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On the Euler Characteristic for Piecewise Linear Manifolds

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ABSTRACT

The Dehn-Sommerville relations and the corresponding equations for the angle sums are used to derive two expressions for the Euler characteristic of a simplicial manifold, firstly in terms of the numbers of even dimensional sub-simplices, and secondly in terms of even-dimensional deficit angles. In each case the coefficients involved are related to the Bernoulli numbers.

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I. Introduction

Consider a simplicial manifold K of dimension $2n$. It is well-known that its Euler characteristic $\chi^{(2n)}$ is given by

$$\chi^{(2n)} = \sum_{i=0}^{2n} (-1)^i N_i^{(2n)} \quad (1.1)$$

where $N_i^{(2n)}$ is the number of i -dimensional cells in K . It is also possible to express $\chi^{(2n)}$ in terms of angles in the simplicial complex. Let (j, k) denote the interior angle at the j -dimensional face σ^j in the k -dimensional simplex σ^k , normalized so that the unit sphere in all dimensions has volume 1 [For example, the interior angle $(1, 3)$ at an edge in a regular tetrahedron would be given by $\cos^{-1}(1/3)$ divided by the normalization factor 2π]. It can then be shown that

$$\chi^{(2n)} = \sum_{\sigma^0} (1 + \sum (-1)^l (0, 2i_1)(2i_1, 2i_2) \dots (2i_{l-1}, 2i_l)) \quad (1.2)$$

where the second sum is over all even dimensional subsimplices such that

$$\sigma^{2i_1} \subset \sigma^{2i_2} \subset \dots \subset \sigma^{2i_l}.$$

(see Banchoff 1982, Cheeger 1983, Cheeger, Müller and Schrader 1984 and references therein).

For a given complex, the numbers $N_i^{(2n)}$ are not all independent; they satisfy a set of linear equations called the Dehn-Sommerville equations (Dehn 1905, Sommerville 1927). These encode the requirement that for a simplicial manifold, the link of any sub-simplex is homeomorphic to a sphere in the appropriate dimension. The equations can be used to eliminate the $N_i^{(2n)}$'s for odd i from the expression for $\chi^{(2n)}$:

$$\chi^{(2n)} = \sum_{i=0}^n a_{i+1} N_{2i}^{(2n)}, \quad a_i = \frac{2}{i} (2^{2i} - 1) B_{2i} \quad (1.3)$$

where $\{B_{2i}\}$ are the even Bernoulli numbers. This is the first main result in this paper. The second is an expression for $\chi^{(2n)}$ in terms of even-dimensional deficit angles (see (2.4) below), which follows from the fact that certain sums of interior angles satisfy equations exactly equivalent to the Dehn-Sommerville ones:

$$\chi^{(2n)} = \sum_{i=0}^{n-1} a_{i+1} \sum_{\sigma^{2i}} \epsilon_{2i}^{(2n)} \quad (1.4)$$

2. The Dehn-Sommerville equations, angle sums and deficit angles.

The Dehn-Sommerville equations (Sommerville 1927) may be written as

$$N_p^{(2n)} = \sum_{i=p}^{2n} (-1)^i \binom{i+1}{p+1} N_i^{(2n)} \quad (2.1)$$

for $p = 0, 1, 2, \dots, 2n$. (For $p = -1$, the equation reduces to (1.1) provided $N_{-1}^{(2n)}$ is interpreted as $\chi^{(2n)}/2$.) Only n of these equations are linearly independent, which means that there is precisely the right amount of information to solve for the odd-dimensional $N_i^{(2n)}$'s in terms of the even ones; in the next section, we show how this leads to formula (1.3) for $\chi^{(2n)}$ in terms of the even $N_i^{(2n)}$'s.

