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GENERALIZED LAGRANGIAN DYNAMICS AND NOETHERIAN SYMMETRIES

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ABSTRACT

We propose a generalization of the usual Lagrangian formulation of dynamics, suggested by the Souriau approach to analytical mechanics. It is a global formulation in which the Lagrangian is defined, in general, only locally. Some examples are given of generalized dynamics in this sense. We analyse in detail the Galileian invariant N -particle system and we also study the case of identical particles.

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1. Introduction

The purpose of this paper is to provide a generalization of the usual Lagrangian formalism, suggested by the reformulation of this formalism due to Souriau [1]. We will consider only the case of a system with a finite number of degrees of freedom.

This formalism consists, essentially, of two steps [1], [2]:

- a kinematical part, in which one builds the so-called "evolution space" E , using for instance the first-order jet construction.

- a dynamical part, in which, starting from a given Lagrangian function L defined on E , one builds a 2-form σ_L on E . This form (called from now on the Lagrange-Souriau form) is globally defined, and so one is able to describe in an intrinsic way the Euler-Lagrange equations of motion.

The fact that σ_L is globally defined suggests investigating the more general case when one has only a locally defined Lagrangian, but the Lagrange-Souriau form is globally defined. For this purpose we will need necessary and sufficient conditions on a form $\sigma \in \wedge^2 E$ such that there exists, at least locally, a Lagrangian function L such that we have locally $\sigma = \sigma_L$. A set of such conditions have been found in [3]. So, the main idea is to raise these conditions to the rank of axioms for any dynamical system.

In Section 2 we will present the general framework and we will provide an example of a simple system for which the Lagrangian can be defined only locally. This system describes a particle in the magnetic field generated by a constant current along a straight line. In Section 3 we analyse the Noetherian symmetries in this formulation, simplifying considerably the previous analyses [4]-[6]. (See also [7].) We will concentrate on the Galilean invariant N -particle system and we will also study the case of identical particles. Section 4 is devoted to some final comments.

2. The generalized Lagrangian formalism

2.1 First we recall briefly how one can construct the evolution space of a mechanical system [2]. Suppose S is a $(N + 1)$ -dimensional manifold interpreted as the "space-time" manifold of the system. The usual case is $S = R \times Q$ where R is the time axis and Q is the configuration manifold of the system. More generally, one can consider that S is fibered over the time axis R . We prefer to work in the general case, when this hypothesis is not

made. We have in mind the case of a relativistic particle when S is the Minkowski space and the choice of time and space is observer-dependent.

Then, by definition, an *evolution space* is an open subbundle E of the projective tangent bundle $PT(S)$.

In local coordinates, the picture is the following one. Say (t, q_A) , $A = 1, \dots, N$ is a local chart on S in a neighbourhood of the point $s_0 \equiv (t_0, (q_A)_0)$. Then a direction in $PT(S)_{s_0}$ is of the form:

$$\frac{\delta}{\delta t} \equiv \left(\frac{\partial}{\partial t} \right)_0 + v_A \left(\frac{\partial}{\partial q_A} \right)_{s_0}. \quad (2.1)$$

(with the summation convention over the dummy indices).

An *evolution* is an immersion $\gamma : R \rightarrow S$. In the local coordinates above, we have:

$$\gamma(t) = (t, q_A(t)). \quad (2.2)$$

There exists a natural lift of γ to a map $\dot{\gamma} : R \rightarrow PT(S)$. In local coordinates:

$$\dot{\gamma}(t) = \left(q_a(t), \frac{dq_A}{dt}(t) \right). \quad (2.3)$$

We will suppose that for a given evolution space E , the evolutions γ are such that $\dot{\gamma}$ takes values in E .

2.2 Suppose that $L : E \rightarrow R$ is a smooth function called *Lagrangian*. Let $\gamma : R \rightarrow S$ be an evolution. Then we say that γ verifies the *Euler-Lagrange equations of motion* if:

$$\frac{\partial L}{\partial q_A} \circ \dot{\gamma} - \frac{d}{dt} \left(\frac{\partial L}{\partial v_A} \circ \dot{\gamma} \right) = 0. \quad (2.4)$$

One can write these equations in a chart-independent form as follows. One first defines on E the *Poincaré-Cartan form* associated to L :

$$\theta_L \equiv Ldt + \frac{\partial L}{\partial v_A} \delta q_A. \quad (2.5)$$

where we have defined:

$$\delta q_A \equiv dq_A - v_A dt. \quad (2.6)$$

Then, one defines the so-called *Lagrange-Souriau form* by:

$$\sigma_L \equiv d\theta_L. \quad (2.7)$$

One can prove then that σ_L is globally defined [3]. Also, one shows easily that γ is a solution of the Euler-Lagrange equations of motion *iff*:

$$i_\gamma \sigma_L = 0. \quad (2.8)$$

This gives the looked-for global form of the Euler-Lagrange equations of motion.

