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Phase Structure of $SU(2)$ -Lattice Gauge Theory with Fluctuating Coupling

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Abstract

We investigate a $SU(2)$ Wilson action with fluctuating coupling. The kinetic gluon energy is modified by a dielectric function which fluctuates from link to link. We show numerically that along a line in the plane of couplings the plaquette, the dielectric function and the Creutz ratios change drastically. We present calculations in mean field theory and in strong coupling approximation which indicate a phase transition in the same region. We discuss the use of such an action as an effective action for QCD.

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1 Introduction

Effective field theories are popular among hadron and nuclear physicists as low energy approximations to QCD. Underlying such theoretical models [1] is the idea that one dominant degree of freedom governs the dynamics of hadrons at the long distance scale. Bag-type chromodielectric [2] and colordielectric models [3] or Skyrme [4] type actions are such type of models. In the first category the colordielectric field plays the role of a confining field for the colored quarks and gluons. In the second approach of Skyrme all colored quarks and gluons have been integrated out to yield a pionic action. Because of the trace anomaly [5] a remnant of the gluon fields must supplement the chiral mesonic fields. The directional average of the color dielectric field strengths may play the role of such a dilaton field.

The couplings between the effective fields are determined by the underlying microscopic QCD theory. The main theoretical problem is to develop techniques which eliminate the high frequency gluon and quark fields in favour of the effective degrees of freedom and their corresponding couplings. For the pure gluon theory first attempts in this direction have been made [6, 7] in constructing a colordielectric lattice theory with nonunitary block links. Similar variables are used in the multigrid approach to avoid critical slow down [8]. In order to model QCD truthfully it is important to know the phase structure of the effective theory. Systems with the same space-time dimensionality and an order parameter of the same dimensionality or symmetry group belong to the same universality class [9].

In this paper we are studying whether the colordielectric theory in the strong coupling domain is separated from the UV-stable fixed point of QCD by a phase transition.

In colordielectric theory the link variables are matrices $\Phi = \chi U$, which are elements of the product of $R^+ \times SU(2)$ where $R^+ = [0, \infty]$ is the positive real half axis. The gluon kinetic energy in QCD is related to the sum over plaquettes $\beta \sum_{\square} (1 - \frac{1}{2} \text{tr} U_{\square})$. In colordielectric theory it is replaced by a sum over products of angle terms $\frac{\theta}{4} \sum_{\square} \text{tr} (A - B)(A - B)^+$ with $A = \Phi_{\mu}(x)\Phi_{\nu}(x + \hat{\mu})$ and $B = \Phi_{\nu}(x)\Phi_{\mu}(x + \hat{\nu})$, where we use the standard notation for link variables: $\Phi_{\mu}(x)$ is a colordielectric link from the point x in direction μ . In the naive continuum limit the above kinetic energy amounts to substituting $\frac{\varepsilon(x)}{2g^2} \text{tr} F_{\mu\nu} F_{\mu\nu}$ for $\frac{1}{2g^2} \text{tr} F_{\mu\nu} F_{\mu\nu}$ where $\varepsilon(x)$ is given by $\chi^4(x)$. The effective coupling constant of the dielectric theory fluctuates with the colordielectric field as $\frac{g^2}{\varepsilon(x)} = \frac{g^2}{\chi^4(x)}$.

The colordielectric model depends on the dynamics of the χ -field, which is an order parameter differentiating between the inside and outside of string or fluxtube configurations. Usually it is assumed that a large value of $\chi(x)$ characterizes the inside of the hadronic "bag" compared to a small value of $\chi(x)$ outside of the bag.

