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**ANALYTICAL SOLUTIONS FOR
COSMOLOGICAL PERTURBATIONS
WITH RELATIVISTIC COLLISIONLESS MATTER**

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ABSTRACT

Thermal field theory is used for the derivation of self-consistent and automatically gauge-invariant equations for cosmological perturbations. Exact, analytical solutions are obtained for scalar, vector, and tensor perturbations in the general case of a two-component system containing a perfect radiation fluid and relativistic collisionless matter. The results for scalar (energy density) perturbations agree with previous numerical integrations of the Einstein-Boltzmann equations, whereas new results are found for the vector and tensor sector. Particularly interesting are the solutions for vector perturbations on superhorizon scales, where a growing mode corresponding to spontaneous generation of vorticity in the collisionless component is found.

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1. Introduction

The theory of linear perturbations of homogeneous isotropic cosmological models as pioneered by Lifshitz [1] in the 40's and worked over many times during the last decades constitutes the basis for most current theories on the formation of large-scale structure of the universe. Simple, analytical solutions for cosmological perturbations have been found for universes filled with a perfect fluid [1, 2]. They exhibit a general growth of density perturbations on superhorizon scales, which once they come within the Hubble horizon either turn into acoustic waves if pressure prohibits further growth, or continue to grow to form nonlinear structures.

A perfect-fluid model for matter that filled the early universe describes well the behaviour of a collision-dominated primordial plasma, but is inadequate for weakly coupled components with a large mean-free path, like neutrinos or other kinds of dark matter. A collisionless plasma can sustain anisotropic stress and thus has more complicated gravitational interactions than a perfect fluid. In order to obtain closed sets of equations for metric and matter perturbations, a more fundamental description of matter is needed. To this end, Peebles and Yu [3] have first used classical kinetic theory, and in the case of purely collisionless matter such equations have been derived in Refs. 4, 5. These equations turn out to be integro-differential equations, and numerical solutions for density perturbations have been obtained in Refs. 4, 6 in longitudinal and synchronous gauges, respectively.

In this work we shall employ finite-temperature field theory [7] to give a self-consistent treatment of (adiabatic) cosmological perturbations in a relativistic plasma of weakly coupled particles, the results of which in the special case of purely collisionless density perturbations have been reported in Ref. 8. In this framework, the response of the matter to metric perturbations is determined by the thermal graviton self-energy, whose leading temperature contributions have been obtained by the present author in Ref. 9.* This way, a purely geometrical treatment can be performed which turns out to lead automatically to a manifestly gauge invariant

*The discussion in Ref. 9 was restricted to the local tangent space and to constant temperature, thus global aspects were disregarded, whereas here we shall consider a specific cosmological model and derive global solutions for linear perturbations.

formulation as developed by Bardeen [2], obviating the notoriously delicate issue of gauge fixing of cosmological perturbations.

Besides elegance, a thermal field theory approach also has the advantage, in principle, of being intrinsically quantum theoretical, whereas a quantum kinetic theory would presumably be more difficult to develop. In the present application, however, we shall be concerned with the leading effects, which come from thermal rather than quantum fluctuations, so that equivalence with classical kinetic theory has to be expected (and will be verified).

The analysis of cosmological perturbations will be performed for a radiation-dominated Friedmann-Robertson-Walker model containing both collisionless and collision-dominated matter, and closed sets of perturbation equations will be derived for scalar, vector, and tensor modes in the metric perturbation, which corresponds to energy density perturbations, rotational perturbations, and gravitational waves, respectively.

Although we shall find integro-differential equations similar to those of Refs. 4-6, we shall be able to give analytical solutions where so far only numerical ones were known, and only in the case of density perturbations. As to the latter, our results are consistent with the existing ones, but our solutions for the vector and tensor modes are new. Apparently because vector and tensor perturbations do not show instabilities in the perfect-fluid case, they have not been studied in the collisionless case before. However, as remarked in Ref. 10, for collisionless particles there is the possibility of generation and decay of vorticity, and, indeed, we find a growing mode in the vector sector on superhorizon scales, which describes spontaneous generation of vorticity. Vector perturbations on subhorizon scales and tensor perturbations, on the other hand, show a purely dissipative behaviour.

This paper is organized as follows. In chapt. 2, the various gauge invariant cosmological perturbations are defined and their interpretation discussed. For a general orientation, the solutions in the perfect-fluid case are recapitulated in chapt. 3 for scalar, vector, and tensor perturbations. In chapt. 4 the general procedure for obtaining self-consistent perturbation equations using thermal field theory is described in some detail, which are then given in chapt. 5 for our cosmological model in the general two-component case of collisionless and collision-dominated relativistic mat-

ter. The solutions are listed and described separately in chapt. 6, with some of the more unwieldy details relegated to the appendix. Chapt. 7 contains our conclusions.

2. Gauge invariant cosmological perturbations

The background spacetime which we shall consider is that of a radiation-dominated spatially flat Friedmann-Robertson-Walker (or, Einstein-de Sitter) model to which all conventional cosmological models reduce at some early stage of their evolution. Its line element is given by*

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 = \sigma(\tau)[d\tau^2 - d\vec{x}^2], \quad (2.1)$$

with scale factor

$$a^2 = \left(\frac{32\pi G\rho_0}{3} \right)^{\frac{1}{2}} t = \sigma = \frac{8\pi G\rho_0}{3} \tau^2, \quad (2.2)$$

where ρ_0 is the energy density when $a = 1$. The conformal time

$$\tau = \int_0^t \frac{dt'}{a(t')} = R_H$$

measures the size of the horizon, R_H , in comoving coordinates. It also gives the inverse Hubble constant at a given time by virtue of

$$\frac{da}{dt} = \frac{\dot{a}}{a} = \frac{\dot{\sigma}}{2\sigma} = \frac{1}{\tau}, \quad (2.3)$$

where here, and in the following, a dot denotes differentiation with respect to τ .

The background energy-momentum tensor is formally that of a perfect fluid

$$T_{\mu\nu}^{(0)} = \frac{\rho_0}{\sigma^2} (u_\mu u_\nu + \frac{1}{3}(u_\mu u_\nu - g_{\mu\nu})), \quad (2.4)$$

with $u_m = \sigma^{1/2}\delta_m^0$ in the comoving coordinate system defined by (2.1). Radiation domination implies that the background pressure assumes its maximal value $\rho_0/3\sigma^2$.

Metric perturbations, and correspondingly perturbations in the energy-momentum tensor, can be classified according to their spatial transformation behaviour

*Our sign conventions are the same as in Ref. 9, 11: metric signature $(+ - - -)$, $R_{\beta\gamma\delta}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha} \delta - \dots$, and $R_{\mu\nu} = R_{\mu\nu}^{\alpha}{}_{\alpha}$. Further, we use units in which $\hbar = c = 1$.

as scalar, vector, and tensor modes [1]. Since the background metric of (2.1) only depends on time, and 3-space being flat, the spatial dependence can be separated and decomposed in spatial Fourier modes. Scalar perturbations can be built from plane waves $e^{i\vec{k}\cdot\vec{x}}$, vector perturbations from $v_i e^{i\vec{k}\cdot\vec{x}}$ with transverse spatial vector $\vec{v}$, $(\vec{v}\cdot\vec{k}) = 0$, and tensor perturbations from $t_{ij} e^{i\vec{k}\cdot\vec{x}}$ with spatially transverse-traceless tensor t , $k_i t_{ij} = 0$ and $t_{ii} = 0$.

2.1. SCALAR PERTURBATIONS

The scalar part of a metric perturbation can be parametrized in terms of four scalar functions,

$$\delta g_{\mu\nu}^{(S)} = \sigma(\tau) \begin{pmatrix} C & D_{,i} \\ D_{,j} & A\delta_{ij} + B_{,ij} \end{pmatrix}, \quad (2.5)$$

of which any two can be gauged away by $\delta g_{\mu\nu} = \delta \xi_{\mu,\nu} + \delta \xi_{\nu,\mu}$ and appropriate functions $\delta \xi_0 = \varphi$ and $\delta \xi_i = \psi_{,i}$.