2.3 Let us study more closely the form σ_L . In local coordinates, we get from (2.5) and (2.7):

$$\sigma_L = \sigma_{AB} dv_A \wedge \delta q_B + \tau_A \delta q_A \wedge dt + \frac{1}{2} \tau_{AB} \delta q_A \wedge \delta q_B. \quad (2.9)$$

where:

$$\sigma_{AB} = \frac{\partial L}{\partial v_A \partial v_B}. \quad (2.10)$$

$$\tau_A = \frac{\partial L}{\partial q_A} - \frac{\partial^2 L}{\partial t \partial v_A} - v_B \frac{\partial^2 L}{\partial v_A \partial v_B}. \quad (2.11)$$

$$\tau_{AB} = \frac{\partial^2 L}{\partial q_A \partial v_B} - \frac{\partial^2 L}{\partial q_B \partial v_A}. \quad (2.12)$$

The problem is now to establish necessary and sufficient conditions on $\sigma \in \wedge^2 E$ such that one can find, at least locally, a function L such that $\sigma = \sigma_L$. A necessary condition following directly from (2.7) is:

$$d\sigma_L = 0. \quad (2.13)$$

Inserting (2.9) we get in local coordinates:

$$\frac{\partial \sigma_{AB}}{\partial v_C} - \frac{\partial \sigma_{AC}}{\partial v_B} = 0. \quad (2.14)$$

$$\frac{\partial \sigma_{AB}}{\partial q_C} - \frac{\partial \sigma_{AC}}{\partial q_B} + \frac{\partial \tau_{BC}}{\partial v_A} = 0. \quad (2.15)$$

$$\frac{\partial \sigma_{AB}}{\partial t} + v_C \frac{\partial \sigma_{AB}}{\partial q_C} - \tau_{BA} + \frac{\partial \tau_B}{\partial v_A} = 0. \quad (2.16)$$

$$\frac{\partial \tau_{AB}}{\partial t} + v_C \frac{\partial \tau_{AB}}{\partial q_C} + \frac{\partial \tau_B}{\partial q_A} - \frac{\partial \tau_A}{\partial q_B} = 0. \quad (2.17)$$

Also, from (2.10) and (2.12) we have the necessary conditions:

$$\sigma_{AB} = \sigma_{BA}. \quad (2.18)$$

$$\tau_{AB} = -\tau_{BA}. \quad (2.19)$$

The last condition is rather natural if one regards (2.9): it can be considered as a part of the definition of σ . But one can show more [3].

a) Let us denote by $\pi : E \rightarrow S$ the canonical projection. We define now:

$$(\wedge^2 E)_{LS} \equiv \{\sigma \in \wedge^2 E | i_{V_1} i_{V_2} \sigma = 0, \forall V_i, \pi_* V_i = 0, i = 1, 2\}$$

Locally $\forall \sigma \in (\wedge^2 E)_{LS}$ is of the form:

$$\sigma = \sigma_{AB} dv_A \wedge \delta q_B + \tau_A \delta q_A \wedge dt + \frac{1}{2} \tau_{AB} \delta q_A \wedge \delta q_B. \quad (2.20)$$

where one may choose τ_{AB} verifying (2.19).

b) Let us define locally the operator $K : \wedge^2 E \rightarrow \wedge^1 E$ by:

$$K\sigma \equiv i_{\frac{\partial}{\partial t}} i_{\frac{\partial}{\partial v_A}} (\delta q_A \wedge \sigma). \quad (2.21)$$

Then, one shows that for $\sigma \in (\wedge^2 E)_{LS}$, K is globally defined, and moreover, the relation:

$$K\sigma = 0. \quad (2.22)$$

has the local form of (2.18).

c) Conversely, one can show that if $\sigma \in (\wedge^2 E)_{LS}$ verifies (2.13) and (2.22), then it can be written, at least locally, as $\sigma = \sigma_L$, for a function L on E (defined, in general, only locally).

Remark 1 One may wonder what is the arbitrariness of this (local) Lagrangian L . Using (2.10)-(2.12) it is easy to show that $\sigma_L = 0$ iff L is a *total derivative* i.e., a Lagrangian of the form:

$$L(t, q, v) = \frac{\partial \Lambda}{\partial t}(t, q) + v_A \frac{\partial \Lambda}{\partial q_A}(t, q). \quad (2.23)$$

for some (locally defined) function Λ on S .

The preceding discussion leads to the following definition:

Definition 1 Let S be a "space-time" manifold. A *Lagrangian system* is a couple (E, σ) where E is an evolution space and $\sigma \in (\wedge^2 E)_{LS}$ verifies (2.13) and (2.22).

