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RADIATED ENERGY LOSS DUE TO BEAMSTRAHLUNG IN HIGH-ENERGY ELECTRON-POSITRON LINEAR COLLIDERS

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ABSTRACT

Using the recently discovered property of independent photon emissions at fixed impact distance, an integro-differential equation is set up to give the radiated energy due to beamstrahlung to all orders in the expansion parameter $\alpha L_b/\ell_c$. Here L_b is the bunch length and ℓ_c is the coherence length. This integro-differential equation is solved for both the Klein-Gordon case and the physically important Dirac case.

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1 Introduction

Beamstrahlung is the coherent radiation of electrons (positrons) traversing the intense field region associated with a bunch of positrons (electrons). High-energy electron-positron colliders have to achieve luminosities which increase as the square of the collision energy. This imposes very high bunch densities and, as a result, an intense radiation during bunch crossing. This has recently been the object of extensive studies [1]-[4].

Of particular interest at the very high energies, considered for future machines, is the so-called deep quantum regime, whereby the radiative length becomes much longer than the classical coherent radiation length. In this case, the relevant expansion parameter is

$$\frac{\alpha}{\pi} \frac{L_b}{\ell_c} \quad (1)$$

where α is the fine-structure constant, L_b is the length of the bunch, assumed to be uniform and cylindrical, and ℓ_c is the new coherent radiation length proper to that regime [1]. This parameter (1) consists of two factors. The first one, α/π , is the usual QED expansion parameter, and is of course small. For beamstrahlung, this first factor is, however, multiplied by the second one, L_b/ℓ_c , which can be fairly large, of order 50, say. The product is thus typically 0.1.

To the lowest-order in this parameter (1), the radiation intensity, which is to this order due to one-photon emission only, takes a particularly simple form

$$\frac{dI}{dx} = \frac{\alpha}{\pi} K \frac{L_b}{\ell_c} \left(\frac{1-x}{x}\right)^{2/3} \quad (2)$$

where x is the fraction of the incident electron energy taken by the radiated photon, and K is a numerical coefficient of order 1. Relation (1) corresponds to the Klein-Gordon case. The Dirac case is only slightly more complicated. However, at very high energy, where helicity flip is negligible, the rate is obtained through the simple multiplication of (1) by a spinology factor [1]

$$\frac{1}{2} \frac{1 + (1-x)^2}{1-x} \quad (3)$$

Actually, even for the uniform bunch, Eq. (2) is not the complete answer. There is an additional term, due to radiation before and after bunch crossing, of the form [1]

$$\frac{\alpha}{\pi} \ln \frac{2(1-x)E}{xm^2L_c} \quad (4)$$

where E is the beam energy, m is the electron mass, and L_c is the classical coherent radiation length. Although this term is sometimes important, as for ribbon beams, in this paper we concentrate on the term given on the right-hand side of Eq. (2).

We stay within the approximation of a cylindrical uniform bunch. Using more realistic shapes implies technical difficulties which can be dealt with [4].

In a recent paper [5], we analyzed the radiation to second order in $\frac{\alpha}{\pi} \frac{L_b}{\ell_c}$ of (1). The result is remarkably simple when one remains in the low disruption approximation [6].

The coherence conditions determine the value of the impact parameter which hardly changes as photons are successively emitted along the trajectory of the electron.

Relation (2) applies to the deep quantum regime which one meets at high energy and high luminosity [1]. The radiative length $L_e = \hbar \frac{\gamma}{m_e}$ is then much larger than the classical coherent radiation length L_c [7]. This is the interesting new regime on which we focus here. One has [1]

$$\ell_c = (L_e L_c^2)^{1/3} \quad (5)$$

If one considers (2) for a fixed impact parameter, or for a particular transverse momentum recoil taken by the bunch traversed by the incident electron, then the radiation rate simply factorizes to all orders in $\frac{\alpha L_b}{\pi \ell_c}$ [5].

For instance, the radiation rate per unit length for a second photon is the same as that calculated for a first photon but for the fact that the electron has lost energy and taken some transverse recoil from the first emission. Relation (2) for $\frac{dI}{dx d\ell}$ applies for the second photon radiation provided that x is now defined in terms of the new electron energy, and so is ℓ_c . The latter quantity is proportional to the cubic root of the energy of the radiating electron [1].

