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Vector Supersymmetry in Topological Field Theory

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Abstract

We consider a type of supersymmetry which is present in the Chern-Simons and BF-type topological field theories in terms of a superconnection formalism. The combined global supersymmetry and BRST symmetry are realized in this superspace when non-covariant constraints on the supercurvature are chosen. This construction extends naturally within the bundle of frames approach to superspace, and we present a formalism which may lead to the construction of vector supergravity theories and their coupling to the topological field theories considered here.

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1 Introduction

Some time ago, an unusual supersymmetry (SUSY) was discovered [1, 2] in the three dimensional Chern-Simons theory when quantized in the Landau gauge. This symmetry is characterized by a Grassmann odd vector parameter and relates the classical and ghost fields in a nontrivial way. This additional symmetry imposes strong constraints on the correlation functions in the theory, and it was shown in [3, 4] how one could deduce the finiteness of Chern-Simons theory by exploiting this symmetry. The BRST-SUSY was then combined [5, 6] with the anti-BRST symmetry, leading to an N=2 BRST-SUSY algebra.

The unusual supersymmetry in Chern-Simons theory proved to be more generic than exceptional, and a similar symmetry was observed in three and four dimensional BF topological field theory when a Landau gauge fixing condition was employed [7]. The algebra in this case has essentially the same structure as in Chern-Simons.

One of the disappointing features of this global vector SUSY is that it is valid only when these theories are considered on a flat spacetime background. This follows trivially from the integrability constraint on the Riemann curvature; a covariantly constant vector ϵ_λ implies $[D_\mu, D_\nu]\epsilon_\lambda = R_{\mu\nu\lambda\sigma}\epsilon^\sigma = 0$. In contrast, the global BRST parameter is a spacetime scalar, and leads to no such restriction on the background. A natural question to pose is whether this supersymmetry can be made local so as to circumvent this constraint. Such a theory would represent a coupling of the gauge fixed Chern-Simons or BF theory to some kind of classical vector supergravity theory.

In this paper, we consider both the Chern-Simons and BF theory in terms of a framework which has been successfully used to construct supergravity theories and their coupling to gauged matter fields. We present a variation on the bundle of frames approach to superspace which is tailored to the situation at hand. In principle, one should be able to consider solutions to the Bianchi identities for pure vector supergravity, as well as its coupling to Chern-Simons and BF theory. The global SUSY is seen as a solution to these equations when a flat spacetime is chosen and the gravitino (vector rather than spinor) field vanishes. This requires choosing non-covariant curvature constraints of the super Yang-Mills connection. We speculate on the form of

the torsion constraints needed to realize local vector SUSY.

We should point out that a number of papers exist which deal with something called “Chern-Simons supergravity” [8]. All of these consider supergroup extensions of the usual Chern-Simons approach to gravity [9], and there is no overlap with the type of SUSY that we are considering here.

2 Review

In this section we will review the key aspects pertaining to the vector supersymmetries which are realized in both the Chern-Simons and BF topological field theories. This will serve to set the stage for the superconnection formalism presented in the succeeding sections.

When the Chern-Simons theory is quantized in the Landau gauge, the action which one takes into the path integral is given by

$$S_g = \int d^3x \text{Tr} \left[-\frac{1}{2} \epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho + \frac{1}{3} A_\mu [A_\nu, A_\rho]) + d \partial_\mu A^\mu + b \partial^\mu D_\mu c \right], \quad (1)$$

and the associated BRST transformations take the form

$$\delta^1 A_\mu = -D_\mu c, \quad \delta^1 c = \frac{1}{2} \{c, c\}, \quad \delta^1 b = d, \quad \delta^1 d = 0. \quad (2)$$

An unusual feature of this particular quantum action is that there is an additional symmetry parametrized by a Grassmann odd vector [1, 2], and these transformations are given by

$$\delta_\rho^1 A_\mu = \epsilon_{\mu\rho\nu} \partial^\nu c, \quad \delta_\rho^1 c = 0, \quad \delta_\rho^1 b = A_\rho, \quad \delta_\rho^1 d = D_\rho c. \quad (3)$$

It is important to retain the multiplier field d which serves to enforce the gauge constraint. All of this is straightforward to verify, and one finds that these transformations form an abelian algebra. It was pointed out in [5] that this theory also admits an anti-BRST symmetry, which is obtained through the following replacement scheme

$$A_\mu \rightarrow A_\mu, \quad c \rightarrow b, \quad b \rightarrow -c, \quad d \rightarrow d - \{b, c\}, \quad (4)$$

and this extends the BRST-SUSY defined above to an N=2 algebra. Denoting the anti-BRST and anti-SUSY transformations by δ^2 , the on-shell algebra then takes the form

$$\{\delta_\rho^i, \delta_\sigma^j\} = \epsilon^{ij} \epsilon_{\rho\sigma\tau} \partial^\tau, \quad \{\delta^i, \delta_\rho^j\} = \epsilon^{ij} \partial_\rho. \quad (5)$$

The algebra is no longer abelian, and the anticommutators close on a translation generator.