In this paper we use a simple potential energy for the χ -field in the vacuum,

$$V(\chi) = \frac{\beta^2}{\kappa^2} \sum_{\mu, x} (\chi_\mu^4(x) - \chi_0^4)^2. \quad (1)$$

In addition we will set $\chi_0^4 = 1$ without reducing the generality of the problem, since a rescaling of χ -fields can be compensated by changing β and κ . With the potential above the variance of the effective gauge coupling ($\beta\chi_\square$) is approximately given by $\kappa/(2\sqrt{2})$

$$\beta \sqrt{\langle \chi_\square^2 \rangle - \langle \chi_\square \rangle^2} \approx \kappa/(2\sqrt{2}) \quad (2)$$

This action is not exactly the action which was found after blockspinning the QCD Wilson action [7], but it shares many of its important features.

The outline of the paper is as follows. In Section 2 the model action and the measured operators are introduced. The relation to the Wilson action is discussed. In Section 3 the numerical results are presented as a function of the two free couplings β and κ . In Section 4 the numerical simulations are compared to weak and strong coupling approximations of the action. Section 5 discusses whether the action can be used as an effective theory which interpolates between the asymptotically free gauge theory at short distances and the strong coupling like behavior of QCD at long distances.

2 The Wilson action with fluctuating coupling

We generalize the Wilson $SU(2)$ -action with unitary link matrices $U = a_0 + i\vec{\sigma}\vec{a}$, $a_0^2 + \vec{a}^2 = 1$, to the case of links $\Phi = b_0 + i\vec{\sigma}\vec{b}$ with fluctuating lengths $\sqrt{b_0^2 + \vec{b}^2} < \infty$. We will also use the notation $\Phi = \chi \cdot U$ with $\chi \geq 0$. The generalized action is assumed to have the following form

$$S = \frac{\beta}{16} \sum_{\substack{\mu, x \\ \nu \perp \mu \\ \mu > 0}} (\Phi_\nu(x) \Phi_\mu(x + \hat{\nu}) - \Phi_\mu(x) \Phi_\nu(x + \hat{\mu})) \cdot (\Phi_\nu(x) \Phi_\mu(x + \hat{\nu}) - \Phi_\mu(x) \Phi_\nu(x + \hat{\mu}))^+ \quad (3)$$

$$+ \frac{\beta^2}{\kappa^2} \sum_{\substack{\mu, x \\ \mu > 0}} ((\frac{1}{2} \text{tr} \Phi_\mu(x) \Phi_\mu^+(x))^2 - 1)^2; \quad (4)$$

Here and in the following equations ν takes the values $\nu = \pm 1, \pm 2, \pm 3, \pm 4$ and is orthogonal to μ .

The partition function $Z = \int \mathcal{D}\Phi e^{-S(\Phi)}$ is defined with the measure $\mathcal{D}\Phi = \prod_{x, \mu} \chi_\mu^3(x) d\chi_\mu(x) dU_\mu(x)$. The relation to the color dielectric action S_{CD} used in ref. 6 is obtained by expanding S of eq. (4):

$$S_{CD} = - \frac{1}{2} \beta' \sum_{\square} \text{tr} \Phi_{\square} + \gamma \sum_{x, \mu} \chi_\mu^4(x) + \delta \sum_{x, \mu} \chi_\mu^8(x)$$

$$+ \frac{\lambda'}{12} \sum_{x,\mu} \chi_\mu^2(x) \cdot \sum_{\nu \perp \mu} (\chi_\nu^2(x) + \chi_\nu^2(x + \hat{\mu})) \quad (5)$$

We identify:

$$\begin{aligned} \beta' &= \beta \\ \gamma &= -2\beta^2/\kappa^2 \\ \delta &= \beta^2/\kappa^2 \\ \lambda' &= \frac{3}{2}\beta \end{aligned} \quad (6)$$

Factorizing the plaquette term $\frac{1}{2}\beta \text{tr} \Phi_\square$ of eq. 5 into a product of four unitary matrices $U_\square$ and four fluctuating χ -fields $\chi_\square$ one sees that the generalized theory corresponds to a theory with fluctuating coupling $\beta\chi_\square$.

