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CRITICAL PROPERTIES OF THE DYNAMICAL RANDOM SURFACE WITH EXTRINSIC CURVATURE

J. Ambjørn, *J. Jurkiewicz*¹ and *S. Varsted*

The Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

A. Irbäck

CERN - CH 1211 Geneva 23, Switzerland

B. Petersson

Fakultät für Physik, Universität Bielefeld
D-4800 Bielefeld 1, Germany

Abstract

We analyze numerically the critical properties of a two-dimensional discretized random surface with extrinsic curvature embedded in a three-dimensional space. The use of the toroidal topology enables us to enforce the non-zero external extension without the necessity of defining a boundary and allows us to measure directly the string tension. We show that a most probably second-order phase transition from the crumpled phase to the smooth phase observed earlier for a spherical topology appears also for a toroidal surface for the same finite value of the coupling constant of the extrinsic curvature term. The phase transition is characterized by the vanishing of the string tension. We discuss the possible non-trivial continuum limit of the theory, when approaching the critical point.

¹Permanent address : Inst. of Phys., Jagellonian Univ., ul. Reymonta 4, PL-30 050 Kraków 16, Poland

1 Introduction

In the last year discretized random surface models [1, 2, 3] received a lot of attention. Viewed as a regularization of the two-dimensional quantum gravity these models permit to formulate and understand some important non-perturbative questions, such as summation over the genera of the surface in the so-called double-scaling limit. From the point of view of the bosonic string theory it turned out, however, to be very difficult to formulate a model, which would give a non-trivial physics for the dimension D of the target space larger than one. The problem is a possible phase transition to a phase of crumpled surfaces with infinitely many spikes, which seems to take place at $D = 1$ and which appears to be a consequence of a non-vanishing string tension in the scaling limit [4].

A possible cure for this problem is a generalization of the model in which an additional term, restricting the outgrowth of spikes, is added to the action [5]. This term depends on the extrinsic curvature of the surface. In the continuum its addition was suggested by Polyakov [6] on the grounds of the general symmetry properties of the model. A term of this kind was also found in the effective bosonic theory obtained from the supersymmetric string theory after integrating out the fermionic part of the action [7]. The thermodynamic properties of such systems are also interesting in other fields of physics, where they can be used to describe the statistical behaviour of fluctuating, flexible membranes [8].

The discretized model with extrinsic curvature and dynamical geometry suggested in [5] was studied numerically in [9, 10]. A second order phase transition was observed for a finite value of the extrinsic curvature coupling. The transition seemed identical to the so-called crumpling transition [11, 12, 13], valid for a fixed, regular triangulation, i.e. fixed connectivity of the random surface². The existence of this phase transition was a surprise, since the original interest for the crumpling transition was generated precisely because any transition of this kind was believed to be absent for membranes and random surfaces with dynamical connectivity [15, 6]. The observed second order transition opens the interesting possibility of defining a new, non-trivial scaling limit at this critical point. To make the scaling limit inter-

²In the beginning there was some discussion on the order of the crumpling transition [14, 16]. In addition to the second order transition reported in [12, 13] also transitions of first and third order were observed, using discretized extrinsic curvature terms which differed somewhat, but had the same continuum limit for smooth surfaces. It is now understood that these alternative versions of extrinsic curvature terms cannot be used, since for generic random surfaces, which are quite far from smooth continuum surfaces, they will create lattice artifacts [18]. This leaves us with the model suggested in [5] and the observed second order transition.

esting from the point of view of string theory one would require that the bare string tension vanishes and that other physical observables scale in a consistent way. An attempt to perform a numerical study of these problems was made in [19], where it was claimed that the mass gap and string tension scale properly. The string tension measurement used a system with the topology of a disc, with a fixed number of points on a boundary and in the interior. Unfortunately the investigation could be criticized for the following points: The definition used for both the mass gap and the string tension assumed a grand canonical ensemble, but a canonical ensemble was used. Further, the string tension was then measured relative to its value at the critical point (assumed to be zero). Therefore the assumption of the vanishing of the string tension was not actually tested. Finally the number of points, which for a finite lattice form a boundary constitutes a big fraction of all the points (a square with 100 points would have 40 points on a boundary). One may therefore suspect large finite size effects and in fact a strong dependence of results on details of assignment of points to a boundary.