2.4 To provide examples of Lagrangian systems for which the Lagrangian is only locally defined, let us consider the case of a particle moving in R^3 . We can take $S \equiv R \times R^3$ with global coordinates $(t, \mathbf{x})$ and we can identify $PT(S) \cong R \times R^3 \times R^3$ with global coordinates $(t, \mathbf{x}, \mathbf{v})$.

Suppose now that the particle moves in an electromagnetic field described by the potentials Φ and $\mathbf{A}$. According to the principle of minimal coupling, the Lagrangian of such a system is:

$$L(t, \mathbf{x}, \mathbf{v}) = \frac{1}{2}m\mathbf{v}^2 - \mathbf{v} \cdot \mathbf{A}(t, \mathbf{x}) + \Phi(t, \mathbf{x}). \quad (2.24)$$

One can compute easily the corresponding Lagrange-Souriau form:

$$\sigma_L = m dv_i \wedge \delta x_i - (E_i + \varepsilon_{ijk} v_j B_k) \delta x_i \wedge dt - \frac{1}{2} \varepsilon_{ijk} B_k \delta x_i \wedge \delta x_j. \quad (2.25)$$

an expression already appearing in [1]. Here:

$$\mathbf{E} \equiv \nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}. \quad (2.26)$$

$$\mathbf{B} \equiv \nabla \times \mathbf{A}. \quad (2.27)$$

We remark that in (2.25) only the observable fields $\mathbf{E}$ and $\mathbf{B}$ appear. This suggests defining by a particle in an electromagnetic field, a Lagrangian system of the type (E, σ) where E (a subbundle of $PT(S)$) is an evolution space and σ is of the form (2.25) for some functions $\mathbf{E}$ and $\mathbf{B}$ defined on S , not necessarily of the form (2.26) and (2.27). We must however impose (2.13) which in this case reduces to the homogeneous Maxwell equations:

$$\nabla \times \mathbf{E} = 0. \quad (2.28)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (2.29)$$

We can consider for instance that:

$$S' \equiv \{(t, \mathbf{x}) \in S \mid x_1^2 + x_2^2 \neq 0\}$$

and:

$$\mathbf{E} = \mathbf{0}. \quad (2.30)$$

$$\mathbf{B} = \frac{I}{2\pi(x_1^2 + x_2^2)}(x_2, -x_1, 0). \quad (2.31)$$

This corresponds to the magnetic field created by a current I travelling along the Ox_3 axis. It is clear that σ given by (2.25) and (2.30),(2.31) is well defined, but L can be defined only locally.

3. Noetherian symmetries in the Lagrangian Formalism

3.1 Let us consider first that we have a globally defined Lagrangian L . Let $\phi \in \text{Diff}(S)$ and denote by $\dot{\phi} \in \text{Diff}(PT(S))$ the natural lift of ϕ to $PT(S)$. We will suppose that ϕ is such that $\dot{\phi}$ is in fact a diffeomorphism of the evolution space E .

Then one can establish that ϕ leaves invariant the action functional:

$$A_L(\gamma) \equiv \int L \circ \dot{\phi}. \quad (3.1)$$

iff:

$$(\dot{\phi})^* \sigma_L = \sigma_L. \quad (3.2)$$

This fact leads us to the following definition:

Definition 2: Let (E, σ) be a Lagrangian system. Then a *Noetherian symmetry* is any diffeomorphism Φ of E such that:

$$\Phi^* \sigma = \sigma. \quad (3.3)$$

If Φ is of the form $\Phi = \dot{\phi}$ for some $\phi \in \text{Diff}(S)$, then we will call Φ a *restricted Noetherian symmetry*.

In what follows we will be interested mainly in restricted Noetherian symmetries. Let us denote by $\text{Sym}(E, \sigma)$ the set of restricted Noetherian symmetries of (E, σ) . This set is a group with respect to the function composition operation. A *group of restricted Noetherian symmetries* is an action of a Lie group G on S : $g \mapsto \phi_g$ such that for any $g \in G$, $\dot{\phi}_g \in \text{Sym}(E, \sigma)$ i.e.:

$$(\dot{\phi}_g)^* \sigma = \sigma. \quad (3.4)$$

In the rest of this section we will study the following classification problem: given the "space-time" manifold S and an action of a Lie group G on it, find all Lagrangian systems having G as a group of restricted Noetherian symmetries. As a strategy, one tries to find the most general $\sigma \in (\wedge^2 E)_{LS}$ verifying (2.13), (2.22) and (3.4). If one can write globally σ in the form (2.20) this amounts to the study of the equations (2.14)-(2.19) and the additional restrictions following from (3.4). In most of the cases this is not very hard to do.

After one has succeeded in solving this problem, one can try to find out if σ is of the form σ_L for some Lagrangian L globally defined on E . Let us note that we need only a

particular solution L of the system $\sigma_L = \sigma$. Any other solution can be obtained by adding a trivial Lagrangian. (This follows from Remark 1).