In the renormalization group approach the authors of ref. 6 found an action similar to the form of eq. 5 starting from the Wilson action.

A typical set of parameters $\beta, \gamma, \delta, \lambda'$ obtained after blocking the $SU(2)$ -action at $\beta = 2.65$ is $\beta = 14.8$, $\gamma = -6.9$, $\delta = 16.5$, $\lambda' = \frac{3}{2}\beta$, which after rescaling the χ -variable yields $\beta = 3.1$ and $\kappa^2 = 13.2$. We note that the above parametrization does not satisfy the two commonly used confinement criteria of color dielectric theories [12]. In the Abelian approximation to dielectric theory one drops the non-quadratic couplings of the gauge fields. Then the dielectric constant $\varepsilon(\chi)$ has to be zero at the minimum of the effective χ -potential. For non-Abelian dielectric lattice theory, Mack [11] proved that a sufficient criterion for confinement is a monotonically increasing potential $V(\chi)$. It can be rather easily shown that even if $V(\chi) \equiv 0$, the λ' -term leads to a strong coupling expansion of the lattice action in eq. 5 which is trivially confining.

We have chosen the parametrization of the action in eq. 4, because the parameters can be adjusted in such a way that the generalized action is arbitrarily close to the Wilson action. For $\kappa^2 \rightarrow 0$ the fluctuations of the effective coupling constant $\beta_{eff} = \beta\chi_\square$ become vanishingly small and χ is fixed to the value of unity. Then the conventional Wilson theory with $\beta_{eff} = \beta$ is reproduced. Increasing the constant κ^2 , stronger fluctuations of the effective coupling $\beta\chi_\square$ are allowed and a new physical situation arises.

To characterize order or disorder in the system, we introduce the expectation values of the Wilson plaquette $\langle \frac{1}{2}\text{tr} \Phi_\square \rangle$, $\langle \chi^2 \rangle$ and an operator I_0 which tests for the roughening transition [13].

$$\begin{aligned} I_0 &= (q_0 - q_w)^{-1} \\ q_0 &= \langle e^{-\frac{\beta}{2}\text{tr} \Phi_\square} \rangle \\ q_w &= \langle \frac{1}{2}\text{tr} \Phi_\square e^{-\frac{\beta}{2}\text{tr} \Phi_\square} \rangle / \langle \frac{1}{2}\text{tr} \Phi_\square \rangle \end{aligned} \quad (7)$$

Notice that q_w receives contributions to the plaquette only from random surfaces which do not contain the original plaquette. In addition we will measure the Creutz ratio $C(2, 2)$ which is defined as

$$C(2, 2) = -\ln \left(\frac{W(I, J)W(I-1, J-1)}{W(I, J-1)W(I-1, J)} \right)_{\substack{I=2 \\ J=2}} \quad (8)$$

where $W(I, J)$ is a Wilson plaquette of lattice dimension I by J formed with Φ links. The χ -factors cancel out in this ratio as well as the perimeter contribution. The numerical results for these quantities are given in the next section.

3 Numerical results

In this section we present results of numerical Monte Carlo simulations of the Wilson $SU(2)$ -model with fluctuating coupling given by eqs. 4 or 5 on a 8^4 lattice. The updating of the gauge fields is done with a standard multi hit Monte Carlo method. The dielectric link fields $\chi \in (0, \infty)$ are chosen with a trial distribution P_t by rewriting

$$P(\chi) = P_t(\chi) e^{-S(\chi) - \ln P_t(\chi)} \quad (9)$$

and taking for $P_t(\chi) = e^{-\chi}$. The acceptance rate varies between 30% to 10%. We give results for two characteristic runs with $\kappa^2 = 5$ and $\kappa^2 = 10$ in tables 1 and 2. The errors in the tables are the mean errors corrected for correlations in the data using the method of Daniell, Hey and Mandula [15].