Depending on the specific choice of gauge, residual gauge invariances may be left, as is the case for the widely used synchronous gauge [1], where $C = D = 0$.

Here we shall follow Bardeen [2],* who has proposed to use manifestly gauge invariant metric potentials instead of fixing the gauge, for which we choose (slightly deviating from Bardeen's variables)

$$\begin{aligned} \Phi &= A + \frac{\dot{\sigma}}{\sigma} \left(D - \frac{1}{2}\dot{B} \right), \\ \Pi &= \frac{1}{2} \left(\dot{B} + \frac{\dot{\sigma}}{\sigma} \dot{B} + C - A \right) - \dot{D} - \frac{\dot{\sigma}}{\sigma} D. \end{aligned} \quad (2.6)$$

Gauge invariance is easily verified under the "scalar" diffeomorphisms $\delta \xi_0 = \varphi$, $\delta \xi_i = \psi_{,i}$:

$$\begin{aligned} \delta A &= -\frac{\dot{\sigma}}{\sigma^2} \varphi, & \delta B &= \frac{2}{\sigma} \psi, \\ \delta C &= \frac{2}{\sigma} \left(\dot{\varphi} - \frac{\dot{\sigma}}{2\sigma} \varphi \right), \\ \delta D &= \frac{1}{\sigma} \dot{\varphi} + \frac{1}{2} \delta \dot{B}. \end{aligned} \quad (2.7)$$

The physical meaning of the gauge invariant potentials Φ and Π can be understood by considering the induced perturbations in the energy-momentum tensor [2].

*Useful reviews and extensions of this approach can be found in Refs. 12, 13, 14.

From the Einstein equations one finds that for a given spatial Fourier mode with wave vector $\vec{k}$

$$\delta = \frac{(k\tau)^2}{3}\Phi, \quad \pi_{anis.} = \frac{(k\tau)^2}{3}\Pi, \quad (2.8)$$

where $\delta = \delta T_0^0 / T^{(0)0}_0$ is the energy density perturbation when measured with respect to a space-like hypersurface representing everywhere the local rest frame of $T_{\mu\nu} = T_{\mu\nu}^{(0)} + \delta T_{\mu\nu}$, and the anisotropic pressure $\pi_{anis.}$ is defined through

$$\delta T^i_j = \frac{1}{3}T^{(0)k}_k \left[\pi_{is.}\delta^i_j + \pi_{anis.}(\delta^i_j - 3\frac{k^ik_j}{k^2}) \right]. \quad (2.9)$$

Since there is no anisotropic pressure in the background, $\pi_{anis.}$ is an unambiguous quantity. The energy-density contrast, however, is gauge dependent, i.e. dependent on the choice of the coordinate system for the perturbed universe which implicitly chooses the points in the unperturbed background at which comparisons are made.

Another obviously gauge dependent quantity is the matter 3-velocity which is given essentially by δT^i_0 . The shear of the matter velocity field, however, is gauge invariant, and is determined by the gauge invariant metric perturbations Φ and Π through [2]

$$V_s = -\frac{k\tau}{2}(\Pi + \frac{1}{2}(\Phi + \tau\dot{\Phi})). \quad (2.10)$$

Like $\pi_{anis.}$, this quantity is hypersurface-independent.

A gauge invariant perturbation of the energy-momentum tensor which cannot be related to the gauge invariant metric perturbations Φ and Π is the so-called entropy perturbation [2] $\eta \equiv \pi_{is.} - \delta$. However, throughout this paper we shall be concerned exclusively with adiabatic cosmological perturbations, for which $\eta \equiv 0$.

2.2. VECTOR PERTURBATIONS

Each spatial Fourier mode of a general vector perturbation with wave vector $\vec{k}$ and anisotropy vector $\vec{v}$, $(\vec{k}\vec{v}) = 0$, can be parametrized by two functions of τ according to

$$\delta g_{\mu\nu}^{(V)} = \sigma(\tau) \begin{pmatrix} 0 & v_i E \\ v_j E & \frac{i}{2}(k_i v_j + v_i k_j) F \end{pmatrix}, \quad (2.11)$$

of which one function can now be gauged away through $\delta\zeta_0 = 0$, $\delta\zeta_i = v_i\varphi$.

A gauge invariant combination of E and F is given by [2]

$$\Psi = \frac{1}{k}(\dot{F} - 2E), \quad (2.12)$$

which through the Einstein equations is associated with vorticity, i.e. intrinsic angular velocity of a fluid element, according to

$$V = \frac{(k\tau)^2}{16}\Psi. \quad (2.13)$$

Another directly measurable quantity is the shear of the matter velocity field, which is determined by

$$V_s = V + \frac{1}{2}\dot{\Psi}. \quad (2.14)$$

2.3. TENSOR PERTURBATIONS

A general tensor perturbation is described by only one function per transverse-traceless tensor t ,

$$\delta g_{\mu\nu}^{(T)} = \sigma(\tau) \begin{pmatrix} 0 & 0 \\ 0 & t_{ij} H \end{pmatrix}, \quad (2.15)$$

which is automatically gauge invariant, because the diffeomorphism vector $\delta\xi_\mu$ cannot accommodate an irreducible spatial tensor.

H is, of course, the amplitude of a gravitational wave with polarization t_{ij} .

3. Perfect-fluid behaviour

Eqs. (2.8, 2.13) already exhaust the gauge invariant content of the Einstein equations for scalar and vector perturbations provided the energy-momentum tensor $T_{\mu\nu}$ is covariantly conserved. The implications of this latter condition on the interplay between metric perturbations and perturbations in the energy-momentum tensor in terms of the gauge invariant quantities introduced above have been thoroughly analyzed in Ref. 2.

However, in order to obtain a closed set of equations for perturbations of the cosmological model, one has to further specify the response of the energy-momentum tensor to metric perturbations. The simplest possibility considered originally by Lifshitz [1] is the case of a perfect fluid, where anisotropies in the stress tensor are constrained to vanish. In this case analytical solutions are found without difficulty, and we shall briefly recall them here for the sake of orientation.

3.1. SCALAR PERTURBATIONS

Since by virtue of (2.8) $\pi_{anis.} \equiv 0$ implies $\Pi \equiv 0$, only one gauge invariant metric perturbation remains, for which covariant conservation of the energy-momentum tensor leads to the ordinary differential equation [2]

$$\Phi''(x) + \frac{4}{x}\Phi'(x) + \frac{1}{3}\Phi(x) = 0, \quad (3.1)$$

where we have introduced the dimensionless variable

$$x \equiv k\tau = \frac{R_H}{\lambda/2\pi}, \quad (3.2)$$

which gives the ratio of the size of the Hubble horizon to the comoving wavelength, λ , of the perturbation. A prime will henceforth denote differentiation with respect to this variable x .

Requiring regular behaviour at the origin $\tau = 0 = x$, eq. (3.1) can be solved in terms of the spherical Bessel function j_1 by

$$\Phi = C x^{-1} j_1(x/\sqrt{3}). \quad (3.3)$$

For $x \ll 1$, i.e. at a time when the wavelength of the perturbation is still larger than the continuously growing horizon, $\Phi \sim \text{const.}$, which by virtue of (2.8) gives

a growing density perturbation $\delta \sim x^2$. This behaviour, which turns out to be the same for all perfect fluids, independent of pressure, can be interpreted as the relativistic analogue of the classical Jeans instability [15], and it provides the basis for all current theories about galaxy formation through adiabatic perturbations [16]. For $x \gg 1$, when the horizon has grown larger than the size of the perturbation, radiation pressure comes into effect and gives rise to a density wave propagating with constant amplitude and speed of sound $c_s^2 = 1/3$. (With $p \ll \rho$, hence $c_s \ll 1$, the power-law growth could continue until $x \sim 1/c_s \gg 1$.)

3.2. VECTOR PERTURBATIONS

In the present radiation-dominated case, covariant conservation of the energy-momentum tensor simply gives

$$V' = 0 \Rightarrow V = \text{const.} \quad (3.4)$$

This just reflects the Helmholtz-Kelvin circulation theorem stating that in a perfect fluid the circulation around a closed curve following the motion of matter is conserved, which also holds in the presence of gravity [17].