3.2 We consider $S = R^4$ with the global coordinates $(t, \mathbf{x})$. Then as in Section 2.4 we can identify $PT(S) \cong R^4 \times R^3$ with global coordinates: $(t, \mathbf{x}, \mathbf{v})$.

For any $R \in SO(3)$ and any $(\tau, \mathbf{a}) \in R^4$ we define:

$$\phi_{R,\tau,\mathbf{a}}(t, \mathbf{x}) = (t + \tau, R\mathbf{x} + \mathbf{a}). \quad (3.5)$$

(The group of all such transformations is sometimes called "the Aristotle group"). We want to classify all Lagrangian systems admitting this group as a group of restricted Noetherian symmetries.

First we particularize the formulae (2.14)-(2.20). The 2-form σ has a global expression of the type:

$$\sigma = \sigma_{ij} dv_i \wedge \delta x_j + \tau_i \delta x_i \wedge dt + \frac{1}{2} \tau_{ij} \delta x_i \wedge \delta x_j. \quad (3.6)$$

where we suppose that:

$$\tau_{ij} = -\tau_{ji}. \quad (3.7)$$

and we have defined as usual:

$$\delta x_i \equiv dx_i - v_i dt. \quad (3.8)$$

Then (2.14)-(2.18) are:

$$\frac{\partial \sigma_{ij}}{\partial v_k} - \frac{\partial \sigma_{ik}}{\partial v_j} = 0. \quad (3.9)$$

$$\frac{\partial \sigma_{ij}}{\partial x_k} - \frac{\partial \sigma_{ik}}{\partial x_j} + \frac{\partial \tau_{jk}}{\partial v_i} = 0. \quad (3.10)$$

$$\frac{\partial \sigma_{ij}}{\partial t} + v_k \frac{\partial \sigma_{ij}}{\partial x_k} + \frac{\partial \tau_j}{\partial v_i} - \tau_{ji} = 0. \quad (3.11)$$

$$\frac{\partial \tau_{ij}}{\partial t} + v_k \frac{\partial \tau_{ij}}{\partial x_k} + \frac{\partial \tau_j}{\partial x_i} - \frac{\partial \tau_i}{\partial x_j} = 0. \quad (3.12)$$

$$\sigma_{ij} = \sigma_{ji}. \quad (3.13)$$

Next, (3.5) yields:

$$\dot{\phi}_{R,\tau,\mathbf{a}}(t, \mathbf{x}, \mathbf{a}) = (t + \tau, R\mathbf{x} + \mathbf{a}, R\mathbf{v}). \quad (3.14)$$

According to Section 3.1 we impose:

$$(\dot{\phi}_{R,\tau,\mathbf{a}})^*\sigma = \sigma. \quad (3.15)$$

Taking $R = id$ it is easy to see that the functions σ_{ij} , τ_i and τ_{ij} do not depend on t and $\mathbf{x}$, i.e. they are functions depending only of $\mathbf{v}$. Then (3.15) imposes also the rotation covariance of these functions.

Now from (3.10) it follows that τ_{ij} are in fact some constants. The condition of rotation covariance imposes then that τ_{ij} is proportional to the Kroneker tensor. But using now (3.7) we get:

$$\tau_{ij} = 0. \quad (3.16)$$

Then (3.11) shows that τ_i are also constants. Taking into account the rotation covariance we get:

$$\tau_i = 0. \quad (3.17)$$

At last, we analyse σ_{ij} . Taking into account (3.13) we must have:

$$\sigma_{ij}(\mathbf{v}) = \delta_{ij}A(\mathbf{v}^2) + v_i v_j B(\mathbf{v}^2). \quad (3.18)$$

for some smooth functions $A, B : R_+ \rightarrow R$. But then (3.9) is equivalent to:

$$B = 2A'. \quad (3.19)$$

So, the most general Lagrange-Souriau form σ compatible with the invariance with respect to the Aristotle group is:

$$\sigma = (\delta_{ij}A + 2v_i v_j A') dv_i \wedge \delta x_j. \quad (3.20)$$

It is easy to see now that we have a globally defined Lagrangian. A possible solution of (2.10)-(2.12) is:

$$L(t, \mathbf{x}, \mathbf{v}) = \frac{1}{2} \int^{\mathbf{v}^2} A(\xi) d\xi. \quad (3.21)$$

Remark 2 If one imposes in addition the invariance with respect to the Galilei transformations:

$$\phi_{\mathbf{u}}(t, \mathbf{x}) = (t, \mathbf{x} + \mathbf{u})$$

then one can conclude that $A = \text{const.}$ and one obtains the usual expression for the non-relativistic Lagrangian. In the same way can be treated the case of Poincaré invariance [8]. In this case one has two candidates for the evolution space [2]. They correspond to the directions in $PT(S)$ inside and respectively outside of the light cone. One gets $A(\xi) = m(1 - \xi)^{-1/2}$ and respectively $A(\xi) = m(\xi - 1)^{-1/2}$ and the usual expressions follow. A similar analysis can be done in other dimensions, e.g. for $S = R^3$ or $S = R^2$. It is easy to see that in these cases the Lagrangian will also be globally defined but it will pick additional "interaction" terms. (See [4] for the case R^2).