In this paper we perform a detailed study of the critical properties of a system with extrinsic curvature for which the non-zero external dimension is introduced without the need of defining a boundary. This can be realized with the help of the non-trivial boundary conditions defined for a closed system with a topology of a torus. In effect we expect to have a method which minimizes both the finite size effects and the problems connected with the definition of the boundary points. Details of the method will be explained in the next section. The method permits a direct measurement of the string tension in a canonical simulation.

We show that the system with a topology of a torus undergoes the same phase transition, as the spherical one. Using the Ferrenberg and Swendsen method [21], we are able to find the critical coupling with much higher accuracy than that reported in the literature. We show that at the phase transition the string tension vanishes. We explain implications of the canonical results on the grand canonical continuum limit of the theory and discuss some critical exponents connected with this limit.

2 The twisted boundary conditions

The model used in our computation is a generalization of the discretized Polyakov surface, used in earlier works. We consider closed surfaces built from triangles with a finite (and fixed in the numerical simulation) number of vertices. As usual we consider triangulations $T \in \mathcal{T}$, restricted to the class, where every vertex of the

lattice has at least three neighbours, no two vertices are connected with more than one link and every link connects two different vertices. The parameter space is assumed to be a metric space and each link has unit length in this space. The possible triangulations of a surface with N vertices and a fixed number of handles correspond to different (discretized) intrinsic geometries. In the following we shall concentrate on surfaces with a topology of a torus. In this case the parameter space can be visualized as a plane periodic in two non-parallel directions. In each period we have N vertices, each vertex having infinitely many copies in other periods. These periods can be numbered by two integers k_1, k_2 . Links can connect points both inside a single period and in the neighbouring periods.

We consider closed two-dimensional surfaces embedded in a D -dimensional Euclidean target space. The embedding is a map, assigning to each link $\langle ij \rangle$ of a lattice in the intrinsic (parameter) space a vector $X_{ij}^\mu, \mu = 1 \dots D$ in the target space. The numerical results presented in this paper are related solely to the case $D = 3$. Three vectors forming a triangle i, j, k should satisfy

$$X_{ij}^\mu + X_{jk}^\mu + X_{ki}^\mu = 0. \quad (1)$$

We assume the map to be periodic. If the surface would be a sphere, (1) would mean that

$$X_{ij}^\mu = X_{ij}^\mu - X_{ij}^\mu, \quad (2)$$

where X_{ij}^μ denotes coordinates of a vertex i in the target space and a sum of vectors X_{ij}^μ along any closed path would be zero. For surfaces with handles the embedding may be non-trivial. On a torus a sum of vectors along a closed path can take values

$$E^\mu(n_1, n_2) = n_1 E_1^\mu + n_2 E_2^\mu \quad (3)$$

with two constant vectors E_1^μ and E_2^μ and integers n_1 and n_2 denoting the number of times the path winds around two axes of the torus. Such a choice of boundary conditions on a torus is the analogue of the twisted boundary conditions used in the case of the gauge theories, where the gauge fields are periodic up to non-trivial gauge transformation. Notice that the vector E^μ is a topological invariant, it remains the same if one or more points along the path are shifted in the target space or if the internal geometry of the surface is changed by a flip of some of its links. Vector E^μ does not depend on the point index i , but rather on the integer distance between the initial and final periods of the path. For nonzero vectors E_1^μ and E_2^μ , (3) means that coordinates of the vertex i are not strictly periodic but depend also on the integer

coordinates k_1, k_2 numbering the particular copy and are given by

$$X_i^\mu(k_1, k_2) = X_i^\mu + k_1 E_1^\mu + k_2 E_2^\mu. \quad (4)$$

A configuration of the surface is defined by specifying a triangulation (links joining the vertices) and by assigning values to vectors X_{ij}^μ satisfying (1) and (3). For each configuration we define the action $S = S_G + \lambda S_C$. S_G is the gaussian part of the action

$$S_G = \frac{1}{2} \sum_{\langle ij \rangle} (X_{ij}^\mu)^2 \quad (5)$$

and S_C is the extrinsic curvature part

$$S_C = \sum_{\langle ij \rangle} (1 - \cos \theta_{ij}), \quad (6)$$

θ_{ij} denoting the angle between the normals to two oriented triangles $\langle ij \rangle$ and $\langle jil \rangle$ with a common link $\langle ij \rangle$.