However, the gauge invariant metric potential $\Psi \propto V/x^2$ diverges for $\tau \rightarrow 0$, which by virtue of (2.14) means that at the initial singularity the shear of the matter velocity field becomes infinite, hence (3.4) with $V \neq 0$ does not correspond to a regular solution of the perturbation equations.

3.3. TENSOR PERTURBATIONS

The absence of anisotropic pressure in a perfect fluid implies that the latter does not react back on gravitational waves which thus are found to propagate in the background model according to the source-free equation

$$H''(x) + \frac{2}{x}H'(x) + H(x) = 0. \quad (3.5)$$

Requiring regularity at $\tau = 0 = x$, this is solved by the spherical Bessel function $j_0(x) = \sin(x)/x$. For $x \ll 1$, this solution represents a constant geometrical mode, which gives rise to a wave propagating with the speed of light after horizon crossing, $x \gg 1$. The asymptotic behaviour $H \sim 1/\tau$ corresponds to an adiabatic loss in energy density stored in gravitational waves due to the expansion of the universe according to $\rho = \rho_0/\sigma^2$, as in eq. (2.4) [10].

4. Self-consistent perturbation equations from thermal field theory

A perfect-fluid description of the matter that filled the early universe is a good approximation for studying cosmological perturbations as long as the primordial plasma is highly collisional at the scale set by the perturbation, i.e. as long as the mean free path of particles $\ell \ll \lambda$. Deviations from perfect-fluid behaviour become important, however, when $\ell > \lambda$ for some constituents of the plasma. This is the case, e.g., for the weakly interacting neutrinos on all scales of interest after the temperature has dropped below $\sim 10^{10}$ K.

The limiting case opposite to perfect-fluid behaviour is the ultra-relativistic regime $k = 2\pi/\lambda \gg \ell^{-1}$, where collisionless processes dominate. There the response of matter to metric perturbations is more complicated, because collisionless matter can sustain anisotropic stress which in turn changes the coupling with gravity. Anisotropic contributions in the perturbed energy-momentum tensor have to be expected even for the scalar sector, since the very presence of a wave vector $\vec{k}$ for each Fourier mode breaks the isotropy originally present in the background, and only in a perfect fluid are they constrained to vanish.

The usual approach to this problem is based on classical kinetic theory in the framework of general relativity [18, 3]. Equations for cosmological density perturbations with collisionless matter have been derived first in Ref. 4 in the longitudinal gauge ($B = D = 0$ in (2.5)), and in Ref. 3 in synchronous gauge ($C = D = 0$). General cosmological perturbations for collisionless matter have been considered in Ref. 5 (longitudinal gauge), however (numerical) solutions of the resulting integro-differential equations have been computed so far only for density perturbations [4, 6, 19].

In the remainder of this paper we shall show that by using a thermal field theory framework, rather than coupled Boltzmann-Einstein equations, one automatically arrives at a gauge invariant formulation and that the resulting equations tend to be simpler than the ones derived before in special gauges.* In the case of massless

* A reformulation of the kinetic theory approach in terms of Bardeen's variables is of course also possible, but not entirely straightforward [20, 19], and, to the best of our knowledge, the closed set of equations we are going to derive has not been obtained previously.

collisionless matter the derivation of the perturbation equations becomes particularly elegant, and the results for density perturbations have been reported already in Ref. 8. Here we shall extend this analysis to include vector and tensor perturbations, and we shall also generalize to the more realistic case of a two-component system consisting of both a perfect radiation fluid and collisionless matter. In each case, we shall be able to give exact, analytical solutions.

Moreover, a completely field theoretic formulation has the advantage in principle of not being restricted to the classical case, whereas a quantum kinetic theory is presumably more difficult to develop. Restricting ourselves to adiabatic cosmological perturbations, it is also no drawback that thermal field theory is well understood only in the case of thermal equilibrium.

In a field theoretic approach both the energy-momentum tensor and its response to perturbations in the metric are determined by an effective action. The energy-momentum tensor is given by the one-point function

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g}} \frac{\delta \Gamma}{\delta g^{\mu\nu}(x)}, \quad (4.1)$$

where Γ contains all contributions to the effective action besides the classical Einstein-Hilbert action. Self-consistency requires that

$$\delta T_{\mu\nu}(x) = \int d^4y \frac{\delta T_{\mu\nu}(x)}{\delta g^{\alpha\beta}(y)} \delta g^{\alpha\beta}(y), \quad (4.2)$$

hence $\delta T_{\mu\nu}$ is determined by the two-point function,

$$\Pi_{\mu\nu\alpha\beta}(x, y) \equiv \frac{\delta^2 \Gamma}{\delta g^{\mu\nu}(x) \delta g^{\alpha\beta}(y)} = \frac{1}{2} \frac{\delta(\sqrt{-g} T_{\mu\nu}(x))}{\delta g^{\alpha\beta}(y)}, \quad (4.3)$$

or in other words, by the graviton self-energy.

A perturbative expansion in Feynman diagrams becomes sensible if $GT^2 \ll 1$, which is fulfilled if the temperature is well below the Planck scale. Neutrinos, for example, become collisionless when this expansion parameter has dropped to a comfortable value of $GT^2 \sim 10^{-66}$. External momenta in (4.3) are given by $k = x/r$, where $\tau \sim \rho^{-1/2} \sim (GT^4)^{-1/2}$, hence $k/T \sim \pi(GT^2)^{1/2} \ll 1$, which means that if the mass of the particles under consideration is much smaller than temperature, we only need the high-temperature limit of (4.3). If all nongravitational interactions of

the particles can be neglected, as is the case for collisionless particles, then we only need one-loop diagrams with two graviton insertions.

The leading temperature corrections of one-loop diagrams to the graviton self-energy have been calculated in Ref. 9 in a local-momentum space representation. We shall not repeat this calculation here, which is somewhat unwieldy but otherwise straightforward given the techniques developed in Ref. 21, but only recall the main features.

First, the result is universal, in that it is the same for any massless matter, including gauge fields and even gravitons, if thermalized. Only the overall factor differs in correspondence with the energy-density which is determined by the number of spin degrees of freedom, n_B and n_F for bosons and fermions, respectively, through

$$\rho = \left(n_B + \frac{7}{8}n_F\right) \frac{\pi^2 T^4}{30}. \quad (4.3)$$

$$\tilde{\Pi}^{\alpha\beta\mu\nu} = \frac{1}{2} \Gamma^{\alpha\beta\mu\nu} - \frac{1}{4} (\delta^\alpha_\rho \delta^\mu_\beta \eta^{\nu\rho} + \delta^\alpha_\rho \delta^\nu_\beta \eta^{\mu\rho} + \delta^\beta_\rho \delta^\mu_\alpha \eta^{\nu\rho} + \delta^\beta_\rho \delta^\nu_\alpha \eta^{\mu\rho}) \eta_{\gamma\delta} \Gamma^{\delta\sigma\rho}, \quad (4.4)$$

In momentum-space, with $g_{\mu\nu} = \eta_{\mu\nu}$, this universal result reads

$$\tilde{\Pi}^{\mu\nu\rho\sigma}(k_0, \vec{k}) = \int \frac{d^4 p}{T} \frac{p^\mu p^\nu p^\rho p^\sigma}{(2\pi)^4 p^2 (p - k)^2}. \quad (4.5)$$

$\int_T$ refers to evaluation at finite temperature, where the time direction is taken to coincide with the plasma rest frame velocity. The latter breaks Lorentz invariance down to invariance under spatial rotations, leading to 14 different structure functions for the thermal graviton self-energy (4.4), which are given by linear combinations of Legendre functions of the second kind $Q_n(\omega)$, $\omega \equiv k_0/|\vec{k}|$, i.e. they are built from polynomials in ω and the logarithmic function

$$\frac{\omega}{2} \ln \frac{\omega+1}{\omega-1}, \quad (4.6)$$

as tabulated in Ref. 9.