The supersymmetry present in 3d BF theory is a trivial extension of the known Chern-Simons result. Indeed, given the fact that the Chern-Simons action for the Poincaré group ISO(2,1) is equivalent to SO(2,1) BF theory, one can simply read off the required transformations from equations (3) and (4). The N=1 algebra of interest is essentially the same as in Chern-Simons, and one need only make the replacement $b \rightarrow b'$ and $d \rightarrow d'$, where b' and d' are respectively the antighost and the multiplier for the gauge fixing of the local B symmetry.

A rather similar story unfolds when one considers BF topological field theory [10, 11]. In four dimensions, the classical action for this theory is simply given by

$$S = -\frac{1}{4} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a B_{\rho\sigma}^a. \quad (6)$$

Unlike the case in lower dimensions, the gauge symmetry here turns out to be reducible, and one must take care in quantizing such a system. In [7] it was shown that a similar gauge choice led to a supersymmetry such as we described above in the Chern-Simons theory. The quantum action can be found in [7], though it is not relevant for our purposes here. The BRST transformations take the form

$$\begin{aligned} \delta A_\mu &= -D_\mu c, \quad \delta c = \frac{1}{2}\{c, c\} \\ \delta \xi_\mu &= D_\mu \phi + \{c, \xi_\mu\}, \quad \delta \phi = [c, \phi] \\ \delta B_{\mu\nu} &= -(D_\mu \xi_\nu - D_\nu \xi_\mu) - [B_{\mu\nu}, c] + \epsilon_{\mu\nu\rho\sigma} [\partial^\rho \bar{\xi}^\sigma, \phi] \\ \delta \bar{\xi}_\mu &= h_\mu, \quad \delta h_\mu = 0, \quad \delta \bar{c} = b, \quad \delta b = 0 \\ \delta \bar{\phi} &= \omega, \quad \delta \omega = 0, \quad \delta e = \lambda, \quad \delta \lambda = 0, \end{aligned} \quad (7)$$

while the vector supersymmetry is given by

$$\delta_\rho A_\mu = -\epsilon_{\mu\rho\sigma\tau} \partial^\sigma \bar{\xi}^\tau, \quad \delta_\rho c = -A_\rho, \quad \delta_\rho \bar{c} = 0$$

$$\begin{aligned}
\delta_\rho b &= \partial_\rho \bar{c} , \delta_\rho B_{\mu\nu} = -\epsilon_{\mu\nu\rho\tau} \partial^\tau \bar{c} \\
\delta_\rho \xi_\mu &= B_{\mu\rho} , \delta_\rho \bar{\xi}^\tau = -\delta_\rho^\tau \bar{\phi} , \delta_\rho h^\mu = \partial_\rho \bar{\xi}^\mu + \delta_\rho^\mu \omega \\
\delta_\rho \phi &= \xi_\rho , \delta_\rho \omega = \partial_\rho \bar{\phi} , \delta_\rho \bar{\phi} = 0 , \delta_\rho \lambda = \partial_\rho e , \delta_\rho e = 0 .
\end{aligned} \tag{8}$$

It is trivial to verify that the algebra

$$\{\delta_\rho, \delta_\nu\} = 0 , \{\delta, \delta_\rho\} = \partial_\rho \tag{9}$$

is satisfied on-shell. This is identical to an N=1 sector of the Chern-Simons theory when we combine BRST with anti-SUSY, or anti-BRST with SUSY. It is this N=1 algebra which we seek to capture in the following construction.

3 Superspace for Vector Supersymmetry

In this section, we review the relevant aspects of the bundle of frames approach to superspace which is tailored to the type of theories considered here. This should serve to fix our notation and conventions. For a more comprehensive introduction aimed at conventional supergravity, see [12], and for an application similar in spirit to the one undertaken here, see [13].