The partition function is

$$Z_N(\lambda, E_1^\mu, E_2^\mu) = \sum_{T \in \mathcal{T}} \int \prod_{i=1}^N dX_i^\mu \delta(\sum X_i^\mu) \exp(-S), \quad (7)$$

where the first sum is over all possible triangulations of the surface with N vertices and the delta function is necessary because of the translational invariance of the action. The partition function defines the free energy $\mathcal{F}(N, \lambda; E_1^\mu, E_2^\mu) = -\log Z_N(\lambda, E_1^\mu, E_2^\mu)$. In (7) the integral is over the positions X_i^μ with vectors E_1^μ and E_2^μ fixed. It is easy to realize that non-zero values of this vectors mean that the surface is spanned in the target space on a frame $E_1 \times E_2$. This frame plays a role of the spanning loop, but the boundary of this loop is translationally invariant. In effect no points have to be defined as boundary points, leaving the boundary as a dynamical object.

In the string tension calculations we use two perpendicular vectors $E_1 = (y, 0, \dots)$ and $E_2 = (0, y, 0, \dots)$ so the loop has a projected area $A_p = y^2$. A canonical string tension $\sigma(\lambda; N, A_p)$ is defined as (cf. [8])

$$\sigma(\lambda; N, A_p) = \frac{\partial \mathcal{F}(N, \lambda, A_p)}{\partial A_p}, \quad (8)$$

where we assume that $\mathcal{F}(N, \lambda; E_1^\mu, E_2^\mu)$ depends on E_1 and E_2 only through the projected area A_p . String tension (8) can be directly measured in the numerical

experiment. To see this observe that if we rescale coordinates $X_i^\mu \rightarrow y X_i^\mu$, due to (5) and (6) the action S is changed as

$$S(X_i^\mu, y) = y^2 S_G(X_i^\mu, y = 1) + \lambda S_C(X_i^\mu, y = 1). \quad (9)$$

Here we used the fact that S_C is scale invariant and S_G is quadratic in X_i^μ . In this formulation $A_p = y^2$ plays a role of the coupling constant. Using (7) we get

$$\sigma(\lambda; N, A_p) = \frac{\langle S_G \rangle}{A_p} > -\frac{1}{2} \frac{d(N-1)}{dy}, \quad (10)$$

where $\langle S_G \rangle$ is the expectation value of the gaussian part of the action with the partition function (7). We expect $\sigma(\lambda; N, A_p)$ to be a function of $r = A_p/N$, N playing a role of the area of the surface [8]. We shall verify this statement numerically.

In deriving (10) we used only the scaling arguments, so the value of the string tension should not depend on the shape of the loop. This we expect to be true only approximately. For elongated loops the behaviour will eventually change since it will be difficult to distinguish the physics associated with the area from the (different) physics associated with the perimeter. This is the reason why in our string tension computations we used square loops.

We performed also the numerical simulations using the boundary conditions with only one non-zero vector E_i , say $E_1 = (y, 0, \dots)$, $E_2 = 0$. These boundary conditions can be viewed as an alternative way of measuring the mass gap, as compared to the standard method, where two points of the spherical surface have fixed relative positions. Using exactly the same arguments as before we can measure a quantity related to the mass gap

$$\frac{\partial \mathcal{F}(N, \lambda, y)}{\partial y} = \frac{2 \langle S_G \rangle}{y} > -\frac{d(N-1)}{dy}. \quad (11)$$