Second, the flat-space result for (4.4) obeys certain Ward identities which are crucial for the following applications. $\tilde{\Pi}^{\alpha\beta\mu\nu}$ satisfies the Ward identity corresponding to general covariance,

$$\tilde{\Pi}_{\alpha\beta\mu\nu}(k) (\eta^{\mu\epsilon} k^\nu + \eta^{\nu\epsilon} k^\mu) = -\frac{1}{2} T_{\mu\nu} (\delta^{\nu\epsilon}_\alpha k^\mu + \delta^{\mu\epsilon}_\beta k^\nu - \delta^{\mu\nu}_\alpha k^\epsilon), \quad (4.7)$$

and it is independent of the gauge fixing conditions which become necessary if the internal lines correspond to gauge fields like Yang-Mills fields or gravitons themselves (after inclusion of the Faddeev-Popov ghost fields, of course). Technically, the reason for this is that in the high-temperature limit vertex functions with external ghost lines, which could complicate things, start contributing only at order T^3 [22].

Another Ward identity which (4.4) satisfies in the high-temperature limit is the one corresponding to conformal invariance,

$$\eta_{\mu\nu} \Pi^{\mu\nu\alpha\beta} = 2T^{\alpha\beta}. \quad (4.8)$$

This might be surprising, as T introduces a mass scale, however in an imaginary-time formalism temperature can be incorporated through boundary conditions only, so that the effective masslessness of the internal particles is not touched.*

Conformal invariance of the high-temperature effective action now allows us to obtain the global form of the required graviton vertex-functions in the case of a conformally flat space $g_{\mu\nu} = \sigma\eta_{\mu\nu}$ by the simple transformation

$$\frac{\delta^n \Gamma}{\delta g^{\alpha\beta}(x_1) \delta g^{\mu\nu}(x_2) \dots} \Big|_{g=\sigma\eta} = \sigma(x_1) \sigma(x_2) \dots \frac{\delta^n \Gamma}{\delta g^{\alpha\beta}(x_1) \delta g^{\mu\nu}(x_2) \dots} \Big|_{g=\eta}. \quad (4.9)$$

In particular, the momentum-space results of Ref. 9 for the graviton self-energy (4.4) translate into

$$\Pi_{\mu\nu\alpha\beta}(x, y) \Big|_{g=\sigma\eta} = \sigma(x) \int d^4 k e^{-ik(x-y)} \tilde{\Pi}_{\mu\nu\alpha\beta}(k_0, \vec{k}) \Big|_{g=\eta} \sigma(y). \quad (4.10)$$

Given this, a closed set of perturbation equations is obtained from a variation of the Einstein equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu} \quad (4.11)$$

*A possible conformal anomaly is also of no concern here, because it pertains to the T^0 -part which is discarded in the high-temperature expansion, and with it all potential difficulties in renormalizing gravity.

under $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$ with eq. (4.2). The self-consistent cosmological perturbations in a conformally flat background are thus determined by

$$\begin{aligned} & \left\{ \frac{1}{2} (g^{\alpha\beta} g^{\mu\nu} - g^{\alpha(\mu} g^{\nu)\beta}) \nabla^2 - \frac{1}{2} (g^{\alpha\beta} \nabla^{\mu} \nabla^{\nu}) + g^{\mu\nu} \nabla^{\alpha} \nabla^{\beta} \right\} \\ & + g^{\alpha(\mu} \nabla^{\nu)} \nabla^{\beta} - R^{\alpha(\mu\nu)} \beta - \frac{1}{2} g^{\alpha\beta} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^{\alpha\beta} + g^{\alpha(\mu} R^{\nu)\beta} \\ & + \frac{1}{4} g^{\alpha\beta} g^{\mu\nu} R - \frac{1}{2} g^{\alpha(\mu} g^{\nu)\beta} R - 16\pi G \sigma^{-1} \tilde{\Pi}^{\alpha\beta\mu\nu}(i\partial_0, i\vec{\partial}) \Big|_{g=\eta} \sigma^{-1} \Big\} \delta g_{\alpha\beta} = 0. \end{aligned} \quad (4.12)$$

(Brackets around indices denote symmetrization including normalization so that e.g. $a^{\mu} a^{\nu} = a^{\mu} a^{\nu}$.) In the radiation-dominated case, which we are considering, eq. (4.12) can be simplified considerably, because $T^{\mu}_{\mu} = 0$ implies $R = 0$, and conformal flatness implies vanishing of the Weyl curvature tensor for the background geometry, so that $R^{\alpha(\mu\nu)}\beta$ is determined by

$$R^{\alpha(\mu\nu)}\beta = g^{\alpha(\mu} R^{\nu)\beta} - \frac{1}{2} (g^{\alpha\beta} R^{\mu\nu} + g^{\mu\nu} R^{\alpha\beta}). \quad (4.13)$$

Thus all (explicit) curvature terms in eq. (4.12) turn out to cancel.

We can now consider scalar, vector, and tensor perturbations in turn by inserting the corresponding metric perturbations (2.5, 2.11, 2.15) into eq. (4.12). Doing this, one indeed finds that they occur only in gauge-invariant combinations for which a convenient basis has been introduced in chapt. 2.

This way one again obtains ordinary differential equations in the variable $x = kt$. However, since the contributions from the thermal graviton self-energy are nonlocal, we encounter formally infinite-order differential operators, which stem from the logarithmic terms (4.6) in the momentum-space graviton self-energy and which are of the form

$$\mathcal{P} \left(\frac{d}{dx} \right) \frac{d}{dx} \operatorname{arccot} \left(\frac{d}{dx} \right) f(x) = \frac{i}{2\pi} \int d\omega e^{-i\omega x} \mathcal{P}(-i\omega) \frac{1}{2} \ln \frac{\omega+1}{\omega-1} \int dx' e^{i\omega x'} f(x'), \quad (4.14)$$

where $\mathcal{P}$ is some finite-order polynomial.

Expressing $\mathcal{P}$ as linear combinations of Legendre polynomials $P_n(\omega)$, the ω -integrand of (4.14) can be rewritten according to

$$P_n(\omega) \frac{1}{2} \ln \frac{\omega+1}{\omega-1} = Q_n(\omega) + \sum_{m=1}^n \frac{1}{m} P_{m-1}(\omega) P_{n-m}(\omega). \quad (4.15)$$

The Legendre polynomials on the r.h.s. are interpreted as finite-order differential operators, whereas the integrals involving the Legendre functions Q_n can be evaluated. Since $Q_n(\omega)$ falls off at infinity like $1/\omega^{n+1}$, the integration contour in ω can be closed around the upper or lower half plane depending on the sign of $x - x'$, and requiring analyticity in the upper half plane, which corresponds to retarded boundary conditions, one finds

$$\frac{1}{2\pi} \int d\omega Q_n(\omega) e^{-i\omega(x-x')} = (-i)^{n+1} j_n(x-x') \theta(x-x'), \quad (4.16)$$

after shrinking the integration contour so as to run around the cut between $\omega = -1$ and $\omega = +1$.

Thus (4.14) leads to convolution integrals of the form

$$\int_{x_0}^x dx' \sum_n \alpha_n j_n(x-x') f'(x'), \quad (4.17)$$

hence the infinite-order differential equations can be transformed into integro-differential equations. The infinite number of initial conditions which the former needed when starting integration at $x = x_0$ can be encoded through

$$f'(x') \rightarrow f'(x') \theta(x' - x_0) + \sum_{n=0}^{\infty} \gamma_n \delta^{(n)}(x' - x_0), \quad (4.18)$$

so that (4.17) becomes

$$\int_{x_0}^x dx' \sum_n \alpha_n j_n(x-x') f'(x') + \varphi(x-x_0), \quad (4.19)$$

with some function $\varphi(x-x_0)$.

The infinite number of initial conditions encountered here corresponds to the infinite number of moments of the initial phase space distribution function one would have to specify in a kinetic theory approach. The simplest possible case is $\gamma_n = 0$ for $n > 0$, implying $\varphi \propto \sum_n \alpha_n j_n$, and this turns out to correspond to an initial phase space distribution for the collisionless particles which is isotropic in the momentum space of the particles, as is usually assumed [5].