The first entry in tables 1 and 2 is the effective coupling constant $\beta_{eff} = \beta \langle \chi \rangle^4$. It varies in the numerical simulations in the interval $1.4 \leq \beta_{eff} \leq 3.8$.

The second rows in the tables show the numerical fluctuation of the effective coupling constant $\sigma(\beta\chi_\square) = \beta \sqrt{\langle \chi^8 \rangle - \langle \chi^4 \rangle^2}$. The observed fluctuations agree approximately (eq.(2)) with $\kappa/(2\sqrt{2}) = 0.8$ at $\kappa^2 = 5$ and $\kappa/(2\sqrt{2}) = 1.1$ at $\kappa^2 = 10$.

The simulations for large $\kappa^2 = 10$ show a drastic change of the $\langle \frac{1}{2} tr \Phi_\square \rangle$ and χ^2 expectation values between $\beta = 5.4$ and 5.5 , whereas the same expectation values have a smoothly varying behavior in the small $\kappa^2 = 5$ simulations. Furthermore, given the free energy

$$F \equiv -\ln \int [\mathcal{D}\Phi] \exp(-S(\{\Phi\})) , \quad (10)$$

we define the dielectric energy and heat capacity by differentiation with respect to $\alpha = \beta^2/\kappa^2$:

Table 1: Operator expectation values for $\kappa^2 = 5$, i.e. weak fluctuations in the effective coupling $\beta\chi^4$.

β	3.7	3.9	4.1	4.15	4.2	4.25	4.45	4.65
$\beta < \chi >^4$	1.432(2)	1.637(2)	1.983(6)	2.22(1)	2.77(1)	2.896(5)	3.273(2)	3.584(3)
$\sigma(\beta\chi_\square)$	0.7619(6)	0.8156(5)	0.8723(7)	0.8820(6)	0.842(2)	0.8367(7)	0.814(1)	0.8723(7)
$< \frac{1}{2}tr\Phi_\square >$	0.1581(5)	0.1906(4)	0.257(2)	0.316(3)	0.457(1)	0.486(1)	0.5526(4)	0.5972(6)
$< \chi^2 >$	0.6622(4)	0.6848(3)	0.7269(9)	0.760(2)	0.8287(6)	0.8425(5)	0.8712(2)	0.8891(3)
E_κ	0.3774(4)	0.3489(3)	0.3001(9)	0.266(1)	0.195(1)	0.1831(4)	0.1534(2)	0.1339(3)
$c_\kappa \cdot 10^5$	0.5(1)	0.6(1)	0.6(2)	0.47(6)	0.30(4)	0.32(4)	0.19(3)	0.17(4)
$Cr(2, 2)$	0.98(2)	0.87(1)	0.637(6)	0.455(6)	0.199(2)	0.1653(9)	0.1239(5)	0.10315
I_0	1.415(5)	1.5934(6)	2.11(2)	2.82(4)	7.4(1)	9.00(8)	15.56(8)	23.7(2)

Table 2: Operator expectation values for $\kappa^2 = 10$, i.e. strong fluctuations in the effective coupling $\beta\chi^4$.

β	5.2	5.35	5.4 ⁻	5.5 ⁻	5.4 ⁺	5.5 ⁺	5.6	5.65
$\beta < \chi >^4$	1.496(2)	1.622(6)	1.675(2)	1.813(6)	3.26(1)	3.534(5)	3.743(6)	3.837(5)
$\sigma(\beta\chi_\square)$	0.9159(7)	0.965(2)	0.9825(6)	1.024(1)	1.0754(9)	1.0713(9)	1.0700(6)	1.069(1)
$< \frac{1}{2}tr\Phi_\square >$	0.1271(4)	0.1416(5)	0.1497(4)	0.1700(4)	0.4521(1)	0.4966(6)	0.5268(6)	0.5399(6)
$< \chi^2 >$	0.5807(4)	0.5927(4)	0.5994(4)	0.6157(3)	0.7961(6)	0.8190(3)	0.8340(3)	0.8404(3)
E_κ	0.4632(4)	0.447(1)	0.4402(3)	0.4211(7)	0.228(1)	0.2058(3)	0.1905(3)	0.1844(4)
$c_\kappa \cdot 10^5$	0.50(7)	0.6(1)	0.81(8)	1.0(1)	0.35(8)	0.55(6)	0.43(6)	0.32(7)
$Cr(2, 2)$	0.96(1)	0.878(7)	0.841(4)	0.745(5)	0.136(1)	0.1131(4)	0.1004(3)	0.0949(1)
I_0	1.454(3)	1.562(9)	1.612(2)	1.772(4)	11.1(2)	15.7(1)	20.4(1)	22.9(2)