3 Numerical results

The numerical simulations of random triangulations are conventionally hampered by difficulties of implementing an efficient vectorization. Following the technique introduced in [22] we have, however, developed a fully vectorizable program [23] in which the vectorization of the code is obtained by simulating in parallel a number of systems (typically 64 for the CRAY). In accordance with the discussion in the preceding section one is tempted to store the lattice configurations, using the link

vectors X_{ij}^μ rather than the vertex coordinates X_i^μ . Such a parametrization is however dangerous since it allows rounding errors to accumulate. In particular, this can lead to a violation of the boundary conditions initially imposed. To avoid this problem we decompose, following (4), the vector X_{ij}^μ as

$$X_{ij}^\mu = X_j^\mu - X_i^\mu + E_{ij}^\mu, \quad (12)$$

where $E_{ij}^\mu = n_{ij}^1 E_1^\mu + n_{ij}^2 E_2^\mu$ and the integers n_{ij}^μ are non-zero when the link $\langle ij \rangle$ connects points in different elementary cells. Vectors E_{ij} are additive, i.e. $E_{ij} + E_{jk} + E_{ki} = 0$ for every triangle $\langle ijk \rangle$. The decomposition (12) is however not unique. One can for arbitrary integers l_i^1 and l_i^2 perform a change

$$\begin{aligned} X_i^\mu &\rightarrow X_i^\mu + l_i^1 E_1^\mu + l_i^2 E_2^\mu, \\ E_{ij}^\mu &\rightarrow E_{ij}^\mu + l_i^1 E_1^\mu + l_i^2 E_2^\mu - l_j^1 E_1^\mu - l_j^2 E_2^\mu. \end{aligned} \quad (13)$$

This transformation can be used to keep the E_{ij} vectors bounded. The updating procedure for the triangulations makes use of the standard concept of the flip [3]. This procedure is known to be ergodic in the space of all triangulations, and it preserves both the topology and the boundary conditions described above. Care has only to be taken to satisfy $T \in \mathcal{T}$, rejecting unwanted triangulations. The updating procedure for the positions is also the standard one: with the coding of the lattice used in the program the vector X_i^μ at a point i is incremented by the random vector. As discussed above this procedure also does not change the topology or the boundary conditions. The only place, where the numerical simulation is sensitive either to the topology or to the particular values of the vectors E_1^μ and E_2^μ is the initial configuration.

We check for critical behaviour by investigating the specific heat

$$C(\lambda) = \frac{\lambda^2}{N} \langle S_C^2 \rangle - \langle S_C \rangle^2, \quad (14)$$

where N is the number of vertices. To accurately determine the location and height of the maximum we continue our results in λ using the multihistogram method by Ferrenberg and Swendsen [21]. This method has over the past few years been widely used in studies of spin systems and gauge theories, and, as we will see below, it can be very useful also in random surface calculations. As input we use data from high-statistics simulations at two different couplings, $\lambda = 1.25$ and 1.5 . Following Ref. [21], these data are used to estimate the density of states $\rho(S_C)$, from which, in principle, $C(\lambda)$ can be obtained at an arbitrary λ .

As a test of our procedure we first consider a spherical surface with $N = 36$ vertices. The result for the specific heat is shown on Fig. 1, where the band indicates the size of statistical errors. Errors quoted here and later were obtained by using the Jackknife method [24], taking the results from the 64 different systems as independent measurements. Our result at $\lambda = 1.5$, $C = 3.24(3)$, can be compared with that of Ref. [20]. Using the same number of sweeps as we have for each of the 64 systems, they get the value $C(1.5) = 3.00(17)$. A closer look at our data suggests that this result is not inconsistent with ours. For 12 of the systems, we actually get a value smaller than 3.00. We find that the maximum in the specific heat is at $\lambda_c^N = 1.369(2)$, which we take as a pseudocritical coupling, with a value $C_{max} = 4.06(2)$.

The same computation for surfaces with a topology of a torus is presented in Fig. 2 for $A_p = 0$ and in Fig. 3 for $A_p/N = 1/16$. The plots show a characteristic peak, with a position weakly depending on A_p . In the thermodynamic limit we expect the same value of the critical coupling λ_c , independent of the topology of the surface. In Fig. 4 we show the position of the maximum of the specific heat as a function of $1/N$ for $r = 0$ and $r = 1/16$. We observe a much weaker finite size dependence than for spherical surfaces. At present our statistics is still not sufficient to present a convincing finite-size scaling analysis of these results, but even with the numbers we have we get an estimate $1.44 < \lambda_c < 1.46$, much more accurate than the known numerical values. In Fig. 5 we plot the maximum value of the specific heat as a function of N for the same values of r . Again we postpone the precise determination of the critical exponent to the future publication [25], noticing the qualitative behaviour, suggesting the second-order phase transition.

