CERN-TH.6146/91

Constraint Dependence of the Instanton Calculations and Exponentiation of Hard-Soft Corrections at High Energies

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Abstract

We study the dependence of the final state exponential factor for the total instanton induced cross section on the constraints used to separate the instanton zero-modes. We find the non-trivial dependence of the semiclassical exponent on the translational zero mode constraint, starting in the order $(E/E_0)^{8/3}$. We argue that, to compensate this dependence in the physical cross section, the hard-soft corrections should enter the exponent at this order. We identify the terms in the hard-soft gauge propagator which may give rise to such behaviour.

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June 1991

1 Introduction

Recently, some progress has been made in understanding the behaviour of the fermion number violating instanton-induced cross sections at high energies. It was observed [1, 2] that, because in the leading order semiclassical approximation the instanton induces point-like vertices, the corresponding amplitudes grow rapidly with energy for a large number of bosons in the final state. Such behaviour of the amplitudes leads to the exponentially growing inclusive cross section [3, 4, 5, 6] and gives rise to speculations that the fermion number violation becomes unsuppressed at sufficiently high energies (see e.g. ref.[7]).

The actual understanding of the behaviour of the total cross section in the one-instanton sector requires a full account of the corrections to the leading semiclassical result. The latter corrections can be divided into hard-hard, soft-soft and hard-soft contributions depending on whether they involve initial particles lines, final lines or both. The soft-soft corrections were shown to exponentiate, so that the total cross section has the form [5, 8, 9],

$$\sigma_{tot} \propto \exp\left\{-\frac{16\pi^2}{g^2} + \frac{1}{g^2}F(E/E_0)\right\}, \quad (1)$$

where $E_0 = \sqrt{6}\pi M_W/\alpha_W \sim E_{sph}$ is the characteristic energy scale of the order of the sphaleron [10] energy. The function $F(E/E_0)$ is given by a series in $(E/E_0)^{2/3}$ which can be evaluated order by order in perturbation theory around the instanton [5]. In the electroweak theory the first two contributions to the function $F(E/E_0)$ are [4, 5, 6, 11, 12, 13, 14]

$$F^{(ss)}(E/E_0) = \frac{9}{8} \left(\frac{E}{E_0}\right)^{4/3} - \frac{9}{16} \left(\frac{E}{E_0}\right)^2 + O\left(\frac{E}{E_0}\right)^{8/3}, \quad (2)$$

where the superscript (ss) indicates the soft-soft contribution. The appearance of the overall factor $1/g^2$ in the exponent in eq.(1) suggests that there may exist a semiclassical-type procedure for evaluating the total cross section in the one-instanton sector.

The hard-hard corrections also form an exponential-like series [15]. Being naively summed up, this series gives a contribution to the function $F(E/E_0)$ in the order $(E/E_0)^{10/3}$. However, because of the alternating signs of different terms, the actual exponentiation is questionable [16]. The behaviour of the soft-hard corrections is

totally unclear at the moment. The issue of whether these two types of corrections exponentiate is most important since, intuitively, these corrections might be expected to suppress the total cross section down to an exponentially small, unobservable magnitude.

In this paper we show that, if the total instanton induced cross section indeed has the semiclassical form (1), there *must* be contributions to the function $F(E/E_0)$ from hard-hard and/or hard-soft corrections. Our arguments are based on the observation that, with only soft-soft contributions taken into account, eq.(1) is ambiguous for the following reasons. The perturbative calculation of $F^{(ss)}(E/E_0)$ requires fixing of the instanton zero modes (collective coordinates). This is usually done by introducing the constraint into the functional integral. Although this procedure is ambiguous, the final answer should not depend on the particular choice of the constraint. However, this may not be true for the *partial* sum of the corrections, say, if only soft-soft terms are included. We shall see below (Sect.2) that the sum of the soft-soft corrections, $F^{(ss)}(E/E_0)$, does depend on the choice of the constraint starting from the order $(E/E_0)^{8/3}$. This fact indicates that in the order $(E/E_0)^{8/3}$ there must be contributions to the exponent in eq.(1) from hard-hard and/or hard-soft corrections which compensate the constraint dependence of $F^{(ss)}(E/E_0)$. Since the hard-hard corrections are expected only in the order $(E/E_0)^{10/3}$ (see above), we attribute the order $(E/E_0)^{8/3}$ to the hard-soft corrections. Thus, we give a strong (though somewhat indirect) argument in favour of the exponentiation of hard-soft corrections. Moreover, we show (Sect.3) that such energy dependence is consistent with the naive exponentiation of the single hard-soft propagator insertion and identify the relevant terms in the hard-soft propagator. Our conclusions are summarized in Sect.4.