5. Gauge-invariant perturbation equations with collisionless matter

We are now in a position to determine the manifestly gauge-invariant perturbation equations as outlined above. We shall also consider the general, more realistic case of a two-component system,* where there is also a perfect-fluid present which is not coupled to the collisionless matter other than gravitationally, and where a fraction α of the energy-density is contained in collisionless matter,

$$\alpha = \frac{\rho_{cl.}}{\rho_{cl.} + \rho_{p.f.}}. \quad (5.1)$$

In this case, an explicit variation of the energy-momentum tensor which describes the perfect-fluid part has to be included in eq. (4.12), so that the basic equations now read

$$\left\{ \frac{1}{2} (g^{\alpha\beta} g^{\mu\nu} - g^{\alpha(\mu} g^{\nu)\beta}) \nabla^2 - \frac{1}{2} (g^{\alpha\beta} \nabla^{\mu} \nabla^{\nu}) + g^{\mu\nu} \nabla^{\alpha} \nabla^{\beta} \right. \\ \left. + g^{\alpha(\mu} \nabla^{\nu)} \nabla^{\beta} - 16\pi G \sigma^{-1} \tilde{\Pi}^{\alpha\beta\mu\nu} (i\partial_0, i\vec{\partial}) \right\} \sigma^{-1} \delta g_{\alpha\beta} = 8\pi G \delta T_{p.f.}^{\mu\nu}. \quad (5.2)$$

In the case of purely collisionless matter, eq. (4.12) only involves gauge-invariant combinations of the metric perturbations. However, in the two-component case, where the perfect-fluid component is not taken care of by an action principle, only certain combinations of (5.2) turn out to be gauge invariant, and, in general, the equations have to be closed by the requirement of covariant conservation of the total energy-momentum tensor.

5.1. SCALAR PERTURBATIONS

In the purely collisionless case, the ten equations (4.12) only contain two independent ones for the two gauge invariant potentials Φ and Π . The first one is a finite-order differential equation resulting from the trace of (4.12),

$$\Phi'' + \frac{4}{x} \Phi' + \frac{1}{3} \Phi = \frac{2}{3} \Pi - \frac{2}{x} \Pi', \quad (5.3)$$

*The extension of Bardeen's approach [2] to more-component systems has been worked out in full generality in Refs. 23, 13. The case we are going to consider, however, is transparent enough so that we actually do not need the machinery developed there.

which is a generalization of eq. (3.1), to which it reduces if $\Pi \equiv 0$. However, collisionless matter can support anisotropic stress, so Π is a dynamical variable, and it is determined by the second equation, which is formally of infinite order,

$$(x^2 + 3)\Phi + 3x\Phi' = 6 \left[1 - 2 \frac{d}{dx} \arccot \left(\frac{d}{dx} \right) \right] (\Phi + \Pi). \quad (5.4)$$

This last equation is obtained directly from the 00-component of (4.12), whereas all other components give equations which follow from eq. (5.3) and (5.4).

As described in the preceding chapter, eq. (5.4) can be transformed into an integro-differential equation,

$$(x^2 - 3)\Phi + 3x\Phi' = 6\Pi - 12 \int_{x_0}^x dx' j_0(x - x') (\Phi + \Pi)'(x') + c j_0(x - x_0), \quad (5.5)$$

where $c (\equiv \gamma_0$ in (4.18)) parametrizes the initial conditions.

In the case of the two-component system, all of the anisotropic stress still resides in the collisionless matter, but the potential for density perturbations Φ is shared between the two components. With $\bar{\Phi} = \Phi_{p.f.} + \bar{\Phi}_{cl.}$, the total density contrast is given by (2.8) and the ones for the individual components by

$$\delta_{cl.} = \frac{1}{\alpha} \frac{x^2}{3} \bar{\Phi}_{cl.}, \quad \delta_{p.f.} = \frac{1}{1 - \alpha} \frac{x^2}{3} \bar{\Phi}_{p.f.} \quad (5.6)$$

Because this introduces one further unknown function, one now needs three equations to form a closed system. It turns out that the trace equation (5.3) remains unchanged, whereas the 00-component of (5.2) would no longer be manifestly gauge invariant (but only after imposing covariant conservation of the total energy momentum tensor). However, two further combinations (which are the same as those which led to (2.8) in the general case) of the 14 equations (5.2) turn out to remain manifestly gauge invariant; these read

$$x^2 \Pi = -12\alpha \left[\int_{x_0}^x dx' j_2(x - x') (\Phi + \Pi)'(x') + c j_2(x - x_0) \right] \quad (5.7)$$

$$x^2 \bar{\Phi}_{cl.} = -6\alpha \left[-\Phi - 2\Pi + 2 \int_{x_0}^x dx' j_0(x - x') (\Phi + \Pi)'(x') \right. \\ \left. + \frac{6}{x} \int_{x_0}^x dx' j_1(x - x') (\Phi + \Pi)'(x') + 2c \left(j_0(x - x_0) + \frac{3}{x} j_1(x - x_0) \right) \right], \quad (5.8)$$

thus closing the system of equations for the gauge-invariant variables in the two-component case, without having to impose covariant conservation of the energy-momentum tensor separately.

Similar integro-differential equations can be found in the literature for particular gauge choices (in Ref. 6 for the synchronous gauge and in Refs. 4, 5 for the longitudinal gauge). However, in each case they are more complicated than the present equations, involving spherical Bessel functions of orders up to 3, whereas our equations in the case of purely collisionless matter only involves j_0 , and in the mixed case stops with j_2 . Nevertheless, our equations are consistent with those derived previously using kinetic theory, as will be shown in the next chapter, where we shall also obtain the solutions.

5.2. VECTOR PERTURBATIONS

In the case of vector perturbations, there is no need to present the results for purely collisionless matter and for the two-component system separately. It turns out that vorticity which may be present in the perfect fluid is conserved independently of the other component, $V_{p,f} = \text{const.}$, and the perturbation equation reads

$$(x^2 + 8\alpha)\Psi(x) = 8\alpha \left[\int_{x_0}^x dx' (j_0(x-x') + j_2(x-x')) \Psi'(x') \right. \\ \left. + c(j_0(x-x_0) + j_2(x-x_0)) \right] + 16(1-\alpha)V_{p,f}, \quad (5.9)$$

where one could substitute $j_0(x) + j_2(x) = 3j_1(x)/x$. Eq. (5.9) can be turned into a pure Volterra integral equation by a partial integration,

$$x^2\Psi(x) = 24\alpha \left[\int_{x_0}^x dx' \frac{j_1(x-x')}{x-x'} \Psi(x') + c \frac{j_1(x-x_0)}{x-x_0} \right] + 16(1-\alpha)V_{p,f}, \quad (5.10)$$

where $\tilde{c} = c - \Psi(x_0)$.

Notice that there is now room for generation and dissipation of vorticity $V \propto x^2\Psi$. Indeed, in collisionless matter the Helmholtz-Kelvin circulation theorem no longer applies, whereas in the perfect-fluid component it still enforces $V'_{p,f} = 0$.

For the purely collisionless case $\alpha = 1$, eq. (5.9) has been derived previously in Ref. 5 in the longitudinal gauge, where fortuitously the metric component F , which is retained after gauge fixing, coincides with the gauge invariant quantity Ψ (cf. eq. (2.12)). Eq. (5.9) agrees with the corresponding equation of Ref. 5, except that in the latter j_1 appears in place of j_2 and the term $8\alpha\Psi(x)$ is missing, which, however, is presumably due to typographical errors.

5.3. TENSOR PERTURBATIONS

In contrast to the case of a perfect fluid, there is a backreaction of collisionless matter to gravitational waves, and this is described by the r.h.s. of

$$x^2 H'' + 2xH' + x^2 H = -24\alpha \left[\int_{x_0}^x dx' \frac{j_2(x-x')}{(x-x')^2} H'(x') + c \frac{j_2(x-x_0)}{(x-x_0)^2} \right]. \quad (5.11)$$

In the case of tensor perturbations, there is no gauge ambiguity, and eq. (5.11) agrees with the one derived in Ref. 5, after making the identification $j_2(x)/x^2 = j_0(x)/15 + 2j_2(x)/21 + j_4(x)/35$.

6. Analytical solutions

The above integro-differential equations can be solved numerically either directly [4, 6], by approximating them by systems of ordinary differential equations [19], or by approximating the solution through a finite sum of spherical Bessel functions, which leads to a finite number of linear algebraic equations [5]. In each case, there is no better way for deciding the quality of the numerical approximations than monitoring their apparent convergence.