Superspace is locally parametrized by a set of even and odd coordinates; (x, θ) . In the case at hand, we will explicitly separate an additional odd coordinate θ which will be related to the BRST symmetry of the theory; $z^\Lambda = (x^\mu, \theta^\alpha, \theta)$. Notice that we use lower case Greek indices from the beginning of the alphabet to label the odd coordinates, while those from the middle label the even directions. A capital Greek index denotes generically any of the coordinates. Over a supermanifold, we can consider a fibre bundle together with a connection $h_\Lambda^{A_0}$ and form the covariant derivative

$$D_\Lambda = \partial_\Lambda - h_\Lambda^{A_0} D_{A_0} . \tag{10}$$

The labels $A_0, B_0, \dots$ are used to denote a vertical direction in the bundle, and the differential operators D_{A_0} represent the structure group

$$[D_{A_0}, D_{B_0}] = f_{B_0 A_0}^{C_0} D_{C_0} . \tag{11}$$

The structure group will be a product of $SO(n)$ for the the local Lorentz transformations, and a compact Yang-Mills group G . It will be convenient

to use $A_0 = [rs]$ when referring to a $SO(n)$ direction, and $I, J, K, \dots$ for the group G . It is also conventional to denote the spin and Yang-Mills connections respectively by $h_\Lambda^{[rs]} = \omega_\Lambda^{[rs]}$ and $h_\Lambda^I = A_\Lambda^I$. The equivariance of a connection is expressed through the relation $[D_\Lambda, D_{A_0}] = 0$, or equivalently

$$D_{A_0} h_\Lambda^{B_0} = f_{A_0 C_0}^{B_0} h_\Lambda^{C_0} . \quad (12)$$

The curvature is defined by the graded commutator

$$[D_\Lambda, D_\Sigma] = -R_{\Lambda\Sigma}^{A_0} D_{A_0} , \quad (13)$$

and it is usually conventional to write $R_{\Lambda\Sigma}^I = F_{\Lambda\Sigma}^I$ for the Yang-Mills curvature.

In addition to the bundle structure that we have so far considered, we will need to supplement this with a collection of local frame fields $h^{A_1} = dz^\Lambda h_\Lambda^{A_1}$ which generalize the notion of vielbein to the case of supermanifolds. An index $A_1, B_1, \dots$ is being used to denote the horizontal directions in the bundle, and objects with these labels transform in definite representations of the structure group,

$$D_{A_0} h^{A_1} = f_{A_0 B_1}^{A_1} h^{B_1} . \quad (14)$$

We will use a lower case Latin index from the beginning of the alphabet to denote an odd direction, and those near the middle ($i, j, k, \dots$) for an even direction. As in supergravity, the even labels transform as vectors under the $SO(n)$ group, $f_{A_0 j}^i = f_{[rs]j}^i = \frac{1}{2} \delta_{[r}^i \delta_{s]j}$. However, instead of the h^a transforming as spinors, we will require them to also transform as vectors. This is simply because we seek to realize a Grassmann odd vector symmetry.

The addition of frame fields in the picture allows us to define torsion

$$T_{\Lambda\Sigma}^{A_1} = D_\Lambda h_\Sigma^{A_1} - (-1)^{\Lambda\Sigma} D_\Sigma h_\Lambda^{A_1} , \quad (15)$$

as well as relate objects with curved labels to those with flat labels, e.g.

$$D_\Lambda = h_\Lambda^{A_1} D_{A_1} = h_\Lambda^i D_i + h_\Lambda^a D_a + h_\Lambda^\theta D_\theta . \quad (16)$$

Notice that we also employ θ as an index to denote a “flat” BRST direction, and this naturally transforms as a scalar under $SO(n)$. It follows that the covariant derivatives with local Lorentz labels satisfy the algebra

$$[D_{A_0}, D_{A_1}] = -f_{A_0 A_1}^{B_1} D_{B_1} . \quad (17)$$

If we also define “flat” curvature and torsion components by

$$\begin{aligned} R_{\Lambda\Sigma}^{A_0} &= (-1)^{B_1\Sigma} h_{\Lambda}^{B_1} h_{\Sigma}^{C_1} R_{B_1 C_1}^{A_0} \\ T_{\Lambda\Sigma}^{A_1} &= (-1)^{B_1\Sigma} h_{\Lambda}^{B_1} h_{\Sigma}^{C_1} T_{B_1 C_1}^{A_1} , \end{aligned} \quad (18)$$

then the entire algebraic picture can be written very compactly as

$$[D_A, D_B] = F_{BA}^C D_C , \quad (19)$$

where $F_{B_1 A_1}^{C_1} = T_{B_1 A_1}^{C_0}$, $F_{B_1 A_1}^{C_0} = R_{B_1 A_1}^{C_0}$, $F_{B_0 A_1}^{B_1} = f_{B_0 A_1}^{B_1}$, and so on.