2 The constraint dependence of $F^{(ss)}(E/E_0)$

In this section we estimate in which order in (E/E_0) the dependence of $F^{(ss)}(E/E_0)$ on the constraint appears. We perform the perturbation expansion around the instanton which is a *fixed* (except for the collective coordinates) field configuration. The function $F^{(ss)}(E/E_0)$ is then expressed through the residues of the instanton field, propagator in the instanton background, etc. [5]. The leading term includes the instanton action and the residue squared of the instanton field. It gives the contribution of the order $(E/E_0)^{4/3}$ in eq.(2). The second term contains the residues of

the propagator and two instanton fields and contributes into the order $(E/E_0)^2$ and higher. The vertex contributions start from the order $(E/E_0)^{10/3}$. We consider the constraint dependence which appears in $F^{(ss)}$ after the saddle-point integrations over the collective coordinates have been performed. So, the dependence we find cannot be washed out by these integrations. Of course, the saddle-point values of the collective coordinates may themselves depend on the constraints. However, due to the saddle-point nature of these values, their variations do not contribute to the first order, and we are left with the explicit variation of the exponent only.

The instanton action and the residue squared have no explicit constraint dependence since we are expanding near the fixed field profile¹. The dependence on the zero mode fixing first appears in the propagator term and is due to the constraint dependence of the propagator itself.² Explicitly, the propagator contribution into $F^{(ss)}(E/E_0)$ is [5]

$$-\frac{1}{8} \int \frac{d\mathbf{k}d\mathbf{q}}{(2\pi)^6 \omega_{\mathbf{k}} \omega_{\mathbf{q}}} e^{-(\omega_{\mathbf{k}} + \omega_{\mathbf{q}})\tau} \times \left\{ [R_{\mu}^a(\mathbf{k}) \epsilon_{\mu}^m(\mathbf{k})]^* \epsilon_{\nu}^m(\mathbf{k}) D_{\nu\lambda}^{ab}(\mathbf{k}, \mathbf{q}) \epsilon_{\lambda}^n(\mathbf{q}) [\epsilon_{\sigma}^n(\mathbf{q}) R_{\sigma}^b(\mathbf{q})]^* + \text{h.c.} \right\}, \quad (3)$$

where $D_{\nu\lambda}^{bc}(\mathbf{k}, \mathbf{q})$ is the residue of the propagator in the instanton background and

$$R_{\mu}^a(\mathbf{k}) = -4\pi^2 i \rho^2 \bar{\eta}_{\mu\nu}^a k_{\nu}, \quad k_{\nu} = (i\omega_{\mathbf{k}}, \mathbf{k}), \quad (4)$$

is the residue of the instanton field on mass shell. The Euclidean polarization vectors $\epsilon_{\mu}^m(\mathbf{k})$ are

$$\epsilon_{\mu}^{1,2}(\mathbf{k}) = (0, \mathbf{e}^{1,2}(\mathbf{k})), \quad \epsilon_{\mu}^3(\mathbf{k}) = \frac{1}{M_W} (i|\mathbf{k}|, \omega_{\mathbf{k}} \frac{\mathbf{k}}{|\mathbf{k}|}).$$

The instanton zero mode parameters (collective coordinates) ρ and τ entering eq.(3) are determined by the leading order saddle point values,

$$\rho \propto \frac{1}{M_W} \left(\frac{E}{E_0} \right)^{2/3}, \quad \tau \propto \frac{1}{M_W} \left(\frac{E}{E_0} \right)^{1/3}. \quad (5)$$

In what follows we restrict ourselves to analyzing the propagator contribution (3) as it is the lowest order constraint-dependent part of $F^{(ss)}(E/E_0)$. Our strategy will be

¹We are grateful to L.Yaffe for a discussion on this point.

²There are also some contributions to $F^{(ss)}$ related to the fact that, in the gauge-Higgs theory, the instanton is not an exact solution to the field equations. These contributions start in the order $(E/E_0)^{8/3}$, but can be shown not to produce the constraint dependence in that order for the constraint variations considered below.

to find the dependence of the residue of the propagator on the constraint and then estimate the change of the contribution (3) under the small change of the constraint.