Let us assume that an electron of energy E has radiated a photon with energy $x E$, thus keeping an energy $(1 - x)E$. The emission of a second photon of energy $x' E$ corresponds then to a rate per unit length of:

$$\frac{dI}{dx'' d\ell} = \frac{\alpha}{\pi} K \frac{1}{\ell_c} \frac{1}{(1 - x)^{1/3}} \left(\frac{1 - x''}{x''} \right)^{2/3} \quad (6)$$

with

$$x'' = \frac{x'}{1 - x} \quad (7)$$

and therefore:

$$\frac{dI}{dx' d\ell} = \frac{1}{1 - x} \frac{dI}{dx'' d\ell} \quad (8)$$

Combining (6) and (8) we get

$$\frac{dI}{dx' d\ell} = \frac{\alpha}{\pi} K \frac{1}{\ell_c} \frac{1}{(1 - x)^{4/3}} \left(\frac{1 - x - x'}{x'} \right)^{2/3} \quad (9)$$

This quantity, which corresponds to the radiation rate of a photon of fractional energy x' by an electron which has a remaining fractional energy $(1 - x)$ is designated by $I_0(x'; 1 - x)$. We write it as:

$$I_0(x'; 1 - x) = g \frac{1}{(1 - x)^{4/3}} \left(\frac{1 - x - x'}{x'} \right)^{2/3} \quad (10)$$

where $g = \frac{\alpha}{\pi} \frac{K}{\ell_c} \sim 0.0045/\ell_c$ [1].

One can thus follow the history of an electron traversing a bunch with successive photon emissions. This simple factorization applies provided that one considers successive orders in $\frac{\alpha}{\pi} \frac{L_b}{\ell_c}$, while neglecting higher orders of $\frac{\alpha}{\pi}$ which are not associated with the large factor $\frac{L_b}{\ell_c}$ [5].

The purpose of this paper is to exploit this property reaching an expression for the full radiated flux to all orders in $\frac{\alpha}{\pi} \frac{L_b}{\ell_c}$.

We carry the calculation for the impact parameter fixed by the global recoil momentum. The full radiated flux is obtained through integration over impact parameters performed after summing over terms of different orders in g .

Whilst we assume a constant bunch density and therefore a well-defined value of ℓ_c [1], the calculation can be extended to more realistic bunch shapes [4]. This implies only technical complications.

2 Radiated flux. The Klein-Gordon case as a paradigm

We denote by $J(x'; 1-x, \ell)$ the emission rate for any number of photons, taking away, globally, an energy $x'E$, which are radiated by an electron of initial energy $(1-x)E$ traversing a remaining bunch length ℓ . Let us increase the bunch length, at the front end, by an infinitesimal quantity $d\ell$.

The emission rate now becomes:

$$\begin{aligned} J(x'; 1-x; \ell + d\ell) &= J(x'; 1-x; \ell) \left(1 - d\ell \int_0^{1-x} d\bar{x} I_0(\bar{x}; 1-x) \right) \\ &+ d\ell \int_0^{x'} d\bar{x} I_0(\bar{x}; 1-x) J(x' - \bar{x}; 1-x - \bar{x}; \ell) \end{aligned} \quad (11)$$

The first term corresponds to the absence of any extra emission over $d\ell$, which could have taken a fractional energy $\bar{x}$ from the electron (10). The second term takes care of some radiation over $d\ell$, a radiation which can take away up to an amount $x'E$ of energy, but then only through one single emission.

We rewrite (11) as a differential equation:

$$\begin{aligned} \frac{\partial J(x'; 1-x, \ell)}{\partial \ell} &= \int_0^{x'} d\bar{x} I_0(\bar{x}; 1-x) J(x' - \bar{x}; 1-x - \bar{x}; \ell) \\ &- J(x'; 1-x; \ell) \int_0^{1-x} d\bar{x} I_0(\bar{x}; 1-x) \end{aligned} \quad (12)$$

This defines J up to a constant. The normalization is provided by the constraint proper to a probability, namely

$$\int_0^{1-x} J(x'; 1-x; \ell) dx' = 1 \quad (13)$$

J includes a term proportional to $\delta(x')$ corresponding to no emission at all.

Of practical interest is the mean energy loss while traversing the length ℓ of bunch. It is

$$A(1-x; \ell) = \int_0^{1-x} J(x'; 1-x; \ell) x' dx' \quad (14)$$

The quantity δ , calculated in Ref. [1] is:

$$\delta = A(1; L_b) \quad (15)$$

The differential equation for A reads (12,14)

$$\begin{aligned} \frac{\partial A(1-x; \ell)}{\partial \ell} &= \int_0^{1-x} x' dx' \left(\int_0^{x'} d\bar{x} I_0(\bar{x}; 1-x) J(x' - \bar{x}; 1-x - \bar{x}; \ell) \right) \\ &- A(1-x; \ell) \int_0^{1-x} d\bar{x} I_0(\bar{x}; 1-x) \end{aligned} \quad (16)$$