In general, there is indeed no way to obtain analytical solutions by a power series ansatz about x_0 , because one would only find equations involving all the infinitely many coefficients at the same time. However, a power series ansatz about $x_0 = 0$, which actually is the most interesting point to start at, leads to recursion relations which can be solved, at least in the present case of effectively massless collisionless matter. Strangely enough, this possibility seems to have been overlooked so far in the existing literature. Moreover, the resulting series turn out to be alternating and rapidly converging with infinite radius of convergence, allowing us to obtain arbitrary precision in their numerical evaluation.

In the following, we shall list these solutions and discuss their physical interpretation, relegating the various recursion relations and their derivation to the appendix.

6.1. SCALAR PERTURBATIONS

With $x_0 = 0$, requiring that Π , Φ , and $\Phi_{p.f.}$ are regular at $x = 0$ fixes the constant c which parametrizes the initial conditions in eqs. (5.7, 5.8) to

$$c = \frac{1}{4} (\Phi(0) + 2\Pi(0)). \quad (6.1)$$

The power series solutions are found to be even in x and alternating, with coefficients decreasing faster than factorials. In the limiting case $\alpha = 0$, where only a perfect fluid is present, we recover the result (3.1), where the coefficients decrease as $\sim 1/(n!)^3$, which becomes even better with α approaching 1, where they go like $\sim 1/(n!)^3$. Thus they are rapidly converging, and the numerical error made by truncating at order n is bounded at worst by $x^n/(n!)^3$.

The first few terms of the power series solution for Π , Φ , $\Phi_{p.f.}$, and $\Phi_{cl.} \equiv \Phi - \Phi_{p.f.}$ for the general case of a two-component system read

$$\begin{aligned} \Pi(x) &= C\alpha(-1 + \frac{295 + 56\alpha}{63(25 + 2\alpha)} \frac{x^2}{2!} \mp \dots), \\ \Phi(x) &= C(5 + 2\alpha - \frac{525 + 580\alpha + 56\alpha^2}{63(25 + 2\alpha)} \frac{x^2}{2!} \pm \dots), \\ \Phi_{p.f.}(x) &= C(1 - \alpha)(\frac{15 + 4\alpha}{3} - \frac{5 + 2\alpha}{15} \frac{x^2}{2!} \pm \dots), \\ \Phi_{cl.}(x) &= C\alpha(\frac{17 + 4\alpha}{3} - \frac{4265 + 1456\alpha + 84\alpha^2}{315(25 + 2\alpha)} \frac{x^2}{2!} \pm \dots). \end{aligned} \quad (6.2)$$

In the limit $\alpha = 0$, Π vanishes and $\Phi = \Phi_{p.f.}$ reduces to the result given in eq. (3.3), whereas in the purely collisionless limit $\alpha = 1$ the first three terms of the power series solutions for Φ and Π , respectively, read

$$\begin{aligned} \Phi(x) &\equiv \Phi_{cl.}(x) = C(7 - \frac{43}{63} \frac{x^2}{2!} + \frac{14249}{80115} \frac{x^4}{4!} \mp \dots), \\ \Pi(x) &= C(-1 + \frac{13}{63} \frac{x^2}{2!} - \frac{857}{11445} \frac{x^4}{4!} \pm \dots). \end{aligned} \quad (6.3)$$

In Ref. 6 the asymptotic behaviour $x \rightarrow 0$ of their (otherwise numerical) results in the synchronous gauge has been determined exactly. Given gauge dependent results, one can project them onto the gauge invariant quantities through the formulae of chapt. 2, and thus one can verify that the ratio of the gauge invariant variables $\Pi(0)/\Phi(0) = -\alpha/(5 + 2\alpha)$ found in our solutions indeed agrees with the results of

Ref. 6.* Notice, however, that there is generally no way to reconstruct gauge dependent solutions from the gauge invariant ones, if a particular gauge has residual gauge invariance.

For a numerical evaluation of our gauge invariant results, the recursion relations given in the appendix are readily put on a computer and the power series summed up to a certain order, where the error is less than the first neglected term. In this way, arbitrary precision can be achieved.

In fig. 1, the results for $\delta \propto x^2 \Phi$ and $\pi_{anis.} \propto x^2 \Pi$ in the purely collisionless case $\alpha = 1$ are plotted in logarithmic scales for x ranging up to 10π , where the perfect-fluid solution ($\alpha = 0$) has been added for comparison. For small values of x , where $\lambda > R_H$, δ as well as $\pi_{anis.}$ grows like x^2 , the only difference to the perfect-fluid case being that the growth of a density perturbation is accompanied by growing anisotropic pressure. After horizon crossing ($x/\pi = 1$) corresponds to the time when half of a wavelength of the perturbation fits inside the horizon, an oscillatory behaviour arises, which in contrast to the perfect-fluid case is strongly damped and the oscillation rate corresponds to speed of light rather than speed of sound $\sqrt{3}$. This can be easily understood by the fact that a perturbation made out of collisionless particles propagates with the speed of their constituents, which is speed of light in the present case of effectively massless particles. Furthermore, collisionless damping occurs through directional dispersion [6, 24]. Neglecting gravity in the collisionless Vlasov equation one finds $\delta \sim \sin(x)/x$ for this phenomenon, and, indeed, for large values of x the amplitude of the solution given in fig. 1 falls off like $1/x$.

In fig. 2, the two-component case is evaluated for $\alpha = 1/2$, where the perfect-fluid component and the collisionless component are found to behave roughly in the same way as above. Because the fluctuations in the collisionless part die out quickly, the dominant effect for large x is the density wave in the perfect-fluid component, which is not influenced much by the former. The oscillations in the collisionless component, however, are somewhat distorted in comparison with the purely collisionless case due

*We have also checked that the same value can be derived from the equations given in Ref. 5 for the longitudinal gauge. Numerical solutions for the latter have been studied in Ref. 4, where, however, the initial anisotropic pressure potential $\Pi(0)$ has been artificially put to zero.

to the presence of relatively large density fluctuations in the perfect fluid part.

Our solutions agree well with the numerical results of [6], with slight deviations for $x \sim 1$. The latter are to be expected, however, because only for large x , i.e. on subhorizon scales, do differences due to different gauges and hypersurfaces disappear [2]. Ref. [6] uses the synchronous gauge and so they are using a definition of density perturbations other than the gauge invariant one set up in chapt. 2.

6.2. VECTOR PERTURBATIONS

As before, the requirement of regularity at $x = 0$ fixes the initial condition parameter. This gives

$$\bar{c} = 2V_{cl}(0) = -\frac{2(1-\alpha)}{\alpha}V_{p.f.} \quad (6.4)$$

in eq. (5.10), so that there may be vorticity separately in the collisionless and in the perfect-fluid component, which conspire to a vanishing total vorticity at the initial singularity. Again, a power series ansatz about $x = 0$ gives rise to a rapidly converging solution, of which the first few terms read

$$\Psi(x) = C\left(1 - \frac{15}{14(15+2\alpha)}\frac{x^2}{2!} + \frac{5(525+16\alpha)}{147(15+2\alpha)(75+4\alpha)}\frac{x^4}{4!} \mp \dots\right). \quad (6.5)$$

Although formally the limit $\alpha \rightarrow 1$ exists, it does not correspond to the purely collisionless case, because by virtue of (6.4) the disappearing perfect-fluid component would still have to contribute a finite amount of vorticity $(1-\alpha)V_{p.f.}$. In fact, with $V_{p.f.} = 0$ the Volterra equation (5.10) lacks an inhomogeneous term, and the only solution is the trivial one, $\Psi \equiv 0$. A nontrivial solution exists for $x_0 \neq 0$, which, however, is very sensitive of the value of x_0 if x_0 is small, and becomes singular for $x_0 \rightarrow 0$.

In fig. 3, the solution (6.5) is plotted for $\alpha = 1/2$. One finds that there is a build-up of total vorticity $V \sim x^2$ which saturates for $x > 1$, i.e. when the horizon has grown larger than the perturbation. However, this generation of vorticity results entirely from the dissipation of the vorticity initially present in the collisionless component, so that in the end only the invariable vorticity of the perfect-fluid part is retained.