3.3 We consider now the general case of N particles with Galilei invariance. We can take as "space-time" manifold:

$$S_N \equiv R \times (R^D \times \dots \times R^D - \Delta)$$

with global coordinates: $(t, \mathbf{x}_1, \dots, \mathbf{x}_N)$ and:

$$\Delta \equiv \{(\mathbf{x}_1, \dots, \mathbf{x}_n) | \exists i \neq j, \mathbf{x}_i = \mathbf{x}_j\}$$

is the so-called "collision set". Then we can take as the evolution space:

$$E_N = PT(S_N) \cong S_N \times (R^D \dots \times R^D)$$

with global coordinates: $(t, \mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{v}_1, \dots, \mathbf{v}_N)$. We keep for the moment the dimension $D \in N$ arbitrary.

The action of the proper orthochronous Galilei group on S is:

$$\phi_{R, \mathbf{u}, \tau, \mathbf{a}}(t, \mathbf{x}_1, \dots, \mathbf{x}_N) = (t + \tau, R\mathbf{x}_1 + t\mathbf{u} + \mathbf{a}, \dots, R\mathbf{x}_N + t\mathbf{u} + \mathbf{a}). \quad (3.22)$$

It is convenient to use relative coordinates:

$$\xi_1 \equiv \mathbf{x}_1 - \mathbf{x}_2, \dots, \xi_{N-1} \equiv \mathbf{x}_{N-1} - \mathbf{x}_N, \xi_N \equiv \frac{1}{N}(\mathbf{x}_1 + \dots + \mathbf{x}_N). \quad (3.23)$$

$$\rho_1 \equiv \mathbf{v}_1 - \mathbf{v}_2, \dots, \rho_{N-1} \equiv \mathbf{v}_{N-1} - \mathbf{v}_N, \rho_N \equiv \frac{1}{N}(\mathbf{v}_1 + \dots + \mathbf{v}_N). \quad (3.24)$$

Then (3.22) becomes:

$$\phi_{R, \mathbf{u}, \tau, \mathbf{a}}(t, \xi_1, \dots, \xi_N) = (t + \tau, R\xi_1, \dots, R\xi_{N-1}, R\xi_N + t\mathbf{u} + \mathbf{a}). \quad (3.25)$$

This lifts to:

$$\begin{aligned} \dot{\phi}_{R,\mathbf{u},\tau,\mathbf{a}}(t, \xi_1, \dots, \xi_N, \rho_1, \dots, \rho_N) = \\ (t + \tau, R\xi_1, \dots, R\xi_{N-1}, R\xi_N + t\mathbf{u} + \mathbf{a}, R\rho_1, \dots, R\rho_{N-1}, R\rho_N + \mathbf{u}). \end{aligned} \quad (3.26)$$

We will require that $\dot{\phi}_{R,\mathbf{u},\tau,\mathbf{a}}$ are Noetherian symmetries:

$$(\dot{\phi}_{R,\mathbf{u},\tau,\mathbf{a}})^*\sigma = \sigma. \quad (3.27)$$

1) First we particularize (2.14)-(2.20). The indices A are of the type $A = (ai)$ where $a = 1, \dots, N$ and $i = 1, 2, 3$. The form σ has the global expression:

$$\sigma = \sigma_{ai,bj} d\rho_{ai} \wedge \delta\xi_{bj} + \tau_{ai} \delta\xi_{ai} \wedge dt + \frac{1}{2} \tau_{ai,bj} \delta\xi_{ai} \wedge \delta\xi_{bj}. \quad (3.28)$$

where we suppose that:

$$\tau_{ai,bj} = -\tau_{bj,ai}. \quad (3.29)$$

and we have denoted:

$$\delta\xi_{ai} \equiv d\xi_{ai} - \rho_{ai} dt. \quad (3.30)$$

Then (2.14)-(2.18) are:

$$\frac{\partial \sigma_{ai,bj}}{\partial \rho_{ck}} - \frac{\partial \sigma_{ai,ck}}{\partial \rho_{bj}} = 0. \quad (3.31)$$

$$\frac{\partial \sigma_{ai,bj}}{\partial \xi_{ck}} - \frac{\partial \sigma_{ai,ck}}{\partial \xi_{bj}} + \frac{\partial \tau_{bj,ck}}{\partial \rho_{ai}} = 0. \quad (3.32)$$