Through a change of integration variables it can be rewritten as:

$$\begin{aligned} \frac{\partial A(1-x; \ell)}{\partial \ell} &= \int_0^{1-x} d\bar{x} I_0(\bar{x}; 1-x) A(1-x - \bar{x}; \ell) \\ &+ \int_0^{1-x} d\bar{x} \bar{x} I_0(\bar{x}; 1-x) - A(1-x; \ell) \int_0^{1-x} d\bar{x} I_0(\bar{x}; 1-x) \end{aligned} \quad (17)$$

where we have made use of (13). We now introduce (10), and do the integrals associated with β functions, to obtain

$$\begin{aligned} \frac{\partial A(1-x; \ell)}{\partial \ell} &= g \left\{ \int_0^{1-x} d\bar{x} \left(\frac{1-x-\bar{x}}{\bar{x}} \right)^{2/3} \frac{1}{(1-x)^{4/3}} A(1-x-\bar{x}; \ell) \right. \\ &\left. + \frac{2\pi}{9\sqrt{3}} (1-x)^{2/3} - \frac{4\pi}{3\sqrt{3}} (1-x)^{-1/3} A(1-x; \ell) \right\} \end{aligned} \quad (18)$$

The quantity A , which is a mean energy loss, is proportional to the electron energy. We can take that factor out, writing

$$A(1-x; \ell) = (1-x) B(1-x; \ell) \quad (19)$$

With an equation for B (18)

$$\begin{aligned} \frac{\partial B(1-x; \ell)}{\partial \ell} &= g(1-x)^{-1/3} \left\{ \frac{2\pi}{9\sqrt{3}} - \frac{4\pi}{3\sqrt{3}} B(1-x; \ell) \right. \\ &\left. + \int_0^1 dy (1-y)^{5/3} y^{-2/3} B((1-x)(1-y); \ell) \right\} \end{aligned} \quad (20)$$

This equation can be simplified further, if one takes into account the fact that B depends actually on one single variable

$$\xi = (1-x)^{-1/3} g\ell \quad (21)$$

This results from the specific property of the radiation rate (2) with a coherent radiation length proportional to the cubic root of the electron energy [1]. We then rewrite (20) as

$$\begin{aligned} \frac{\partial \bar{B}(\xi)}{\partial \xi} &= \frac{2\pi}{9\sqrt{3}} - \frac{4\pi}{3\sqrt{3}} \bar{B}(\xi) \\ &+ \int_0^1 dy (1-y)^{5/3} y^{-2/3} \bar{B}((1-y)^{-1/3} \xi) \end{aligned} \quad (22)$$

The boundary conditions are such that

$$\begin{aligned}\bar{B}(0) &= 0 \\ \bar{B}(\infty) &= 1\end{aligned}\tag{23}$$

The latter condition corresponds to total radiation of the electron energy. The derivative at the origin is equal to $\frac{2\pi}{9\sqrt{3}}$. This is the linear approximation corresponding to the distorted wave Born approximation calculation (2) [1]. The derivative vanishes asymptotically. Indeed,

$$\int_0^1 dy (1-y)^{5/3} y^{-2/3} = \frac{4\pi}{3\sqrt{3}} - \frac{2\pi}{9\sqrt{3}}\tag{24}$$

Equation (22) is a regular equation. However, it resists any simple expansion since a small argument ξ in two of the terms can still correspond to a very large argument in the third one, where it is multiplied by $(1-y)^{-1/3}$, which can reach very high values under the integral. It has therefore to be solved numerically. This is the price to pay to assess how much one departs from the linear approximation (2) when the bunch length increases and the electron gradually loses all its energy through successive radiations.

One can simplify (22) a little, using (24) when considering instead of B , the energy kept by the electron. One introduces

$$F(\xi) = 1 - \bar{B}(\xi)\tag{25}$$

which varies from 1 to 0, with:

$$\frac{\partial F(\xi)}{\partial \xi} = \int_0^1 dy (1-y)^{5/3} y^{-2/3} F((1-y)^{-1/3}\xi) - \frac{4\pi}{3\sqrt{3}} F(\xi)\tag{26}$$

This is our master equation.

A linear approximation in ξ cannot be accurate because of the large value of the argument reached under the integral. Yet it already shows a very important cancellation between the two terms on the right-hand side of (26). Whilst the first derivative at the origin is equal to -0.4 the second derivative is equal to 0.12. Note that a typical value for ξ is 0.2 [1]. This is the effect noticed in the calculation of the one- and two-photon rates [5]. The linear approximation of the energy loss remains a good estimate of the global energy loss up to important bunch lengths.