$$\begin{aligned}
E_\kappa &\equiv \frac{\partial}{\partial \alpha} F \\
&= < \sum_{x,\mu} (\chi_\mu^4(x) - 1)^2 >
\end{aligned} \tag{11}$$

$$\begin{aligned}
c_\kappa &\equiv -\frac{\partial^2}{\partial \alpha^2} F \\
&= + < \left(\sum_{x,\mu} (\chi_\mu^4(x) - 1)^2 \right)^2 > - < \sum_{x,\mu} (\chi_\mu^4(x) - 1)^2 >^2 .
\end{aligned} \tag{12}$$

The heat capacity gives a rather clear maximum for $\kappa^2 = 10$, whereas for $\kappa^2 = 5$ it only shows a bump (c.f. fig. 1). This again indicates a phase transition for $\kappa^2 = 10$. It is expected that there is no phase change for $\kappa^2 \rightarrow 0$, because the fluctuating coupling theory freezes to the Wilson theory which does not have a phase transition in β . Allowing larger fluctuations in the effective coupling $\beta\chi^4$ may induce in the Wilson theory a phase transition. At $\beta_{eff} \approx 2$ and $\kappa^2 = 10$ we find two metastable configurations for $\beta = 5.4$ and $\beta = 5.5$ indicated in table 2 by $\beta = 5.4^\mp$ and $\beta = 5.5^\mp$, where the superscript minus (plus) indicates the phase with smaller (larger) expectation value $< \chi >$ of the dielectric field at the phase transition. The metastable states persist for more than 25000 updates as shown in fig. 2. Similar signatures of metastability for $\kappa^2 = 10$ and $\kappa^2 = 36$ are produced

when we change the coupling constant β by $\Delta\beta = 0.03$ after every 16th Monte Carlo step from the disordered (small β) phase to the ordered phase and back. The resulting hysteresis curves are shown in fig. 3. With decreasing κ^2 the area under the hysteresis curve becomes smaller. Since the mean field analysis in section 4 gives a first order transition line near the same (β, κ^2) values as the Monte Carlo simulations, we think that the drastic changes we observe are not an artefact of our small lattice size. In fig. 7 we show the points determined from the center of the hysteresis loops at $\kappa^2 = 5$, $\kappa^2 = 10$ and $\kappa^2 = 36$. We conjecture that these points are part of a phase transition line with a first order transition and an end point of second order. We will elaborate more on this point in section 5. The 8^4 -lattice is too small to measure the string tension. But in the Creutz ratios $C(2,2)$ of table 2 we see a strong decrease from a large value $C(2,2) = 0.841$ to a small value of $C(2,2) = 0.136$ in the two phases at $\beta = 5.4$. This agrees with the intuitive notion of a strong coupling phase with a large string tension below the transition and a weak coupling phase above the transition line.

The last variable $I_0 = \frac{1}{q_0 - q_w}$ (cf. eqs. 7) tests roughening near the phase transition. It changes from 1.612 for $\beta = 5.4^-$ to 11.1 for $\beta = 5.4^+$ at $\kappa^2 = 10$. In the case $\kappa^2 = 5$, I_0 varies much more smoothly. With the transition to the weak coupling phase the strings populating the vacuum become longer and they map out rougher surfaces.