The angle sum $S_p^{(q)}$ is defined to be the sum of all interior angles at p -dimensional subsimplices in a simplex of dimension q :

$$S_p^{(q)} \equiv \sum_{\sigma^p \subset \sigma^q} (p, q) \quad (2.2)$$

with $S_q^{(q)} \equiv 1$. The angle sum $S_p^{(q)}$ is related to $\Sigma_p^{(2n)}$ in Sommerville (1927) by a p -independent normalization factor

$$\Sigma_p^{(2n)} = k^{2n} (2n+1) \frac{\pi^{(2n+1)/2}}{\Gamma[(2n+3)/2]} S_p^{(2n)} \quad (2.3)$$

where k is the space-constant. Now the $\Sigma_p^{(2n)}$'s satisfy exactly the same linear relations as the $N_p^{(2n)}$'s (Sommerville 1927) and thus the angle sums $S_p^{(2n)}$ must satisfy the Dehn-Sommerville

equations (2.1), with the convention this time that

$$S_{-1}^{(2n)} = 0$$

This means that we are able to obtain a relation between the angle sums $S_p^{(2n)}$ for even p , analogous to the expression for $\chi^{(2n)}$ in terms of $N_p^{(2n)}$ for even p .

Combining the new formula for $\chi^{(2n)}$ with the angle sum relation leads to an expression for $\chi^{(2n)}$ in terms of higher dimensional deficit angles. The (normalized) k -dimensional deficit angle at a simplex σ^j is defined by

$$\epsilon_j^{(k)} = 1 - \sum_{\sigma^k \supset \sigma^j} (j, k) \quad (2.4)$$

Let us illustrate how this works by looking at the case of four dimensions.

From equation (1.2), the Euler characteristic in four dimensions is given in terms of interior angles by

$$\chi^{(4)} = \sum_{\sigma^0} \left[1 - \sum_{\sigma^2 \supset \sigma^0} (0, 2) - \sum_{\sigma^4 \supset \sigma^0} (0, 4) + \sum_{\sigma^4 \supset \sigma^2 \supset \sigma^0} (0, 2)(2, 4) \right] \quad (2.5)$$

We simplify this by using the simplest case of the angle-sum relations, that the angles of a triangle add to π (divided by the normalization factor of 2π). This gives, after appropriate changes of the orders of summation,

$$\sum_{\sigma^0} \sum_{\sigma^2 \supset \sigma^0} (0, 2) = \sum_{\sigma^2} \sum_{\sigma^0 \subset \sigma^2} = \sum_{\sigma^2} \frac{\pi}{2\pi} \quad (2.6)$$

and

$$\begin{aligned} \sum_{\sigma^0} \sum_{\sigma^2 \supset \sigma^0} \sum_{\sigma^4 \supset \sigma^2} (0, 2)(2, 4) &= \sum_{\sigma^2} \sum_{\sigma^4 \supset \sigma^2} \left(\sum_{\sigma^0 \subset \sigma^2} (0, 2) \right) (2, 4) \\ &= \frac{1}{2} \sum_{\sigma^2} \sum_{\sigma^4 \supset \sigma^2} (2, 4) \end{aligned} \quad (2.7)$$

The expression (2.5) for $\chi^{(4)}$ may then be written as

$$\begin{aligned}\chi^{(4)} &= \sum_{\sigma^0} \left(1 - \sum_{\sigma^4 \supset \sigma^0} (0, 4) \right) - \frac{1}{2} \sum_{\sigma^2} \left(1 - \sum_{\sigma^4 \supset \sigma^2} (2, 4) \right) \\ &= \sum_{\sigma^0} \epsilon_0^{(4)} - \frac{1}{2} \sum_{\sigma^2} \epsilon_2^{(4)}\end{aligned}\tag{2.8}$$

where $\epsilon_0^{(4)}$ is the 4-dimensional deficit angle at a vertex (the “solid deficit” in volume there) and $\epsilon_2^{(4)}$ is the deficit angle at a triangle (the normalized version of the usual deficit angle in 4-dimensional Regge calculus [Regge 1961]). Of course the Gauss-Bonnet theorem in two dimensions is the classic example of the Euler characteristic written in terms of deficit angles

$$\chi^{(2)} = \sum_{\sigma^0} \epsilon_0^{(2)}\tag{2.9}$$

where $\epsilon_0^{(2)}$ is the deficit angle at a vertex.