$$\frac{\partial \sigma_{ai,bj}}{\partial t} + \rho_{ck} \frac{\partial \sigma_{ai,bj}}{\partial \xi_{ck}} + \frac{\partial \tau_{bj}}{\partial \rho_{ai}} - \tau_{bj,ai} = 0. \quad (3.33)$$

$$\frac{\partial \tau_{ai,bj}}{\partial t} + \rho_{ck} \frac{\partial \tau_{ai,bj}}{\partial \xi_{ck}} + \frac{\partial \tau_{bj}}{\partial \xi_{ai}} - \frac{\partial \tau_{ai}}{\partial \xi_{bj}} = 0. \quad (3.34)$$

$$\sigma_{ai,bj} = \sigma_{bj,ai}. \quad (3.35)$$

Now if we take in (3.27), $R = id$, it follows that the functions $\sigma_{\dots}$ and $\tau_{\dots}$ must not depend on t , ξ_N and ρ_N . It will be convenient to use the shorthand notations:

$$\xi' \equiv (\xi_1, \dots, \xi_{N-1}); \rho' \equiv (\rho_1, \dots, \rho_{N-1}). \quad (3.36)$$

and denote the indices $a, b, \dots$ running from 1 to $N - 1$ by greek letters $\alpha, \beta, \dots$. Moreover, from (3.27) we get that the functions $\sigma_{\dots}$ and $\tau_{\dots}$ are rotation covariant.

We will use then (3.31)-(3.35). The computations are elementary, so we indicate only the main points. A useful remark is that the variable ρ' takes values in a Euclidean space, so we can apply the Poincaré lemma.

a) From (3.31) and (3.35) we can show that there exists a function K depending only of ξ' and ρ' such that:

$$\sigma_{\alpha i, \beta j} = \frac{\partial K}{\partial \rho_{\alpha i} \partial \rho_{\beta j}}. \quad (3.37)$$

Moreover, we have:

$$\frac{\partial \sigma_{\alpha i, N j}}{\partial \rho_{c k}} = \frac{\partial \sigma_{N j, \alpha i}}{\partial \rho_{c k}} = 0. \quad (3.38)$$

i.e., $\sigma_{\alpha i, N j}$ and $\sigma_{N j, \alpha i}$ are functions only of ξ' .

b) We analyse (3.32). We have two possibilities.

b1) $a = N$. In this case one gets supplementary informations on $\sigma_{\dots}$. Namely:

$$\frac{\partial \sigma_{N k, \alpha i}}{\partial \xi_{\beta j}} - \frac{\partial \sigma_{N k, \beta j}}{\partial \xi_{\alpha i}} = 0. \quad (3.39)$$

and:

$$\frac{\partial \sigma_{N i, N j}}{\partial \xi_{\alpha k}} = 0. \quad (3.40)$$

Combined with (3.38), it shows that $\sigma_{N i, N j}$ are in fact some constants. Taking into account the rotation covariance and (3.35) it follows that there exists $M \in \mathbb{R}$ such that:

$$\sigma_{N i, N j} = M \delta_{ij}. \quad (3.41)$$

b2) $a < N$. In this case (3.32) can be regarded as a system for the functions $\tau_{\alpha i, \beta j}$. One gets:

$$\tau_{\alpha i, \beta j} = \frac{\partial K}{\partial \xi_{\alpha i} \partial \rho_{\beta j}} - \frac{\partial K}{\partial \xi_{\beta j} \partial \rho_{\alpha i}} + t_{\alpha i, \beta j}. \quad (3.42)$$

where $t_{\alpha i, \beta j}$ are functions only of ξ' and verifying (see (3.29)):

$$t_{\alpha i, \beta j} = t_{\beta j, \alpha i}. \quad (3.44)$$

Also we have:

$$\tau_{\alpha i, N j} = -\tau_{N j, \alpha i} = \frac{\partial \sigma_{N j, \gamma k}}{\partial \xi_{\alpha i}} \rho_{\gamma k} + t_{\alpha i, N j}. \quad (3.45)$$

where $t_{\alpha i, N j}$ are functions only of ξ' .

Finally:

$$\frac{\partial \tau_{Ni, Nj}}{\partial \rho_{\gamma k}} = 0$$

i.e., $\tau_{Ni, Nj}$ are functions only of ξ' .

c) We study now (3.33). Again we have two cases:

c1) $a = N$. We get that in fact:

$$t_{\alpha i, \beta j} = 0$$

so, instead of (3.45) we have:

$$\tau_{\alpha i, Nj} = -\tau_{Nj, \alpha i} = \frac{\partial \sigma_{Nj, \gamma k}}{\partial \xi_{\alpha i}} \rho_{\gamma k}. \quad (3.46)$$

Also, we get that in fact:

$$\tau_{Ni, Nj} = 0. \quad (3.47)$$

c2) $a < N$. We have a system for the functions $\tau_{\alpha i}$ with the following solution:

$$\tau_{\alpha i} = \frac{\partial K}{\partial \xi_{\alpha i}} - \frac{\partial^2 K}{\partial \rho_{\alpha i} \partial \xi_{\beta j}} \rho_{\beta j} + t_{\alpha i, \beta j} \rho_{\beta j} + t_{\alpha i}. \quad (3.48)$$

where the functions $t_{\alpha i}$ depend only on ξ' .