4 Analytical weak and strong coupling calculations

a) The mean field approximation

In the spirit of mean field theory we consider those field configurations as important where each link interacts with all other links via some background fields $h = (h_1, h_2)$. The action $S\{\Phi\}$ is approximated by an action $S_h\{\Phi\}$ containing the background fields. The mean field method is in general quite adequate for gauge theories in the weak coupling approximation. It is well known that it is a saddle point approximation and underestimates the importance of fluctuations. Therefore the phase structure of the simple mean field approximation may be misleading. But in many multiparameter plaquette actions it gives a good educated guess of the phase diagram.

We rewrite the partition function Z belonging to the action $S\{\Phi\}$ using the approximate background field action $S_h\{\Phi\}$:

$$\begin{aligned} Z &= \int \mathcal{D}\Phi e^{-(S\{\Phi\} + S_h\{\Phi\} - S_h\{\Phi\})} \\ &= \int \mathcal{D}\Phi e^{-S_h} \cdot \frac{\int \mathcal{D}\Phi e^{-S + S_h} e^{-S_h}}{\int \mathcal{D}\Phi e^{-S_h}} \\ &= Z_h \langle e^{-S + S_h} \rangle_h \end{aligned} \quad (13)$$

Using the convexity of the exponential function $\langle \exp X \rangle \geq \exp \langle X \rangle$, one obtains an upper limit for the free energy $F = -\ln Z$

$$F \leq F_h + \langle S - S_h \rangle_h \equiv F_{mf}(h) \quad (14)$$

The optimal background fields h are determined by minimising $F_{mf}(h)$. A possible phase transition is indicated when the minimum F_{mf} jumps between different values of h for a given set of coupling constants β and κ^2 .

We choose as background fields

$$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 6\beta M^3 \\ -2\lambda' A \end{pmatrix} \quad (15)$$

These are the terms to which a given link $\Phi_\nu(x)$ is coupled via the plaquette and angle term. The approximate action $S_h\{\Phi\}$ is given as:

$$S_h\{\Phi\} = \sum_{x,\nu} \left[-\frac{\beta}{2} \text{tr}(6\hat{M}^3 \Phi_\nu(x)) + \frac{\beta^2}{\kappa^2} (\chi_\nu^4(x) - 1)^2 + 2\lambda' A \chi_\nu^2(x) \right] \quad (16)$$

The evaluation of $\langle S - S_h \rangle_h$ is simplified by assuming that $\hat{M} = M \cdot 1$ and consequently $\langle \Phi \rangle = k \cdot 1$; $k \in R$. The mean field free energy F_{mf}/N_l per link is obtained

by a one link expectation value

$$\begin{aligned} \frac{F_{mf}}{N_t} = & -\frac{1}{N_t} \ln \int \mathcal{D}\Phi e^{-S_h\{\Phi\}} \\ & - \beta \langle \frac{1}{2} tr \Phi \rangle_h (\frac{3}{2} \langle \frac{1}{2} tr \Phi \rangle_h^3 - 6M^3) \\ & + \lambda' \langle \chi^2 \rangle_h (\langle \chi^2 \rangle_h - 2A). \end{aligned} \quad (17)$$

The condition for an extremum of F_{mf} becomes

$$\left(6\beta(\langle \frac{1}{2} tr \Phi \rangle_h^3 - M^3), 2\lambda'(-\langle \chi^2 \rangle_h + A) \right) \begin{pmatrix} \frac{\partial \langle \frac{1}{2} tr \Phi \rangle_h}{\partial h_1} & \frac{\partial \langle \frac{1}{2} tr \Phi \rangle_h}{\partial h_2} \\ \frac{\partial \langle \chi^2 \rangle_h}{\partial h_1} & \frac{\partial \langle \chi^2 \rangle_h}{\partial h_2} \end{pmatrix} = 0, \quad (18)$$