The Bianchi identities follow from the super Jacobi identity

$$\begin{aligned} 0 &= (-1)^{AC} [[D_A, D_B], D_C] + (-1)^{BC} [[D_C, D_A], D_B] \\ &+ (-1)^{BA} [[D_B, D_C], D_A] , \end{aligned} \quad (20)$$

which represents the associativity of the differential operators, and they take the form

$$\begin{aligned} 0 &= \sum_{(cyclic)} (-1)^{A_1 B_1} [D_{A_1} T_{B_1 C_1}^{D_1} - T_{C_1 A_1}^{E_1} T_{B_1 E_1}^{D_1} + R_{C_1 A_1}^{E_0} f_{E_0 B_1}^{D_1}] \\ 0 &= \sum_{(cyclic)} (-1)^{A_1 B_1} [D_{A_1} R_{B_1 C_1}^{D_0} - T_{C_1 A_1}^{E_1} R_{B_1 E_1}^{D_0}] . \end{aligned} \quad (21)$$

The sum here is over the three cyclic permutations of the lower indices.

The algebra which represents all of the local symmetries of the theory is given by

$$\begin{aligned} \delta h_{\Lambda}^A &= D_{\Lambda} \epsilon^A - h_{\Lambda}^{B_1} \epsilon^C F_{B_1 C}^A \\ \delta F_{BC}^A &= \epsilon^E D_E F_{BC}^A , \end{aligned} \quad (22)$$

where all of the parameters have a “flat” index. Note that $A, B, \dots$ with no subscript denotes either a vertical or horizontal direction.

In the next section, we will see that a flat background supergeometry supports a solution to the Bianchi identities, and realizes the Chern-Simons and BF theories together with their symmetries.

4 Global BRST-SUSY and Superconnections

In this section, we will embed the theories under discussion in the superconnection framework outlined above. Applying this to the case at hand, we begin by demanding that the vielbein describes a flat geometry, $h_\mu^i = \delta_\mu^i$, and that the gauge fields which could be introduced for the SUSY and BRST symmetries are trivial, $h_\mu^a = 0$, $h_\mu^\theta = 0$. The fundamental torsion constraint is modelled on the algebra we seek to realize, and this takes the form

$$T_{a\theta}^i = \delta_a^i ; \quad (23)$$

all other torsion components vanish.

Let us now turn to the details of the Chern-Simons theory. Our goal is to realize the combined BRST and anti-SUSY in this theory. A glance then at the known transformation law of A_μ , together with (22) applied to the Yang-Mills connection,

$$\delta A_\mu^I = -h_\mu^i \epsilon^\theta F_{i\theta}^I - h_\mu^i \epsilon^a F_{ia}^I \quad (24)$$

immediately dictates two of the super Yang-Mills curvature components. In writing the above equation, we have cast out the term involving the local Yang-Mills parameter ϵ^I . This follows, since we will choose non-covariant constraints on the supercurvature, and these explicitly break the local Yang-Mills symmetry. What remains of the original Yang-Mills symmetry will be retained in the BRST transformations. The full set of constraints turns out to be given by

$$\begin{aligned} F_{\theta\theta} &= 0 , \quad F_{ab} = 0 \\ F_{i\theta} &= D_i c , \quad F_{ia} = -\epsilon_{iaj} \partial^j b \\ F_{a\theta} &= A_i \delta_a^i , \end{aligned} \quad (25)$$

where $A_i \equiv h_i^\mu A_\mu$. One checks the consistency of this set through an analysis of the Bianchi identities (21). The appearance of the connection as a supercurvature component is perhaps most surprising, but can be seen through an analysis of the $(\theta, \theta, a)^I$ identity. The transformation laws of the fields entering these curvature components are given by

$$\begin{aligned} D_\theta c &= \frac{1}{2} \{c, c\} , \quad D_\theta b = d , \quad D_\theta d = 0 , \\ D_a c &= -A_i \delta_a^i , \quad D_a b = 0 , \quad D_a d = \delta_a^i \partial_i b , \end{aligned} \quad (26)$$

where $\partial_i \equiv h_i^\mu \partial_\mu$. In particular, the identities with index structure $(\theta, \theta, i)^I$ and $(a, b, i)^I$ yield respectively $D_\theta c$ and $D_a b$; closure of the algebra then fixes the remaining transformations. Finally, an analysis of the $(\theta, a, i)^I$ identity yields the following curvature constraint,

$$F_{ij} = -\epsilon_{ijk}(\partial^k d - \{\partial^k b, c\}) , \quad (27)$$

and this can be interpreted as an equation of motion arising from the quantum action S_q . For 3d BF theory, the analysis is similar and leads to the replacement $b \rightarrow b'$ and $d \rightarrow d'$ in the above equations.

