure discussed in section 2.2. Unlike the critical string case, there are BRST cohomology classes with ghost numbers not restricted to being $\pm 1/2$ (cf. (2.19)). The non-vanishing classes are labelled by

$$H_{d_\beta \pm 1/2} \quad \beta \leq Q/2$$

$$H_{-d_\beta \pm 1/2} \quad \beta > Q/2 \quad (4.6)$$

Recall that d_β is the distance from the tip in the tower of embeddings (2.18) of the state chosen. The condition (4.4) is however independent of whether we work with critical or non-critical strings. Its origin is purely geometrical. Therefore if we restrict ourselves to microscopic states i.e. $\beta \leq Q/2$, the only possible non-vanishing amplitudes are those restricted to the standard dressed primary fields of the minimal CFT $\Psi_{m',m}$, $1 \leq m \leq p-1$, $1 \leq m' \leq p'-1$ or its boundaries of the form $(o, m), (m', o)$. We will call this domain in the Kac table the minimal conformal grid. The theorem of Lian and Zuckerman was strictly proved for fields in the minimal grid; however there does not seem to be any obstruction to extending it to boundary states, and we will assume this to be the case. All we need is the ghost number of the operator $\Psi_{m',0}$. According to the theorem of Lian and Zuckerman, these fields all have ghost number equal to $-1/2$ while the other fields associated to the null vectors of the CFT and with $\beta \leq Q/2$ always have ghost number $\geq 1/2$. Therefore the correlators involving two fields in the minimal grid and one of the null descendants do not satisfy (4.4) and they vanish. A similar argument applies to the three-point function involving two null descendants

and one ordinary field. More precisely we can say that any correlator involving any (different from zero) number of null descendants satisfying the condition $\beta \leq Q/2$ vanishes identically as a consequence of ghost number counting.

We now reconsider the study of the fusion rules started in the previous section. There, it was shown using Dotsenko's prescription [12] that one is led generically to the irrational fusion rules. If we apply these fusion rules we are sometimes driven outside the minimal grid. However, the ghost-induced decouplings truncate the fusion rules to:

$$1 + |\beta - \delta| \leq \alpha \leq \min\{\beta + \delta - 1, \kappa\}, \quad \alpha + \beta + \delta = \text{odd} \quad (4.7)$$

where $\kappa \equiv (p', p)$. These are not exactly the BPZ rules, but only the irrational ones restricted to the minimal conformal grid. It is remarkable that (4.7) do not give an associative OPE. So the ghosts appear to spoil the associativity of the irrational OPE, as inherited from the matter sector. The only associative truncation of the irrational fusion rules is BPZ's.

As a matter of fact, using the enhanced symmetry of the rational theories, one can choose screening conventions (i.e. renormalization prescriptions) such that BPZ's are obtained. The point is that the Liouville "energies" (β), only depend on the combination $|np' - n'p|$; thus they are invariant under the symmetry $\alpha = (n', n) \rightarrow \alpha^* = (p' - n', p - n)$ (in this case $\alpha^* \equiv \bar{\alpha}$ because $p\alpha_+ + p'\alpha_- = 0$). In consequence we can avoid the chiral distinction for rational theories and set up the prescription:

$$L_{\alpha\beta\delta} = L_{\alpha}^{\beta^*\delta^*} \quad (4.8)$$

ensuring that the Liouville part is insensitive to the $\star$ transformation. If we select now the screening convention for matter as the one involving a minimal number of screenings, then we truncate to BPZ's:

$$1 + |\beta - \delta| \leq \alpha \leq \min\{\beta + \delta - 1, 2\kappa - 1 - \beta - \delta\} \quad \alpha + \beta + \delta = \text{odd} \quad (4.9)$$

In other words, the rational limit we are taking in the matter part picks up the linear combination:

$$\theta(\vec{N}^* - \vec{N})D_{\beta\delta}^{\alpha} + \theta(\vec{N} - \vec{N}^*)D_{\beta^*\delta^*}^{\alpha} \quad (4.10)$$

where

$$\vec{N} = \frac{\beta + \delta - \alpha - 1}{2}, \quad \vec{N}^* = \frac{2\kappa - \beta - \delta - \alpha - 1}{2} \quad (4.11)$$

and $\theta(\vec{V})$ vanish if some component of $\vec{V}$ is negative and equals one otherwise. This prescription gives the same result as taking the symmetric product: $(D_{\beta\delta}^{\alpha} D_{\beta^*\delta^*}^{\alpha})^{1/2}$.

Of course, this truncation is not unique. The existence of many different operators sharing the same scaling dimension in the rational limit, allows also different screening prescriptions, each one representing a particular convention of contact terms at the boundaries of moduli space (or analytic transformations of couplings in the language of matrix models). Thus, all possible fusion rules in the conformal frame (Liouville $\times$ matter), are certain restrictions of the irrational ones.