A genuine spontaneous generation of vorticity in collisionless matter is possible with more general initial conditions. With $V_{p.f.} = 0$, the simplest nontrivial case

is $\gamma_0 \neq 0$, $\gamma_1 = 0$, $\gamma_2 \neq 0$ in (4.18). The requirement of regularity of Ψ at $x_0 = 0$ determines the ratio γ_2/γ_0 , and the initial condition term in eq. (5.10) is substituted according to

$$\begin{aligned} \bar{c}(j_0(x) + j_2(x)) &\rightarrow \bar{c}(j_0(x) + j_2(x) + 5(j_0''(x) + j_2''(x))) \\ &= 12\bar{c}\frac{j_3(x)}{x}. \end{aligned} \quad (6.6)$$

Now the power series solution for Ψ reads

$$\Psi(x) = C\left(1 - \frac{5(7+2\alpha)}{21(15+2\alpha)}\frac{x^2}{2!} + \frac{5(11025+4746\alpha+296\alpha^2)x^4}{1617(15+2\alpha)(75+4\alpha)}\frac{x^4}{4!} \mp \dots\right), \quad (6.7)$$

which is plotted in fig. 4 for the purely collisionless case $\alpha = 1$. (In the two-component case, $\alpha < 1$, the behaviour of the solution (6.7) is not very different.) For small values of x (superhorizon scales), a growing vorticity $V = V_{cl}$ is found. V reaches its maximum at horizon crossing $x \approx 1$, after which it decays like $\sim x^{-2}$ while undergoing oscillations which correspond to the propagation of subhorizon-scale vector perturbations at the speed of light.

Also given in figs. 3 and 4 is the shear of the matter velocity field, which is associated with the respective solutions according to eq. (2.14) and which in contrast to the perfect-fluid case is well-behaved at the initial singularity. On superhorizon scales, the shear is of the same order of magnitude as the maximal vorticity arising at horizon crossing.

We thus conclude that for vector perturbations the general phenomenon is dissipation of vorticity in the collisionless component on subhorizon scales, whereas on superhorizon scales, through appropriate, anisotropic initial conditions, one can have spontaneous generation of vortex motion.

6.3. TENSOR PERTURBATIONS

Now, regularity at $x = 0$ fixes $c = 0$ in (5.11), and the power series ansatz about $x = 0$ again gives an everywhere rapidly converging solution.

The first few terms are

$$H(x) = C\left(1 - \frac{5}{15+4\alpha}\frac{x^2}{2!} + \frac{5(105-2\alpha)}{7(15+4\alpha)(25+2\alpha)}\frac{x^4}{4!} \mp \dots\right), \quad (6.8)$$

which reduces to $j_0(x)$ in the limit $\alpha = 0$.

In fig. 5, the two limiting cases $\alpha = 0$ and $\alpha = 1$ are plotted. In the collisionless case one finds that, with the same initial value of H , the amplitude of the gravitational wave after horizon crossing is smaller than in the perfect-fluid case. For large values of x , the two solutions are quite similar, both decreasing in amplitude as $1/x$. In the general two-component case the solution for the gravitational wave lies in between the solutions given in fig. 5.

Like in the case of vector perturbations, one could also consider more general, anisotropic initial conditions. The simplest case of a nonvanishing initial condition term in eq. (5.11) is $c = \gamma_0 \neq 0$, $\gamma_1 = 0$, $\gamma_2 \neq 0$ in (4.18), where the requirement of regularity of $H(x)$ at $x = 0$ fixes γ_2/γ_0 . By this the initial condition term is found to be

$$24\alpha c \left[\frac{j_2(x)}{x^2} + 7\frac{j_2'(x)}{x^2} \right] = 144\alpha c \frac{j_4(x)}{x^2}. \quad (6.9)$$

This time, c is not fixed by the normalization of the solution, but parametrizes a family of solutions all of which show the same behaviour for large values of x and which vary with respect to the amount of $H(0)$ necessary to produce the same amplitude of gravitational waves at a given later time.

7. Conclusion

Using thermal field theory, we have derived self-consistent equations for cosmological perturbations in the presence of relativistic collisionless matter. We have found that these equations are manifestly gauge invariant and also somewhat simpler than the corresponding equations derived from kinetic theory; nevertheless they are consistent with previous results. Furthermore, we were able to find exact, analytical solutions in the form of rapidly converging power series with infinite radius of convergence.

The scalar (density) perturbations, which have been investigated before by means of numerical integration of the coupled Einstein-Boltzmann equations, exhibit the usual gravitational instability on superhorizon scales, whereas on subhorizon scales the collisionless component is strongly damped through directional dispersion. Interaction of the latter with collision-dominated matter only somewhat distorts the collisionless oscillations.

On the other hand, the solutions we have obtained for the vector and tensor modes are new. The tensor perturbations which correspond to primordial gravitational waves are quite similar to those of the well-known perfect-fluid case, the only difference being that during horizon-crossing there is a stronger fall-off in amplitude.

The solutions for vector perturbations, however, exhibit phenomena which are not familiar from the perfect-fluid case. Whereas in the latter no solution which is regular at the initial singularity exists, in the presence of collisionless matter we have found regular solutions describing spontaneous generation of vorticity on superhorizon scales and dissipation of it on subhorizon scales. (For a perfect fluid, both phenomena are excluded by virtue of the Helmholtz-Kelvin circulation theorem, which no longer applies for collisionless matter.) Thus, given appropriate initial conditions (either encoded in the initial singularity or left over from an inflationary epoch), one can have large-scale vortex motion in neutrinos or other forms of collisionless (dark) matter, a possibility which does not seem to have been considered before in the framework of adiabatic cosmological perturbations and which certainly deserves further investigation.

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Appendix A. Recursion relations

The integro-differential equations (5.5-5.11) can be solved by a power series ansatz for the unknown functions, if $x_0 = 0$. Inserting the power series representation of the spherical Bessel functions

$$j_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+k}}{(2n)!(2n+1)(2n+3)\cdots(2n+2k+1)} \quad (A.1)$$

leads to integrals

$$\int_0^x dx' (x-x')^m x'^n = \frac{m!n!}{(n+m+1)!} x^{n+m+1}, \quad (A.2)$$

where n and m are natural numbers. If the unknown functions are analytical at $x = 0$, or have poles of finite order, only a finite number of their coefficients contributes in the integro-differential equations at each power of x . For $x_0 \neq 0$, this is no longer true.

Scalar perturbations

With the ansatz

$$\Pi(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{a_n (2n)!}, \quad \Phi(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{b_n (2n)!}, \quad (A.3)$$

and defining $d_n \equiv 2(b_n - a_n)$ the finite-order differential equation (5.3) gives

$$3[(2n+1)b_n + d_n] = (2n-1)[3b_{n-1} - d_{n-1}], \quad n \geq 1 \quad (A.4)$$

which can be solved for the b 's,

$$3(2n+1)b_n = 3b_0 - d_0 - \sum_{m=1}^{n-1} (2m+4)d_m - 3d_n, \quad (A.5)$$

so that the recursion relations derived from eq. (5.5) (or eq. (5.7) in the case of a two-component system) can be formulated in terms of the coefficients d_n only. For the general, two-component case, one finds

$$\begin{aligned} \left[2n + 3 + \frac{4\alpha}{5(n+1)} \right] d_n &= \frac{d_0}{5} + \alpha \left(\frac{5+4\alpha}{3} - \frac{15\alpha}{(2n+3)(2n+5)} \right) \\ &- \frac{4}{3} \sum_{m=1}^{n-1} (m+2)d_m \\ &- \frac{12\alpha}{n+1} \sum_{m=1}^{n-1} \frac{n-m+1}{[2(n-m)+3][2(n-m)+5]} d_m. \end{aligned} \quad (A.6)$$

In the purely collisionless limit $\alpha = 1$, a simpler form can be obtained which allows one to infer the asymptotic behaviour of the coefficients d_n (for the opposite limit $\alpha = 0$ we already now the form of the coefficients from eq. (3.3)), which is

$$\sum_{m=1}^{n/2} \left[2m + 4 + \frac{18}{(n+1)(n-2m+5)} \right] d_m = \frac{3d_0}{4} \left(1 - \frac{5}{(n+1)(n+5)} \right). \quad (A.7)$$

From the fact that the r.h.s. increases with n for $d_0 > 0$, whereas the coefficients of d_m in the sum on the l.h.s. decrease, one infers that all d_m have to be positive.