The zero mode fixing is usually done by inserting the following identity into the functional integral,

$$1 = J_w \int d\xi \delta \left\{ \int d^4x w(x) \psi_{\mu A}^a(x) (A_\mu^a(x) - A_{\mu,inst}^a(x)) \right\}, \quad (6)$$

where $w(x)$ is an arbitrary weight function and $\psi_{\mu A}^a(x)$ are the zero modes labelled by index A . The presence of the δ -function in the functional integral modifies the equation for the propagator. At given $w(x)$ the latter equation reads

$$\Delta_{\mu\nu}^{ab}(x) D_{\nu\lambda}^{bc}(x, y) = \delta^{ac} \delta_{\mu\lambda} \delta(x - y) - w(x) \Omega_{AB}^{-1} \psi_{\mu A}^a(x) \psi_{\lambda B}^c(y).$$

Here Ω_{AB} is the zero mode normalization matrix,

$$\Omega_{AB} = \int d^4x w(x) \psi_{\mu A}^a(x) \psi_{\mu B}^a(x).$$

The propagator also obeys the orthogonality conditions

$$\int dx w(x) \psi_{\nu A}^b(x) D_{\nu\lambda}^{bc}(x, y) = 0.$$

Consider the variation of the propagator when changing the weight function $w(x)$. The change $\delta w(x)$ induces the change of the propagator, $\delta D(x, y)$. The equation for $\delta D(x, y)$ is

$$\begin{aligned} \Delta_{\mu\nu}^{ab}(x) \delta D_{\nu\lambda}^{bc}(x, y) &= -\delta w(x) \Omega_{AB}^{-1} \psi_{\mu A}^a(x) \psi_{\lambda B}^c(y) \\ &+ \left[\Omega_{AC}^{-1} \int dz \delta w(z) \psi_{\nu C}^a(z) \psi_{\nu D}^a(z) \Omega_{DB}^{-1} \right] w(x) \psi_{\mu A}^b(x) \psi_{\lambda B}^c(y), \end{aligned} \quad (7)$$

while the orthogonality conditions give

$$\int dx \delta w(x) \psi_{\nu A}^b(x) D_{\nu\lambda}^{bc}(x, y) + \int dx w(x) \psi_{\nu A}^b(x) \delta D_{\nu\lambda}^{bc}(x, y) = 0. \quad (8)$$

The solution to eqs.(7) and (8) reads

$$\delta D_{\nu\lambda}^{bc}(x, y) = -\Omega_{AB}^{-1} \theta_{\nu A}^b(x) \psi_{\lambda B}^c(y) - \Omega_{AB}^{-1} \psi_{\nu A}^b(x) \theta_{\lambda B}^c(y), \quad (9)$$

where

$$\theta_{\nu A}^b(x) = \int dz \delta w(z) D_{\nu\mu}^{ba}(x, z) \psi_{\mu A}^a(z), \quad (10)$$

and $D_{\mu\nu}^{ab}(x, y)$ is supposed to be symmetric.

It may be seen that the leading contribution to the variation of eq.(3) is due to the translational zero modes, at least for our choice of δw below. So, in what follows we consider only translational zero modes and use Greek letters for $A, B, \dots = 0, \dots, 3$. $D_{\mu\nu}^{ab}$ in eq.(10) means the Levine-Yaffe propagator [17]. The Levine-Yaffe constraints can be cast into the form eq.(6) by a suitable choice of the weight function w . In fact, using the constraint functions [17] f_A instead of $w\psi_A$ in the above formulae, one achieves a generalization of the formalism, which allows one to vary only some of the constraints (f 's) while keeping others unchanged. This would allow us to concentrate on any particular zero mode from the very beginning.

Now we calculate the on-shell residue of δD . The latter residue is the product of the residues of the zero mode and $\theta_{\nu\rho}^b$. The translational zero modes are proportional to the field strengths [18, 17], $\psi_{\lambda\sigma}^c \propto F_{\lambda\sigma}^c$, and the (normalized) zero mode residue is

$$k^2 \int d^4y e^{iky} \psi_{\lambda\sigma}^c(y) = \text{const. } \rho^2 (\bar{\eta}_{\lambda\mu}^c k_\mu k_\sigma - \bar{\eta}_{\sigma\mu}^c k_\mu k_\lambda) , \quad (11)$$

where, as before, $k_\mu = (i\omega_{\mathbf{k}}, \mathbf{k})$. Hereinafter *const.* always denotes a real *numerical* (that is not containing g and ρ) constant.