5. CONCLUSIONS

In this paper we have studied the fusion rules of a minimal CFT coupled to gravity. Since the notion of operator product expansions in this context is not completely clear, we have taken a practical definition of the fusion rules. They are the selection rules determining when three-point functions of dressed operators are different from zero. Using Dotsenko's prescription we showed that if we take a reasonable choice for the matter three-point functions, we are led to the irrational fusion rules yielding vanishing one point functions (except for the identity operator) and diagonal two-point functions. This differs from Dotsenko's analysis [12] in that we analytically continue to include a negative number of screenings only in the Liouville part of the correlator and not in the matter contribution. This applies as long as the ghosts contributions are ignored. When the ghosts are included, and in order to produce a measure in the moduli space of surfaces with punctures (string scattering amplitudes) we found that when we restrict the gravitational dressing to satisfy the condition $\beta \leq Q/2$ the mismatch of ghost numbers implies the vanishing of any correlation function involving a field outside the minimal grid on any Riemann surface. With the previous definition of fusion rules this result truncates the irrational fusion rules to be always inside the minimal grid or at its boundary (the states analogous to the leg factors in the $c = 1$ case). As noted in the previous section this truncation violates associativity of the fusion algebra as one can easily see by working out a few simple examples. This is not necessarily a disaster. In the rational case, and in the Coulomb gas representation of matter correlators, there are many fields with the same conformal dimension. This introduces an inherent ambiguity in the method used to identify the three and higher n-point correlators. Different prescriptions differ on the definition of the operators and contact terms. We showed that one can choose a prescription which restores the associativity of the fusion rules and the standard rational rules for the minimal models. In particular, for this prescription the states on the boundary of the minimal grid also decouple. These results are reminiscent of the Virasoro constraints in the one matrix models [3,4]. One can read these constraints as the way to eliminate redundant operators.

In the k -th critical point of the one-matrix model the Virasoro constraints allow one to write all correlation functions in principle in terms of correlators involving only the fields $\sigma_1, \dots, \sigma_{k-2}$. The first redundant state, σ_{k-1} , which can be identified in the continuum with a boundary operator [28] in pure gravity, has the same gravitational scaling dimension as the first state in the boundary of the Kac table, and it can be eliminated by shifting the free energy of the matrix model. In the prescription we have chosen to compute three-point functions, one has already implemented the vanishing of correlators involving redundant operators. It would be interesting to exhibit explicitly in the continuum that the different changes of contact interactions lead to analogues of the Virasoro constraints.

To conclude we should mention that one can obtain non-vanishing correlators of fields outside the minimal grid if some of them are chosen with a dressing satisfying $\beta > Q/2$. These correlators have not yet been investigated, but we suspect that they are described in terms of total derivatives in moduli space. Further work is necessary to elucidate their significance.

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6. APPENDIX

We collect here some useful formulae needed in the text. If $\Gamma(x)$ is Euler's Gamma function, we can define the function

$$\Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \quad (6.1)$$

which appears almost everywhere in the computation of correlation functions using the Coulomb gas representation [18]. The identities for $\Gamma(x)$ imply

$$\Delta(1+x) = -x^2 \Delta(x) \quad \Delta(x) \Delta(-x) = -x^{-2} \quad (6.2)$$

$$\Delta(1-x) = \Delta^{-1}(x) \quad \Delta(1+x) \Delta(1-x) = -x^2 \quad (6.3)$$

$$\Delta(n+x) = (-1)^n \prod_{j=0}^{n-1} (j+x)^2 \Delta(x) \quad (6.4)$$

The screened three-point function for three arbitrary primary fields was computed in [18] with the result

$$C_{s's \ n'n}^{p'p} = \mu_{l'l} \prod_{i=0}^{l'-2} \prod_{j=0}^{l-2} \frac{1}{[s' - 1 - i - \rho(s - 1 - j)]^2 [n' - 1 - i - \rho(n - 1 - j)]^2 [p' + 1 + i - \rho(p + 1 + j)]^2} \\ \prod_{i=0}^{l'-2} \Delta(s - \rho'(s' - 1 - i)) \Delta(n - \rho'(n' - 1 - i)) \Delta(-p + \rho'(p' + 1 + i)) \\ \prod_{j=0}^{l-2} \Delta(s' - \rho(s - 1 - j)) \Delta(n' - \rho(n - 1 - j)) \Delta(-p' + \rho(p + 1 + i)) \quad (6.5)$$

where

$$\mu_{l'l} = \rho^{4(l-1)(l'-1)} \prod_{i=1}^{l'-1} \prod_{j=1}^{l-1} \frac{1}{(i - \rho j)^2} \prod_{i=1}^{l'-1} \Delta(i\rho') \prod_{j=1}^{l-1} \Delta(j\rho) \quad (6.6)$$

and l, l' are given in terms of the screening numbers by:

$$l = N + 1 = \frac{s + n - p + 1}{2}, \quad l' = N' + 1 = \frac{s' + n' - p' + 1}{2} \quad (6.7)$$