The measurements of the canonical string tension $\sigma(\lambda, r)$ are given on Fig. 6 for $N = 64, 144, 256$. For fixed values of λ we expect the points for various N to fall on the same, universal curve as a function of $r = A_p/N$. Deviations from this universal behaviour can be interpreted as finite size effects. Fig. 6 shows that finite size effects become important for $r \rightarrow 0$ and $\lambda \rightarrow \lambda_c$. For large enough value of the ratio r we expect the distribution to be dominated by the gaussian part of the action. In this limit $\sigma(\lambda, r)$ should approach a limit independent of λ . This is exactly what we observe. This limit has however no relation to the physical string tension and the continuum limit of the theory, as will be explained in the next section. For smaller values of r the canonical string tension has a strong λ dependence, with a value for $r \rightarrow 0$ seemingly approaching zero when $\lambda \rightarrow \lambda_c$ (and $N \rightarrow \infty$). In practice we are not able to measure the string tension for very small r , where from (10) we predict

the numerical error to grow as $1/r$. The small r region has a direct relation to the physical properties of the continuum limit of the theory. To obtain the critical exponents of the model it is important to control the finite size effects, taking first the limit of N large. We shall discuss implications of the observed behaviour in terms of the physical (scaling) string tension in the next section. The numerical results presented above are for couplings in the crumpled phase or close to the transition. We did not include results for larger values of λ . In this domain, we expect that the measured function $\sigma(\lambda, r)$ behaves in a similar way, with the difference that it should become negative at small r . This can be attributed to the repulsion of the vertices, due to the fact that their number N is fixed. For large r , $\sigma(\lambda, r)$ should approach again the gaussian limit, but for some value of r it should be zero. This value of r corresponds to the equilibrium configuration. Such a behaviour is typical for a smooth phase with a vanishing physical string tension [8]. We have obtained preliminary results which seem to confirm this picture, but at the same time we observe large finite size effects and long thermalization times. We shall present results concerning this domain in the next publication.

4 Discussion

The critical behaviour presented above can be used to understand the properties of the system in the continuum limit. In this limit it is convenient to consider a grand canonical rather than the canonical ensemble, and to go to the open, framed system, using the terminology of [8]. Let us introduce the Legendre transform of $\mathcal{F}(N, \lambda, A_p)$

$$\mathcal{G}(\mu, \lambda, A_p) = N\mu + \mathcal{F}(N, \lambda, A_p) \quad (15)$$

with the chemical potential (cosmological constant) μ satisfying

$$N = \frac{\partial \mathcal{G}}{\partial \mu}. \quad (16)$$

For large A_p we expect

$$\mathcal{G}(\mu, \lambda, A_p) = \bar{\sigma}(\lambda, \mu) A_p, \quad (17)$$

where

$$\bar{\sigma}(\lambda, \mu) = \frac{\partial \mathcal{F}}{\partial A_p} = \sigma(\lambda, r) \quad (18)$$

with N in $r = A_p/N$ expressed in terms of μ , using (16). Function $\mathcal{G}$ is defined for $\mu > \mu_c(\lambda)$ and the continuum limit corresponds to $\mu_R = \mu - \mu_c(\lambda) \rightarrow 0$. In this

limit we expect the general behaviour

$$\bar{\sigma}(\lambda, \mu) = \sigma_0(\lambda) + d(\lambda) \mu_R^{2\nu} \quad (19)$$

with some critical exponent ν . From (15) and (16) we deduce

$$N \sim \mu_R^{2\nu-1} A_p, \quad (20)$$

or

$$\sigma(\lambda, r) = \sigma_0(\lambda) + \sigma_1(\lambda) r^{2\nu/(1-2\nu)}. \quad (21)$$