The residue of $\theta_{\nu\rho}^b$ depends on the choice of the $\delta w(z)$. A priori, the function $\delta w(z)$ is arbitrary except that it should not contain dimensional parameters other than the instanton size ρ . This is necessary to preserve the suppression of the corrections by powers of (E/E_0) , which is related to the fact that the residue of the propagator contains the single dimensional parameter ρ [5]. Clearly, for our purpose, we should choose a function obeying this restriction, which would give the lowest possible power of (E/E_0) for the variation of eq.(3). In fact, the arguments can be given [12] that the order $(E/E_0)^2$ is independent of the constraint. Thus, the lowest order where the constraint dependence can appear is $(E/E_0)^{8/3}$. As we will see below, this dependence is actually achieved with the function $\delta w(z)$ of the form

$$\delta w(z) = \text{const. } \rho^4 \delta^4(z_\mu - c\rho n_\mu) , \quad (12)$$

where n_μ is a unit vector, and c is a dimensionless constant. If $n_0 \neq 0$, the problem of analytical continuation of δD to the Minkowskian signature, needed for calculating the residues, becomes complicated. So, in what follows, we set $n_0 = 0$. With this choice of $\delta w(z)$ eq.(10) becomes

$$\theta_{\nu\rho}^b(x) = \text{const. } \rho^4 D_{\nu\mu}^{ba}(x, c\rho n) \psi_{\mu\rho}^a(c\rho n) . \quad (13)$$

The residue of $\theta_{\nu\rho}^b(x)$ is determined by its asymptotics at large x . Hence, we need the propagator D in the regime $x \gg c\rho, \rho$. For the Levine-Yaffe propagator this asymptotics is [17]

$$D_{\mu\nu}^{ab}(x, c\rho n) = \frac{\delta^{ab}\delta_{\mu\nu}}{4\pi^2 x^2(1 + 1/c^2)} + \frac{\delta_{\mu\nu}\delta^{ab} + \epsilon^{abc}\bar{\eta}_{\alpha\beta}^c(\delta_{\alpha\mu} - 2n_\alpha n_\mu)(\delta_{\beta\nu} - 2n_\beta n_\nu)}{4\pi^2 x^2 c^2(1 + 1/c^2)^2} + O(1/x^3). \quad (14)$$

Further, it is our freedom to take c large, provided that still $c\rho \ll x$. Then, eq.(14) reduces to the free propagator, up to the terms of order $1/c^2$. Making use of eq.(13), eq.(14) and the explicit form of the zero modes, we find³

$$k^2 \int d^4x e^{ikx} \theta_{\nu\rho}^b(x) \Big|_{k_0=i\omega} = \text{const.} \cdot \rho^2 (-\bar{\eta}_{\nu\rho}^b + 2\bar{\eta}_{\nu\lambda}^b n_\lambda n_\rho - 2\bar{\eta}_{\rho\lambda}^b n_\lambda n_\nu) + O(1/c^2). \quad (15)$$

Thus, the change of the residue of the propagator under variation of the constraint is

$$\delta D_{\nu\lambda}^{bc}(\mathbf{k}, \mathbf{q}) = \text{const.} \cdot \rho^4 \Omega_{\rho\sigma}^{-1} \left\{ \bar{\eta}_{\lambda\alpha}^c q_\alpha q_\sigma (-\bar{\eta}_{\nu\rho}^b + 2\bar{\eta}_{\nu\beta}^b n_\beta n_\rho - 2\bar{\eta}_{\rho\beta}^b n_\beta n_\nu) + \bar{\eta}_{\nu\alpha}^b k_\alpha k_\rho (-\bar{\eta}_{\lambda\sigma}^c + 2\bar{\eta}_{\lambda\beta}^c n_\beta n_\sigma - 2\bar{\eta}_{\sigma\beta}^c n_\beta n_\lambda) \right\}. \quad (16)$$

It is now straightforward to substitute eq.(16) into eq.(3). At $\Omega_{\rho\sigma} = \delta_{\rho\sigma}$, the summation over the polarizations and contraction with the instanton residues, eq.(4), gives for the quantity in braces in eq.(3)

$$\text{const.} \cdot \rho^8 \omega_{\mathbf{k}} \omega_{\mathbf{q}} (3\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + 3\omega_{\mathbf{q}}^2 + \mathbf{q}^2)(-3 + 2n^2),$$

where we have used $n_0 = 0$. After that, the integration over $\mathbf{k}, \mathbf{q}$ in eq.(3) gives

$$\text{const.} \cdot \left(\frac{\rho}{\tau} \right)^8.$$

Making use of eq.(5) we finally obtain

$$\delta F^{(ss)} = \text{const.} \cdot \left(\frac{E}{E_0} \right)^{8/3}. \quad (17)$$

This is the first of the claimed results.