As in (3.10) the unconjugated structure constants are defined by:

$$C_{p'p \ s's \ n'n} = C_{s's \ n'n}^{-p'-p} \quad (6.8)$$

and now the associated screening numbers are:

$$\tilde{l} = \tilde{N} + 1 = l + p, \quad \tilde{l}' = \tilde{N}' + 1 = l' + p' \quad (6.9)$$

The proof of (3.16) is somewhat tedious. One calculates first $C_{\alpha\beta\delta}/C_{\beta\delta}^\alpha$ using the listed properties of the Δ functions and the relations:

$$\prod_{j=0}^{l-2} f(j) = \prod_{j=0}^{l-2} f(l-2-j)$$

$$\prod_{j=0}^{\tilde{l}-2} f(j) = \prod_{j=0}^{l-2} f(j) \prod_{k=0}^{p-1} f(j+l-1) \quad (6.10)$$

After some work one finds that the (n', n) and (s', s) dependence in the Δ functions completely cancels. Then, the remaining explicit (p', p) dependence of $C_{\alpha\beta\delta}$ cancels against $\mu_{p'l}$ up to leg factors in (p', p) . The same happens to $\mu_{\tilde{l}'\tilde{l}}$ and finally, the leg factors conspire to give $C_{p'p \ p'p}^{11}$.

The relation (3.15) shows a similar (although a bit simpler) cancellation pattern. Here it can be deduced as a corollary from (3.16).

REFERENCES

- [1] E. Brezin and V. Kazakov : Phys. Lett. **B 236** (1990) 144; M. Douglas and S. Shenker: Nucl. Phys.**B 335** (1990) 635; D.J. Gross and A.A. Migdal: Phys. Rev. Lett. **64** (1990) 717.
- [2] N. Seiberg: *Notes on Quantum Liouville Theory and Quantum Gravity*. RU-90-29. Talk presented at the 1990 Yukawa International Seminar. Kyoto.
- [3] M. Fukuma, R. Kawai and R. Nakayama: KEK-TH-251, (1990).
- [4] R. Dijkgraaf, E. Verlinde and H. Verlinde: Princeton Preprint, IASSNS-HEP-90/48 (1990).
- [5] E. Witten: Nucl. Phys. **B 340** (1990) 281; R. Dijkgraaf and E. Witten: Nucl. Phys. **B 342** (1990) 486.
- [6] P. Di Francesco and D. Kutasov: Nucl. Phys.**B 342** (1990) 589.
- [7] G. Moore N. Seiberg, and M. Staudacher: Rutgers and Yale preprints, RU-91-11, YCTP-P11-91.
- [8] M. Bershadsky and I. Klebanov: Princeton Preprint, PUPT-1241 (1991).
- [9] B. Lian and G. Zuckerman: Phys. Lett. **B 254** (1990) 417.
- [10] M. Goulian and M. Li: Phys. Rev. Lett. **66**-16 (1991) 2051.
- [11] Y. Kitazawa: Harvard Preprint HUTP-91/A013.
- [12] V.S. Dotsenko: Paris preprint, PAR-LPTHE 91-18.
- [13] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov: Nucl. Phys.**B 241** (1984) 333.
- [14] T. Banks and M. Peskin Nucl. Phys.**B 264** (1986) 513.
- [15] I. Frenkel, H. Garland and G. Zuckerman: Proc. Natl. Acad. Sci. USA **83** (1986) 8442.
- [16] B.L. Feigin and D.B. Fuks, Moscow Preprint 1983.
- [17] For details and references see for example: A. Rocha in the Proceedings of the 10th John Hopkins Workshop on Current Problems in Particle Theory. V. Rittenberg ed. World Scientific, Singapore 1986.
- [18] V.S. Dotsenko and V.A. Fateev: Nucl. Phys.**B 240** (1984) 312; Nucl. Phys.**B 251** (1985) 691; Phys. Lett **B 154** (1985) 291.

- [19] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov: Mod. Phys. Lett. **A 3** (1988) 819.
- [20] F. David: Mod. Phys. Lett. **A 3** (1988) 1651.
- [21] J. Distler and H. Kawai: Nucl. Phys.**B 321** (1989) 509.
- [22] P. Di Francesco and D. Kutasov: Phys. Lett. **B 261** (1991) 385.
- [23] A. Gupta, S.P. Trivedi and M.B. Wise: Nucl. Phys. **B 340** (1990) 475.
- [24] D.J. Gross and I. Klebanov: Nucl. Phys. **B 359** (1991) 3.
- [25] A.M. Polyakov: Mod. Phys. Lett. **A 6** (1990) 635.
- [26] L. Alvarez-Gaumé, C. Gómez, G. Moore and C. Vafa: Nucl. Phys.**B 303** (1988) 455.
- [27] L. Alvarez-Gaumé, C. Gómez, P. Nelson, G. Sierra and C. Vafa: Nucl. Phys.**B 311** (1988/89) 333.
- [28] E. Martinec, G. Moore and N. Seiberg: Rutgers, Yale and Chicago preprint RU-14-91, YCTP-P10-91, EFI-91-14.