In the derivation of (21) we assumed that the finite size effects can be neglected. We can get a numerical estimate of the small r behaviour of $\sigma(\lambda, r)$ extrapolating from the larger r values, where our results are more reliable and where the finite size effects become smaller. Results seem to indicate that $\sigma(\lambda, r)$ behaves linearly in r , but we can not exclude higher power of r , probably even r^2 . To determine the exponent more accurately one would need the measurement of the derivative

$$r^2 \frac{d\sigma(\lambda, r)}{dr} = \left(\frac{D(N-1)}{2N} - \frac{<S_G^2> - <S_G>^2}{N} \right), \quad (22)$$

which is not sensitive to the value of $\sigma_0(\lambda)$. Results from a large statistics simulation will be presented in the future publication [25]. The numerically observed behaviour of $\sigma(\lambda, r)$ gives an estimate

$$1 \leq \frac{2\nu}{1-2\nu} \leq 2, \quad (23)$$

or

$$\frac{1}{4} \leq \nu \leq \frac{1}{3}. \quad (24)$$

The behaviour of $\bar{\sigma}(\lambda, \mu)$ given by (19) does not guarantee that the continuum theory will have a finite physical string tension. In the continuum limit a physical area scales as

$$A_{phys} \sim \mu_R^{2\nu} A_p \quad (25)$$

and the physical string tension as

$$\sigma_{phys} \sim \frac{\bar{\sigma}(\lambda, \mu)}{\mu_R^{2\nu}} = \frac{\sigma_0(\lambda)}{\mu_R^{2\nu}} + d(\lambda). \quad (26)$$

The physical string tension would be infinite, except at the critical point $\lambda \rightarrow \lambda_c$, where the coefficient $\sigma_0(\lambda)$ vanishes. Assume that

$$\sigma_0(\lambda) \sim (\lambda_c - \lambda)^\alpha \quad (27)$$

We do not expect a scaling of $d(\lambda)$ for $\lambda \rightarrow \lambda_c$. In the continuum limit we have $\mu_R \rightarrow 0$ and $\lambda \rightarrow \lambda_c$ with

$$\lambda_c - \lambda \sim \mu_R^2. \quad (28)$$

Critical exponents α and ρ satisfy

$$\alpha\rho = 2\nu, \quad (29)$$

In (29) we assumed that both terms in (26) contribute at the same order.

Exponent α is directly measured in our experiment. The data are consistent with $\alpha = 1$. From (29) we get that $\rho = 2\nu$.

As we mentioned in the introduction an additional requirement for an interesting scaling limit is the correct scaling of other physical observables. An observable independent of the string tension is the mass gap, defined in the grand canonical ensemble by the exponential decay of the two point function. In the standard approach one fixes two points, separated by a distance y , and sums over all surfaces passing through these points. Here we shall consider a different definition, making again use of the twisted boundary conditions. This time we consider a system with a zero projected area A_p , with only one non-zero vector E_i . We take $E_1 = (y, 0, \dots)$, $E_2 = 0$. These boundary conditions permit to measure a two loop functions, where two identical loops are separated in a translationally invariant way by a distance y and where we sum over the loop size. The two loop function should have the same scaling behaviour as the two point function.

Let us introduce as in (15) and (16) the Legendre transform of the free energy

$$\mathcal{F}(N, \lambda, y) \quad \mathcal{G}(\mu, \lambda, y) = N\mu + \mathcal{F}(N, \lambda, y). \quad (30)$$

We expect

$$\mathcal{G}(\mu, \lambda, y) \sim m(\lambda, \mu)y \quad \text{for } y \rightarrow \infty \quad (31)$$

and we have

$$N = \frac{\partial \mathcal{G}}{\partial \mu} = \frac{\partial m(\lambda, \mu)}{\partial \mu}. \quad (32)$$

Let us now assume that the mass scales at the critical point $\mu_c(\lambda)$. Opposite to the situation for the string tension we expect the bare mass $m(\lambda, \mu)$ to scale for $\mu_R = \mu - \mu_c(\lambda) \rightarrow 0$ for all λ . Assume the scaling is of the form

$$m(\lambda, \mu) \sim c(\lambda)\mu_R^{\nu(\lambda)} \quad (33)$$

Using (30)-(33) one shows that

$$\frac{\partial \mathcal{F}}{\partial y} \sim D(\lambda)t^{\nu(\lambda)/(1-\nu(\lambda))}, \quad (34)$$

where $t = y/N$. This formula allows us to determine the critical exponent $\nu(\lambda)$ from a canonical simulation. For our specific model we can again measure the derivative of $\mathcal{F}$ with respect to y as the expectation value of the gaussian action (11). On Fig. 7 we show the log-log plots of $-\partial \mathcal{F}/\partial y$ vs. t for two λ values. From (11) we predict the numerical error to behave as $1/t^3$ for small t . For larger t values we see the plots to fall on universal lines, with the slope independent of λ . Preliminary data give an estimate $\frac{1}{4} \leq \nu(\lambda) \leq \frac{1}{3}$, which is to be compared with (24). This suggests that the scaling exponent $\nu(\lambda_c)$ agrees with the ν for the string tension at λ_c and that we may have a consistent scaling of the mass and the string tension. It thus seems to be possible to define a continuum string theory by approaching the critical point $(\lambda_c, \mu(\lambda_c))$ as defined above (cf. (28)).