One can also easily obtain a constraint on F , namely its integral over ξ , taking the Laplace transform of (26)

$$\phi(t) = \int_0^\infty e^{-t\xi} F(\xi) d\xi\tag{27}$$

This gives

$$t\phi(t) - 1 = \int_0^\infty F(u) du \int_0^1 (1-y)^2 y^{-2/3} e^{-tu(1-y)^{1/3}} dy - \frac{4\pi}{3\sqrt{3}} \phi(t)\tag{28}$$

with a value of $\phi(0)$ equal to

$$\int_0^\infty F(\xi) d\xi = 2.1\tag{29}$$

This provides a measure for testing the numerical approximation of F given in Fig. 1. It approaches its asymptotic zero value as an exponential $e^{-\lambda\xi}$ with $\lambda = \frac{4\pi}{3\sqrt{3}}$, as read from (26) since the argument in the second term is always larger than ξ .

The numerical solution of (26) is shown on Fig. 1 (solid curve)

3 The realistic Dirac case

The Dirac case corresponds to only minor complications when one considers only high energies and neglect helicity flip. The radiation intensity is then the product of (2) and (3) [1].

The equation for A (18) now reads

$$\begin{aligned} \frac{\partial A(1-x;\ell)}{\partial \ell} = & g \left\{ \int_0^{1-x} d\bar{x} \frac{1}{(1-x)^{4/3}} \times \right. \\ & \times \left(\frac{(1-x-\bar{x})^{2/3}}{\bar{x}} \right) \frac{(1-x-\bar{x})^2 + (1-x)^2}{2(1-x)(1-x-\bar{x})} A(1-x-\bar{x};\ell) + \\ & \left. + \frac{32\pi}{81\sqrt{3}} (1-x^{2/3}) - \frac{14\pi}{9\sqrt{3}} (1-x)^{-1/3} A(1-x;\ell) \right\} \end{aligned} \quad (30)$$

whilst the equation for $\bar{B}(\xi)$ (22) now is

$$\frac{\partial \bar{B}(\xi)}{\partial \xi} = \left\{ \frac{32\pi}{81\sqrt{3}} - \frac{14}{9\sqrt{3}} \bar{B}(\xi) + \int_0^1 dy \frac{(1-y)^{5/3}}{y^{2/3}} \frac{(1-y)^2 + 1}{2(1-y)} \bar{B}((1-y)^{-1/3}\xi) \right\} \quad (31)$$

One then introduces F (25) to write the master equation as:

$$\frac{\partial F}{\partial \xi} = \int_0^1 dy \frac{(1-y)^{5/3}}{y^{2/3}} \frac{(1-y)^2 + 1}{2(1-y)} F((1-y)^{-1/3}\xi) - \frac{14\pi}{9\sqrt{3}} F(\xi) \quad (32)$$

The slope at the origin is now larger in absolute value. Whilst we had $-\frac{2\pi}{9\sqrt{3}}$ in the Klein-Gordon case, we now have $-\frac{32\pi}{81\sqrt{3}}$, hence the factor 16/9 which was previously obtained in the distorted wave Born approximation [1],[2].

The asymptotic form corresponds to an exponential with $e^{-\lambda\xi}$ with $\lambda = \frac{14\pi}{9\sqrt{3}}$ which is 7/6 times the value in the Klein-Gordon case. The numerical solution is shown as a dashed curve on Fig. 1.

With similar caveat about a linear approximation, one can again use it in (32) to estimate the second derivative. One finds a similar cancellation at the level of the second derivative. Whilst the first derivative is now -0.71, the second derivative is only 0.29. The linear approximation remains valid up to long bunch lengths, when one calculates the total radiated energy.

One can again use the Laplace transform (28) to obtain

$$\int_0^\infty F(\xi) d\xi = 1.2 \quad (33)$$

Whilst the linear approximation remains good over a large interval of ξ it should be stressed that our approach assumes a small disruption [6]. As the electron radiates more and more of its energy its bending will increase and one can no longer assume that the impact parameter remains practically the same as more and more energy is radiated. Nevertheless since the radiated energy is proportional to the incident energy at fixed x value, most of the radiated energy will originate from radiation from still energetic electrons. Its dominant part will therefore come from a regime where the disruption remains small.

We may therefore consider our results are valid for all x values even though values close to 1 bring us into the large disruption regime.

4 Conclusions

One can conclude that radiation yields calculated in the distorted wave approximation, to lowest order in $\alpha \frac{L_b}{\ell_c}$ are good approximations to the full radiated energy. Despite the fact that the expansion parameter is of order 0.1, the inclusive radiated flux is already accurately obtained through the linear approximation results previously obtained. The full radiated energy can be calculated as a function of the bunch length L_b the relevant parameter being gL_b (10), with $x = 0$ in (21) for an impinging electron. It is obtained from a numerical integration of the master equation.

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FIGURE CAPTION

The average fraction F of energy retained by the electron (or positron) after traversing a normalized bunch length ξ of the positron (or electron) bunch, where $\xi = 0.0045 L_b/\ell_c$. The dashed curve is for the realistic Dirac case, while the solid curve is for the Klein-Gordon case.

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