3. Statement and proof of the main results.

We obtain the following relationships between the Euler characteristic, the numbers of sub-simplices, the angle sums and the deficit angles for a simplicial manifold in $2n$ -dimensions:

$$(i) \quad N_{2i-1}^{(2n)} = \frac{1}{2} \sum_{j=0}^{n-i} a_{j+1} \binom{2(i+j)+1}{2i} N_{2(i+j)}^{(2n)}\tag{3.1}$$

$$(ii) \quad \chi^{(2n)} = \sum_{i=0}^n a_{i+1} N_{2i}^{(2n)}\tag{3.2}$$

$$(iii) \quad 0 = \sum_{i=0}^n a_{i+1} S_{2i}^{(2n)}\tag{3.3}$$

$$(iv) \quad \chi^{(2n)} = \sum_{i=0}^{n-1} a_{i+1} \sum_{\sigma^{2i}} \epsilon_{2i}^{(2n)}\tag{3.4}$$

where the coefficient a_i are defined by:

$$(v) \quad a_i = 1 - \frac{1}{2} \sum_{j=1}^{i-1} \binom{2i-1}{2j-1} a_j, \quad i > 1,\tag{3.5}$$

with $a_1 = 1$. The a_i 's are universal (i.e., independent of n) and are related to the Bernoulli numbers B_i by

$$(vi) \quad a_i = \frac{2}{i}(2^{2i} - 1)B_{2i} \quad (3.6)$$

We shall now prove these relations using the Dehn-Sommerville equations (2.1) and the corresponding equations for the angle sums.

Proof of (i). Suppose that (i) holds for $i > I$. It certainly holds for $i = n$, because then it reduces to

$$N_{2n-1}^{(2n)} = \frac{(2n+1)}{2} N_{2n}^{(2n)} \quad (3.7)$$

which follows immediately from the Dehn-Sommerville relation (2.1) with $p = 2n - 1$. We shall now prove that (i) holds for $i = I$.

From the Dehn-Sommerville equations with $p = 2I - 1$,

$$2N_{2I-1}^{(2n)} = \sum_{i=I}^n \binom{2i+i}{2I} N_{2i}^{(2n)} - \sum_{i=I+1}^n \binom{2i}{2I} N_{2i-1}^{(2n)}.$$

In the second term, we use (i), assumed true for $i > I$, to give

$$2N_{2I-1}^{(2n)} = \sum_{i=I}^n \binom{2i+1}{2I} N_{2i}^{(2n)} - \frac{1}{2} \sum_{i=I+1}^n \binom{2i}{2I} \sum_{j=0}^{n-1} a_{j+1} \binom{2(i+j)+1}{2i} N_{2(i+j)}^{(2n)}$$

In the first term, put $i = I + j$. In the second term, use the relation

$$\binom{p}{q} \binom{q}{r} = \binom{p}{r} \binom{p-r}{q-r} \quad (3.8)$$

and then put $i = I + 1 - k + l, j = k - 1$. This leads to

$$\begin{aligned} 2N_{2I-1}^{(2n)} &= \sum_{j=0}^{n-1} \binom{2(I+j)+1}{2I} N_{2(I+j)}^{(2n)} \\ &\quad - \frac{1}{2} \sum_{l=1}^{n-I} \sum_{k=1}^l a_k \binom{2(I+l)+1}{2I} \binom{2l+1}{2(l-k+1)} N_{2(I+l)}^{(2n)} \end{aligned}$$

Replacing l by j in the second term, and separating the $j = 0$ part in the first sum, we obtain

$$2N_{2I-1}^{(2n)} = \binom{2I+1}{2I} N_{2I}^{(2n)} + \sum_{j=1}^{n-I} \left(1 - \frac{1}{2} \sum_{k=1}^j a_k \binom{2j+1}{2k-1} \right) \binom{2(I+j)+1}{2I} N_{2(I+j)}^{(2n)}$$

The definition of a_i in (v) makes it possible to simplify this to

$$\begin{aligned} 2N_{2I-1}^{(2n)} &= \binom{2I+1}{2I} N_{2I}^{(2n)} + \sum_{j=1}^{n-I} a_{j+1} \binom{2(I+j)+1}{2I} N_{2(I+j)}^{(2n)} \\ &= \sum_{j=0}^{n-I} a_{j+1} \binom{2(I+j)+1}{2I} N_{2(I+j)}^{(2n)} \end{aligned}$$

as required, and (i) has been proved by induction on i .