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References

- [1] J. Ambjørn, B. Durhuus and J. Fröhlich, *Nucl. Phys. B* 257[FS14] (1985) 433.
- [2] F. David, *Nucl. Phys. B* 257[FS14] (1985) 543.
- [3] V.A. Kazakov, I.K. Kostov and A.A. Migdal, *Phys. Lett.* 157B (1985) 295.
- [4] J. Ambjørn and B. Durhuus, *Phys. Lett.* 188B (1987) 253.
- [5] J. Ambjørn, B. Durhuus, J. Fröhlich and T. Jonsson, *Nucl. Phys. B* 290[FS20] (1987) 480.
- [6] A. Polyakov, *Nucl. Phys. B* 268 (1986) 406.
- [7] P. Wiegman, *Nucl. Phys. B* 323 (1989) 330.
- [8] F. David and S. Leibler, "Vanishing tension of fluctuating membranes", *Saclay preprint SPH/T/90/187*, December 1990.
- [9] S.M. Catterall, *Phys. Lett.* 220B (1989) 207.
- [10] C.F. Baillie, D.A. Johnston and R.D. Williams, *Nucl. Phys. B* 335 (1990) 469.

- [11] D.R. Nelson and L. Pelti, *J. Physique* 48 (1987) 1085.
- [12] Y. Kantor and D.R. Nelson, *Phys. Rev. Lett.* 58 (1987) 2774.
- [13] J. Ambjørn, B. Durhuus and T. Jonsson, *Nucl. Phys. B* 316 (1989) 526.
- [14] M. Baig, D. Espriu and J.F. Wheeler *Nucl. Phys. B* 316 (1989) 587.
- [15] W. Helfrich, *J. Physique* 46 (1985) 1263.
- [16] J. Kogut, A. Moreo and S. Das, *Nucl. Phys. B* 327 (1989) 530.
- [17] R. Renken and J. Kogut, *Nucl. Phys. B* 354 (1991) 328.
- [18] R. G. Harnish and J.F. Wheeler, *Nucl. Phys. B* 350 (1991) 861.
- [19] S.M. Catterall, *Phys. Lett. B* 243 (1990) 121.
- [20] C.F. Baillie, D.A. Johnston and R.D. Williams, *Mod. Phys. Lett. A* 5 (1990) 1671.
- [21] A.M. Ferrenberg and R.H. Swendsen, *Phys. Rev. Lett.* 63 (1989) 1195.
- [22] J. Ambjørn, D. Boulatov and V.A. Kazakov, *Mod. Phys. Lett. A* 5 (1990) 771.
- [23] A. Irbäck, J. Jurkiewicz and S. Varsted, in preparation.
- [24] See for example R.G. Miller, *Biometrika* 61 (1974) 86.
- [25] J. Ambjørn, J. Jurkiewicz, S. Varsted, A. Irbäck and B. Petersson, in preparation.

Fig. 1 Specific heat $C(\lambda)$ for spherical surface with $N = 36$.

Fig. 2 Specific heat $C(\lambda)$ for surfaces with the topology of a torus and $r = 0$.

Fig. 3 Specific heat $C(\lambda)$ for surfaces with the topology of a torus and $r = 1/16$.

Fig. 6 String tension $\sigma(\lambda, r)$ as a function of r for $N = 64, 144$ and 256 .

Fig. 7 Mass gap measurement: $-\partial\mathcal{F}/\partial y$ vs. y/N .

Fig. 4 Pseudocritical/crossover coupling λ_c^N as a function of N for fixed $r = A_p/N = 0$ and $1/16$.

Fig. 5 Maximum value of the specific heat as a function of N for fixed $r = A_p/N$.