Eq. (A.7) can be further recast into

$$\frac{10n^2 + 25n + 19}{5} d_n = \frac{36}{2n-1} \left[\sum_{m=0}^{n-1} \frac{(2n-m+2)d_m}{[2(n-m)+3][2(n-m)+5]} - \frac{19(n+1)d_0}{12(2n+3)(2n+5)} \right], \quad (A.8)$$

from which one can estimate that $d_n \sim n^{-4}$ for large n . For the coefficients a_n and b_n , which for $d_0 > 0$ all are positive, this implies a behaviour $\sim n^{-3}$. With decreasing parameter α this gradually changes into a n^{-1} -behaviour.

Having found the coefficients b_n, d_n by a recursive evaluation of (A.6) and (A.4), eq. (5.8) determines the coefficients of $\Phi_{cll} = \sum_{n=0}^{\infty} (-1)^n q_n x^{2n} / (2n)!$ in the two-component case through

$$q_n = \frac{6\alpha}{(2n+2)(2n+1)} \left\{ b_{n+1} - d_{n+1} + \frac{1}{2}(d_0 - b_0) \frac{2n+8}{(2n+3)(2n+5)} + \sum_{m=0}^n \left[\frac{1}{2m+1} + \frac{1}{(2m+3)(2n+3)} \right] d_{n-m+1} \right\}. \quad (A.9)$$

The power series $\Phi_{p,f} = \sum_{n=0}^{\infty} (-1)^n p_n x^{2n} / (2n)!$ is determined by $p_n = b_n - q_n$, and also all the p_n and q_n are found to have the same sign as d_0 .

Vector perturbations

With the ansatz

$$\Psi(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{c_n (2n)!} \quad (A.10)$$

eq. (5.10) leads to the recursion relation

$$\left[(n+1)(2n+1) + \frac{4\alpha}{5} \right] c_n = c_0 f_n - 12\alpha \sum_{m=1}^n \frac{1}{(2m+3)(2m+5)} c_{n-m}, \quad (A.11)$$

where

$$f_n = \frac{3(5+4\alpha)}{(2n+3)(2n+5)}. \quad (A.12)$$

With the initial condition term (6.6) and $V_{p,f} = 0$, the f_n are instead given by

$$f_n = \frac{21(5+4\alpha)(n+1)}{(2n+3)(2n+5)(2n+7)}. \quad (A.13)$$

In both cases, the coefficients c_n all have the same sign and their magnitude decreases like n^{-4} for large n .

With

$$H(x) = \sum_{n=0}^{\infty} (-1)^n h_n \frac{x^{2n}}{(2n)!} \quad (\text{A.14})$$

eq. (5.11) leads to

$$\left[2n(2n+1) + \frac{8\alpha}{5} \right] h_n = 2n(2n-1)h_{n-1} - 24\alpha \sum_{m=1}^{n-1} \frac{1}{(2m+1)(2m+3)(2m+5)} h_{n-m}. \quad (\text{A.15})$$

The coefficients h_n all are of the same sign and are bounded by their $\alpha = 0$ -values,

$$|h_n| \leq \frac{1}{2n+1} |h_0|. \quad (\text{A.16})$$

REFERENCES

- [1] E.M. Lifshitz, *J. Phys. U.S.S.R.* **10** (1946) 116;
E.M. Lifshitz and I.M. Khalatnikov, *Adv. Phys.* **12** (1963) 185
- [2] J.M. Bardeen, *Phys. Rev. D* **22** (1980) 1882
- [3] P.J.E. Peebles and J.T. Yu, *Astrophys. J.* **162** (1970) 815
- [4] A.I. McCone, Ph.D. thesis, Univ. of Maryland, Tech. Rept. UMD-70-054
- [5] J.M. Stewart, *Astrophys. J.* **176** (1972) 323
- [6] P.J.E. Peebles, *Astrophys. J.* **180** (1973) 1;
J.R. Bond and A.S. Szalay, *Astrophys. J.* **274** (1983) 443
- [7] N.P. Landsman and Ch.G. van Weert, *Phys. Rep.* **145** (1987) 141, and references therein
- [8] U. Kraemmer and A. Rebhan, "Self-consistent cosmological perturbations from thermal field theory", CERN-preprint CERN-TH.6052/91, to appear in *Phys. Rev. Lett.*
- [9] A. Rebhan, *Nucl. Phys. B* **351** (1991) 706
- [10] Ya.B. Zel'dovich and I.D. Novikov, *Relativistic Astrophysics*, Vol. 2 (U. o. Chicago Press, Chicago, 1983)
- [11] N.D. Birrell and P.C.W. Davies, "Quantum Fields in Curved Space" (Cambridge Univ. Press, Cambridge, 1982)
- [12] R. Brandenberger, R. Kahn and W.H. Press, *Phys. Rev. D* **28** (1983) 1809
- [13] H. Kodama and Y. Sasaki, *Prog. Theor. Phys. Suppl.* **78** (1984) 1
- [14] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, preprint BROWN-HET-780 (1990)
- [15] J. Jeans, *Philos. Trans. R. Soc. London A* **199** (1902) 491

- [16] P.J.E. Peebles, *The Large-Scale Structure of the Universe* (Princeton University Press, Princeton, 1980)
- [17] B.J.T. Jones, *Rev. Mod. Phys.* **48** (1976) 107.
- [18] J. Ehlers, in *General Relativity and Cosmology, Proceedings of the Int. School of Physics E. Fermi, Course XLVII*, ed. R.K. Sachs (Academic Press, New York, 1971)
- [19] R.K. Schaefer, *Int. J. Mod. Phys. A* **6** (1991) 2075
- [20] M. Kasai and K. Tomita, *Phys. Rev. D* **33** (1986) 1576
- [21] I.E. Dzyaloshinski, *Zh. Eksp. Fiz.* **42** (1962) 1126;
R.D. Pisarski, *Nucl. Phys. B* **309** (1988) 476;
J. Frenkel and J.C. Taylor, *Nucl. Phys. B* **334** (1990) 199
- [22] R. Kobes, G. Kunstatter and A. Rebhan, *Nucl. Phys. B* **355** (1991) 1
- [23] L.F. Abbott and M.B. Wise, *Nucl. Phys. B* **237** (1984) 226
- [24] I. Wasserman, *Astrophys. J.* **248** (1981) 1;
G. Börner, *The Early Universe* (Springer-Verlag, Berlin, 1988) p. 362

FIGURE CAPTIONS

1. The evolution of density perturbations (full line) and of anisotropic pressure (dashed line) in the purely collisionless case as a function of $x/\pi \equiv kr/\pi = R_H/(\lambda/2)$. The perfect-fluid solution for density perturbations is given by the dotted line ($\pi_{\text{anis.}} \equiv 0$ in that case), and the normalization is chosen so that the latter peaks at $\delta = 1$.
2. Density perturbations in the two-component case with $\alpha = 1/2$. The density perturbation in the collisionless component is given by the full line, and that of the perfect-fluid one by the dot-dashed line.
3. The evolution of vector perturbations in a two-component system with $\alpha = 1/2$, where $V_{\text{eff.}}(0) = -V_{p.f.} = 1$ so that initially the total vorticity vanishes, $V(0) = 0$. The full line shows the dissipation of $V_{\text{eff.}}$, which results in the generation of total vorticity as given by the dot-dashed line. The total shear is given by the dashed line.
4. Spontaneous generation of large-scale vorticity in collisionless matter (full line), and its subsequent dissipation after horizon crossing. This is accompanied by shear in the matter velocity field, as given by the dashed line.
5. Primordial gravitational waves with $H(0) = 1$ in the purely collisionless case (full line) and in the perfect-fluid case (dotted line).

Fig. 2

Fig. 1

Fig. 3

Fig. 4

Fig. 5