Proof of (ii). From the definition of $\chi^{(2n)}$ (the Dehn-Sommerville relation with $p = -1$).

$$\begin{aligned} \chi^{(2n)} &= \sum_{i=0}^n N_{2i}^{(2n)} - \sum_{i=1}^n N_{2i-1}^{(2n)} \\ &= \sum_{i=0}^n N_{2i}^{(2n)} - \frac{1}{2} \sum_{l=1}^n \sum_{j=0}^{n-l} a_{j+1} \binom{2(i+j)+1}{2i} N_{2(i+j)}^{(2n)} \end{aligned}$$

where we have used (i) in the second term. We can now put $i = k - l + 1, j = l - 1$ in the second term, to give

$$\chi^{(2n)} = \sum_{i=0}^n N_{2i}^{(2n)} - \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^k a_l \binom{2k+1}{2(k-l+1)} N_{2k}^{(2n)}$$

Relabelling k and l by i and j , and separating out the $i = 0$ from the first sum, we may write this as

$$\chi^{(2n)} = N_0^{(2n)} + \sum_{i=1}^n \left(1 - \frac{1}{2} \sum_{j=1}^i a_j \binom{2i+1}{2j-1} \right) N_{2i}^{(2n)}.$$

From (v), this becomes

$$\begin{aligned} \chi^{(2n)} &= N_0^{(2n)} + \sum_{i=1}^n a_{i+1} N_{2i}^{(2n)} \\ &= \sum_{i=0}^n a_{i+1} N_{2i}^{(2n)}, \end{aligned}$$

as required, where we have used $a_1 = 1$.

Proof of (iii). This follows immediately from the proof of (ii) when we recall that the $S_i^{(2n)}$'s satisfy the same equations as the $N_i^{(2n)}$'s, with the understanding that

$$N_{-1}^{(2n)} = \frac{\chi^{(2n)}}{2}; \quad S_{-1}^{(2n)} = 0.$$

Proof of (iv). The definition of the deficit angles (2.4) is used to express the right-hand side of (iv) in terms of angle-sums:

$$\begin{aligned} \sum_{i=0}^{n-1} a_{i+1} \sum_{\sigma^{2i}} \epsilon_{2i}^{(2n)} &= \sum_{i=0}^{n-1} a_{i+1} \sum_{\sigma^{2i}} \left(1 - \sum_{\sigma^{2n} \supset \sigma^{2i}} (2i, 2n) \right) \\ &= \sum_{i=0}^{n-1} a_{i+1} \left(\sum_{\sigma^{2i}} 1 - \sum_{\sigma^{2n}} \sum_{\sigma^{2i} \subset \sigma^{2n}} (2i, 2n) \right) \\ &= \sum_{i=0}^{n-1} a_{i+1} \left(N_{2i}^{(2n)} - \sum_{\sigma^{2n}} S_{2i}^{(2n)} \right) \\ &= \sum_{i=0}^{n-1} a_{i+1} N_{2i}^{(2n)} - \sum_{\sigma^{2n}} \left(\sum_{i=0}^{n-1} a_{i+1} S_{2i}^{(2n)} \right) \end{aligned}$$

Using (iii) in the second term, we obtain

$$\sum_{i=0}^{n-1} a_{i+1} \sum_{\sigma^{2i}} \epsilon_{2i}^{(2n)} = \sum_{i=0}^{n-1} a_{i+1} N_{2i}^{(2n)} + \sum_{\sigma^{2n}} a_{n+1} S_{2n}^{(2n)}$$

. Now $S_{2n}^{(2n)} = 1$ by definition, and so

$$\sum_{\sigma^{2n}} S_{2n}^{(2n)} = N_{2n}^{(2n)},$$

which gives

$$\sum_{i=0}^{n-1} a_{i+1} \sum_{\sigma^{2i}} \epsilon_{2i}^{(2n)} = \sum_{i=0}^n a_{i+1} N_{2i}^{(2n)}$$

The result then follows from (i).