Also:

$$\tau_{Ni} = -\frac{\partial \sigma_{Ni, \beta j}}{\partial \xi_{\gamma k}} \rho_{\beta j} \rho_{\gamma k} + t_{Ni}. \quad (3.49)$$

with t_{Ni} some ξ' -dependent functions.

d) It remains to use (3.34). Inserting in it the previous expressions, one gets that:

$$\frac{\partial t_{Ni}}{\partial \xi_{\alpha j}} = 0$$

so t_{Ni} are some constants. Using the rotation covariance, we get that in fact:

$$t_{Ni} = 0$$

so, instead of (3.49) we have:

$$\tau_{Ni} = -\frac{\partial \sigma_{Ni, \beta j}}{\partial \xi_{\gamma k}} \rho_{\beta j} \rho_{\gamma k}. \quad (3.50)$$

Also, we get:

$$\frac{\partial t_{\alpha i, \beta j}}{\partial \xi_{\gamma k}} + \frac{\partial t_{\beta j, \gamma k}}{\partial \xi_{\alpha i}} + \frac{\partial t_{\gamma k, \alpha i}}{\partial \xi_{\beta j}}. \quad (3.51)$$

$$\frac{\partial t_{\alpha i}}{\partial \xi_{\beta j}} - \frac{\partial t_{\beta j}}{\partial \xi_{\alpha i}} = 0. \quad (3.52)$$

Finally we use the rotation covariance. It is rather easy to show that one can arrange such that K is a rotation covariant function i.e.

$$K(R\xi', R\rho') = K(\xi', \rho'). \quad (3.53)$$

by a convenient redefinition of the functions $t_{\dots}$. With this choice the (new) functions $t_{\dots}$ must also be rotation covariant.

Let us collect the above results in an explicit expression of the 2-form σ . It useful to define the following function:

$$L(\xi, \rho) \equiv \frac{1}{2} M \xi_N^2 + K(\xi', \rho') + \sigma_{Ni, \beta j}(\xi') \rho_{Ni} \rho_{\beta j}. \quad (3.54)$$

Then it follows that σ has the following generic form:

$$\sigma = \sigma_L + t_{\alpha i} \delta \xi_{\alpha i} \wedge dt + \frac{1}{2} t_{\alpha i, \beta j} (\rho_{\beta j} \delta \xi_{\alpha i} \wedge dt + \delta \xi_{\alpha i} \wedge \delta \xi_{\beta j}). \quad (3.55)$$

The only restrictions that we have are (3.39), (3.44), (3.51), (3.52) and the condition of rotation covariance on the functions $\sigma_{Ni, \beta j}$, $t_{\alpha i, \beta j}$, $t_{\alpha i}$ and K .

3.4 We analyse here the possible obstructions to the existence of a globally defined Lagrangian. From (3.55) it is clear that this obstruction is contained in the last two terms.

First let us define the following forms in the variable ξ' :

$$\tau_1 \equiv t_{\alpha i} d\xi_{\alpha i}. \quad (3.56)$$

$$\tau_2 \equiv t_{\alpha i, \beta j} d\xi_{\alpha i} \wedge d\xi_{\beta j}. \quad (3.57)$$

and we remark that the content of (3.44) (3.51) and (3.52) is:

$$d\tau_i = 0. \quad (3.58)$$

for $i = 1, 2$, i.e. these forms are closed. Also the rotation covariance of the functions $t_{\dots}$ translates into the rotation invariance of these forms. We have now two cases.

1) If $D > 2$ then the domain of the variables ξ' is simply connected. So, according to the Poincaré lemma the forms τ_i are exact: $\tau_i = d\theta_i$ ($i = 1, 2$) for a function θ_1 and a 1-form θ_2 both in the variable ξ' . Now it is rather easy to prove that θ_i can be chosen rotation invariant ($i = 1, 2$). Also σ can be written in the form σ_L with L of the form (3.54) by a redefinition of the function K .

So, for $D > 2$ the most general Lagrangian dynamics invariant with respect to the Galilei group always admits a globally defined Lagrangian which can be chosen of the form (3.54).