Turning now to the situation in the four dimensional BF theory, a similar analysis begins with the same curvature constraints (25) with the exception of the F_{ia} component, and this is given by

$$F_{ia} = \epsilon_{iajk} \partial^j \bar{\xi}^k . \quad (28)$$

These are augmented by the additional rules

$$\begin{aligned} D_\theta c &= \frac{1}{2} \{c, c\} , \quad D_\theta \bar{\xi}_k = h_k , \quad D_\theta h_k = 0 \\ D_\theta \bar{\phi} &= \omega , \quad D_\theta \omega = 0 , \end{aligned} \quad (29)$$

which are the BRST transformations, together with the SUSY rules,

$$\begin{aligned} D_a c &= -A_i \delta_a^i , \quad D_a h_k = \delta_a^l \partial_l \bar{\xi}_k + \delta_{ak} \omega , \\ D_a \bar{\xi}_k &= -\delta_{ak} \bar{\phi} , \quad D_a \omega = \delta_a^k \partial_k \bar{\phi} , \quad D_a \bar{\phi} = 0 . \end{aligned} \quad (30)$$

Again, a constraint on the remaining curvature component follows from the Bianchi identities, and this is given by

$$F_{ij} = \epsilon_{ijkl} \partial^k h^l - \epsilon_{ijkl} \{\partial^k \bar{\xi}^l, c\} . \quad (31)$$

It is worth noting that the fields which are naturally incorporated into this superconnection formalism provide a minimal set which realizes the basic N=1 algebra. However, in order to construct the quantum action in this case, the full set of fields, as given in (7), is required.

5 Towards Vector Supergravity

The next step is to search for solutions to the Bianchi identities in which the symmetries become local. As in supergravity theories, we will at least need to include the vielbein h_μ^i and the “gravitino” h_μ^a in the picture. This requires relaxing the torsion constraints that we employed in the previous section, and one anticipates that other independent fields will arise as torsion components. There is no need for a theory which would realize a local BRST symmetry, although the formalism allows one to consider such animals, and we would then require the gauge field h_μ^θ to vanish. The transformation law for this field is contained in (22),

$$\delta h_\mu^\theta = D_\mu \epsilon^\theta - h_\mu^{B_1} \epsilon^C T_{B_1 C}^\theta , \quad (32)$$

and consistency demands that the right hand side vanish. This translates into constraints on the T^θ components of torsion. In a covariant formulation, where one does not gauge fix the local symmetries, one would require all T^θ components (except $T_{\theta\theta}^\theta$ which does not enter (32)) to vanish. However, it may be essential to choose torsion constraints which break the local symmetries, as we have seen in the Yang-Mills sector. The original global SUSY would then be seen as a remnant of this gauge fixing. A glance at the BRST transformation law of the vielbein (22),

$$\delta h_\mu^i = \epsilon^\theta \delta_a^i h_\mu^a + \dots , \quad (33)$$

encourages this line of thought. It might then be important to consider the weaker solution,

$$T_{iA_1}^\theta = -h_i^\mu h_\mu^a T_{aA_1}^\theta , \quad (34)$$

which is reminiscent of the required $F_{a\theta}$ constraint in the Chern-Simons and BF theories. Beyond this constraint, and the requirement that $T_{a\theta}^i = \delta_a^i$ to achieve the global result, there is little further guideline for finding a solution to the Bianchi identities. Fortunately, the absence of spinors significantly simplifies the situation, and one can reasonably envisage a thorough analysis of these equations.

6 Discussion

In this paper, we have reconsidered the BRST-SUSY in certain topological field theories in terms of a superconnection formalism. The Chern-Simons and BF theories can be embedded in this framework when non-covariant constraints on the supercurvature are chosen. As discussed in the introduction, the vector SUSY we have been considering is only manifestly valid on flat spacetimes. It is puzzling that a topological field theory admits a symmetry of this type, and it is natural to seek an extension of the theory which circumvents this restriction. Our construction really begs the question of whether there are more general solutions to the Bianchi identities which would realize local supersymmetry, and supplies a framework for approaching this problem. If this approach turns out to be fruitful, the solution to this puzzle will involve a coupling of these theories to some form of topological gravity [9, 14]. We hope to report on a further study of these matters.

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