Proof of (vi). We show that $\frac{2}{i} (2^{2i} - 1) B_{2i}$ satisfies the same defining recurrence relation (v) as a_i and can therefore be identified with it.

We use some relations for the Bernoulli numbers taken from Nielsen (1923), together with the fact that Nielsen's B_n is $(-1)^{n+1} B_{2n}$ in more usual notation. After this substitution formula (4), on page 172, gives

$$2(2^{2n} - 1)B_{2n} + \sum_{s=1}^{n-1} \binom{2n}{2s} (2^{2n-2s} - 2) B_{2n-2s} = 1 \quad (3.9)$$

and formula (10), on page 152, is equivalent to

$$\sum_{s=1}^{n-1} \binom{2n}{2s} B_{2n-2s} = n - 1 \quad (3.10)$$

Add (3.9) and (3.10) to give

$$2(2^{2n} - 1)B_{2n} + \sum_{s=1}^{n-1} \binom{2n}{2s} (2^{2n-2s} - 1) B_{2n-2s} = n$$

We now put $i = n - s$ and use

$$\binom{2n}{2(n-i)} = \frac{n}{i} \binom{2n-1}{2i-1} \quad (3.11)$$

to obtain

$$\frac{2}{n} (2^{2n} - 1) B_{2n} = 1 - \frac{1}{2} \sum_{i=1}^{n-1} \binom{2n-1}{2i-1} \frac{2}{i} (2^{2i} - 1) B_{2i}$$

which is exactly of the required form. The defining relation (3.5) holds for $i > 1$. For $i = 1$, $a_1 = 1$, and B_2 can be shown from the simple definition of the Bernoulli numbers to be $1/6$, so (vi) holds also in this special case.

The relationship between the a_i coefficient and the Bernoulli numbers was suggested by John Conway. In fact, the Bernoulli numbers (although not explicitly identified as such) have appeared before (Wall 1964) as coefficients in linear relations between the various Dehn-Sommerville equations.

We conclude this section by stating a number of other recurrence relations satisfied by the a_i :

$$a_i = \sum_{j=1}^i \binom{2i-1}{2j-2} a_j \quad (3.12)$$

$$2 = \sum_{j=1}^i \binom{2i}{2j-1} a_j \quad (3.13)$$

$$a_i = 2 - \sum_{j=1}^i 2^{2(i-j)} \binom{2i-1}{2j-2} a_j \quad (3.14)$$

These are most easily proved from formulae involving the Bernoulli numbers.

4. Conclusion

Using the Dehn-Sommerville equations, we have shown how the Bernoulli numbers enter into expressions for the Euler characteristic in terms of either the number of even-dimensional sub-simplices or the higher dimensional deficit angles. It is a feature of the Bernoulli numbers that they appear in expressions which are in some sense approximations (for example in finite difference approximations to derivatives). Their appearance here may reflect the idea that a piecewise linear space is in a sense an approximation to a space with continuously distributed curvature.

One word of warning: it is not a simple matter to work out the higher dimensional deficit angles for a discrete space-time where the simplices are not equilateral. For example, evaluation of $\epsilon_0^{(4)}$ requires the computation of the $(0, 4)$ angles which are solid angles at the vertices of 4-simplices. Each calculation of a $(0, 4)$ involves finding the volume of a spherical tetrahedron, for which there is no explicit formula (see for example Richmond 1902).

Our original motivation for exploring possible new formulae for the Euler characteristic was the hope of finding a combinatoric expression for the Hirzebruch signature in four

dimensions by operating on the formula for $\chi^{(4)}$. It seems that the significant part of $\chi^{(4)}$ from the point of view of the signature is the part involving the $\epsilon_0^{(4)}$ deficit angles, and so a deeper understanding of the $(0,4)$ angles is essential before further progress can be made.

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