2) $D = 2$. In this case the domain of the variable ξ' is no longer simple connected, so in principle the forms τ_i ($i = 1, 2$) could be non-exact. We succeeded in completely analysing only the case of two particles $N = 2$. In this case $\xi' = (\xi_1)$. We have to study the functions $\sigma_{2i,1j}$, t_{1i} and $t_{1i,1j}$.

a) from the requirement of rotation covariance we have the following generic form for $\sigma_{2i,1j}$:

$$\sigma_{2i,1j}(\xi) = \delta_{ij}A(\xi^2) + \xi_i\xi_jE(\xi^2) + \varepsilon_{ij}C(\xi^2)$$

for some smooth functions $A, E, C : R_+ \rightarrow R$. Using now (3.39) we get $E = 2A'$ and $C = \text{const.}$, i.e.:

$$\sigma_{2i,1j}(\xi) = \delta_{ij}A(\xi^2) + 2\xi_i\xi_jA'(\xi^2) + \varepsilon_{ij}C. \quad (3.59)$$

b) from the same requirement of rotation covariance we have:

$$t_{1i}(\xi) = \xi_i D(\xi^2). \quad (3.60)$$

But in this case (3.52) is automatically fulfilled, and moreover τ_1 is exact. So, according to the discussion from a), the second term in (3.55) gives a global contribution to L which can be absorbed by a redefinition of K .

c) finally, the rotation covariance and (3.44) gives:

$$t_{1i,1j}(\xi) = \varepsilon_{ij}B(\xi^2). \quad (3.61)$$

Again (3.51) is automatically fulfilled.

So, finally, we find out that the most general form for σ is in this case:

$$\sigma = \sigma_L + \frac{1}{2}B(\delta\xi_{11} \wedge \delta\xi_{12} + \rho_{1i}\delta\xi_{1i} \wedge dt). \quad (3.62)$$

where L is of the following form:

$$L(t, \xi_1, \xi_2, \rho_1, \rho_2) = \frac{1}{2}M\rho_2^2 + K(\xi_1, \rho_1) + \sigma_{2i,1j}(\xi_1)\rho_{2i}\rho_{1j}. \quad (3.63)$$

with $\sigma_{2i,1j}$ given by (3.59).

So, in this case the obstruction to a global Lagrangian is contained in the last term of (3.62). Indeed, if we take for instance:

$$B(\zeta) = \frac{b}{\zeta}; (b \in R^*)$$

then the second term in (3.62) can be written in the form σ_L only locally. This second term describes also a magnetic type interaction (like in Section 2).

3.5 Let us try to describe in our formalism a system of identical particles.

It is clear that on S_N given at the beginning of Section 3.4 we have a natural action of the permutation group P_N of N objects:

$$p(t, \mathbf{x}_1, \dots, \mathbf{x}_N) = (t, \mathbf{x}_{p^{-1}(1)}, \dots, \mathbf{x}_{p^{-1}(N)}). \quad (3.64)$$

For a system of identical particles it is natural to take as the "space-time" manifold:

$$S'_N \equiv S_N/P_N$$

In this case, one can show that the corresponding evolution space is:

$$E'_N = E_N/P_N$$

where the action of P_N on E_N is the natural lift of the action (3.64). Next, one sees that the action (3.22) of the Galilei group induces a natural action on S'_N .

So, we can study the same problem as before, namely try to find the most general Lagrange-Souriau form σ' verifying a condition of the type (3.27). Let us denote by $\pi : E_N \rightarrow E'_N$ the canonical projection. Then, we define:

$$\sigma \equiv \pi^* \sigma'. \quad (3.65)$$

and one can easily show that σ verifies (3.27) and also:

$$(\dot{p})^* \sigma = \sigma. \quad (3.66)$$

for any $p \in P_N$.

So, to solve this new problem, one simply takes the solution of the previous analysis (3.54)+(3.55) and imposes the additional constraint (3.66).

For instance, in the case $D = 2$, $N = 2$ studied in Section 3.4, one can show that it is possible to redefine L such that:

$$K(-\xi_1, -\rho_1) = K(\xi_1, \rho_1). \quad (3.67)$$

and:

$$\sigma_{2i,1j} = 0. \quad (3.68)$$

In this case, there exists a unique 2-form σ' on E'_N such that (3.66) is true.

5. Conclusions

We have presented here a generalization of the usual Lagrangian formalism and presented some examples of physical interest when this generalization is really needed.

It is also clear that this formalism is the most simple one for analysing Noetherian symmetries for Lagrangian systems. Also, we note that in this formalism there are no problems to apply some quantization procedure. Indeed, one can first construct the Hamiltonian formalism as in [1], by factorizing the kernel of σ , and after that one can apply for instance geometric quantization. It would be interesting to find a way of quantization directly from (E, σ) , without the help of the Hamiltonian formalism.

Finally, we intend to apply the same ideas in classical field theory.

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