³In fact, the choice $c \lesssim 1$ leads to the same expression for $\theta_{\nu\rho}^b$, except for the overall constant.

3 The hard-soft corrections

The appearance of the constraint dependence for the final state exponent in the order $(E/E_0)^{8/3}$ implies, as discussed in the Introduction, that the soft-hard corrections exponentiate and depend on the constraint in the same order. In fact, our previous formulae can be used to establish this fact explicitly.

On general grounds, the on-shell residue of the gauge propagator $D_{\mu\nu}^{ab}(\mathbf{k}, \mathbf{p})$ ($\mathbf{k}$ and $\mathbf{p}$ now being the soft and hard momenta, respectively) has the form

$$D_{\mu\nu}^{ab}(\mathbf{k}, \mathbf{p}) = f_{1,\mu\nu,\alpha\beta}^{ab}(kp) p_\alpha p_\beta + f_{2,\mu\nu,\alpha\beta}^{ab}(kp) p_\alpha k_\beta + \dots, \quad (18)$$

where (kp) is the four-dimensional scalar product, and dots stand for the terms without powers of p_α . The coefficients f_1 and f_2 can be estimated by the following argument, largely due to Mueller [19]. (This argument also rules out the higher powers of p_α in eq.(18).) Consider the residue eq.(18) for the Lorentz transformed values of the four-momenta, so that the new values k', p' are both large, but $(k'p') = (kp)$. Then $p'_\alpha p'_\beta \sim p'_\alpha k'_\beta$, and one may restrict the various contributions to the propagator, using the known asymptotics of the propagator in this regime [15]. It follows that f_2 is of order $\rho^6(pk)$, while f_1 is suppressed, compared to f_2 , by at least one power of the large parameter $\rho^2(pk)$, $f_1 = O(\rho^4)$. Of course, this reasoning does not prove that the function f_1 is non-zero. Note, however, that, remarkably, our variation of the residue eq.(16) contributes to the function f_1 , precisely in order ρ^4 . Thus, eq.(16) shows that the function f_1 is non-zero and, moreover, it is constraint-dependent.

With $f_1 \sim \rho^4$, it is the first term in eq.(18) that dominates the propagator in the hard-soft regime, $p \sim E$, $k \sim E/n$, n being the total multiplicity, $n \sim (1/g^2)(E/E_0)^{4/3}$ [4, 5, 6]. The total cross section corrected by a single soft-hard propagator insertion is then estimated as

$$\sigma \sim \sigma_0 \left(1 + ng^2 \frac{D(\mathbf{p}, \mathbf{k})}{R(\mathbf{p})R(\mathbf{k})} \right) \sim \sigma_0 \left(1 + \text{const.} \frac{1}{g^2} \left(\frac{E}{E_0} \right)^{8/3} \right). \quad (19)$$

The naive exponentiation of the correction factor eq.(19) gives the claimed $(E/E_0)^{8/3}$ exponential dependence for hard-soft processes. Let us stress that we do not rely on this naive exponentiation when we argue that the soft-hard corrections come to the exponent. Rather, our conclusion rests on the established constraint dependence of the soft-soft factor. The considerations of the hard-soft propagator are only to show that the term in the propagator which might produce the exponent $(E/E_0)^{8/3}$ is indeed there.

4 Conclusion

We have studied the dependence of the final state exponential factor for the total instanton induced cross section, on the constraints used to separate the instanton zero-modes. We found the non-trivial dependence of the exponent on the translational zero mode constraint, starting in the order $(E/E_0)^{8/3}$. We argue that, to compensate this dependence in the physical cross section, the hard-soft corrections should enter the exponent at this order. We identify the terms in the hard-soft gauge propagator which may give rise to such behaviour.

One remark in conclusion. There is a simple physical argument why the constraint dependence for the soft-soft factor does, in fact, appear. The expected independence of the *full* result on the constraint procedure is understood, physically, as a consequence of the conservation laws (energy-momentum conservation, etc.). For the soft-soft factor alone, some conservation laws are obscure since, in the purely semiclassical approach, the final state contains a large classical field with non-zero energy while the two-particle initial state does not. So, we think that it is the semiclassical energy non-conservation which shows up as the constraint dependence in the order $(E/E_0)^{8/3}$.

The authors are grateful to the organizers and the participants of the CERN workshop on non-perturbative electroweak physics, for a possibility of exchanging ideas on the subject. Our special thanks are to Al Mueller and Valery Rubakov who made a number of important points more clear to us. It is a pleasure to thank the CERN Theory Division where this work was done, for hospitality and support. S.K. was supported in part by the INFN Eloisatron Project.

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