and eq.18 leads to the following requirements

$$g_1(M, A) = \langle \frac{1}{2} tr \Phi \rangle_h - M = 0; \quad (19)$$

$$g_2(M, A) = \langle \chi^2 \rangle_h - A = 0; \quad (20)$$

Since the expectation values $\langle 1/2 tr \Phi \rangle_h$ and $\langle \chi^2 \rangle_h$ depend on the background fields M and A , eqs.(19) and (20) are selfconsistency equations for M and A . They always have a trivial solution given by $M = 0$, $A \neq 0$. For $\beta \rightarrow 0$ this solution is the only one. For larger β also non-trivial solutions exist. To see how these solutions appear let us plot $g_1(M, A) = 0$ and $g_2(M, A) = 0$ in the (M, A) plane for $\kappa^2 = 5.0$ and $\beta = 4.4$ (see fig.4a). In this case we clearly have only the trivial solution $M = 0$ and $A = 0.55$. If we keep κ^2 fixed and let β increase we get to the situation shown in fig.4b for $\beta = 4.7$, i.e. we see that a new non-trivial solution appears. Increasing β further gives two non-trivial solutions (see figs.4c,d).

In fig.5 we plot the free energies F_i of three different solutions (M_i, A_i) , $i = 0, 1, 2$, for fixed $\kappa^2 = 5.0$ and varying β . The trivial solution $M_0 = 0$ and $A_0 = 0$ still has a lower free energy F_0 at $\beta \leq 5.67$ than the two non-trivial solutions which appear at $\beta = 4.8$. Judging from the mean field free energy we would locate the phase transition at $\beta = 5.67$. The non-trivial solution (M_1, A_1) with small $\langle \chi^2 \rangle_h = A$ has always a higher free energy than the solution (M_2, A_2) and represents a local maximum. The situation of the fluctuating coupling theory is very similar to the mean field solutions of pure Wilson gauge theory [16]

b) Strong coupling expansion for small β

For small $\beta \rightarrow 0$ the mean field approximation for the gauge fields $\Phi = M \cdot 1$ breaks down, only $M = 0$ solves the consistency equation. In reality the gauge fields U of $\Phi = \chi U$ fluctuate strongly. In this region one can expand the integration over gauge fields U as a function of β and still keep the mean field approximation for the dielectric fields χ . We split the action as in eq. 5 into a term $S_\square$ containing the plaquettes and S_χ containing only the χ -fields:

$$S \equiv S_\square + S_\chi$$

$$\begin{aligned}
&= -\frac{1}{2} \sum_{\square} \beta \chi_{\square} \text{tr} U_{\square} + \frac{\beta^2}{\kappa^2} \sum_{x, \mu} (\chi_{\mu}^4(x) - 1)^2 \\
&\quad + \frac{3}{2} \beta \sum_{x, \mu} \frac{1}{12} \chi_{\mu}^2(x) \sum_{\nu \perp \mu} (\chi_{\nu}^2(x) + \chi_{\nu}^2(x + \hat{\mu}))
\end{aligned} \tag{21}$$

$$(22)$$

The integration over the U -links is done as in conventional strong coupling theory via the character expansion. We define the effective gauge coupling

$$\beta_{eff}(\chi_{\square}) = \beta \chi_{\square} \tag{23}$$

Then the plaquette part of the action can be expanded in a sum over characters $\hat{\chi}_j(U_{\square})$. Conventionally the character $\hat{\chi}_j(U)$ of a $(2j+1)$ -dim representation of U is defined as the trace of U in the representation D_j ,

$$\hat{\chi}_j(U) = \text{tr} D_j(U) \tag{24}$$

(e.g. $\text{tr} D_0 = 1$, $\text{tr} D_{1/2}(U) = 2 \cos(\theta/2)$ etc.) The characters $\hat{\chi}_j$ should not be mixed up with the link field $\chi_{\mu}(x)$. The expansion of the action becomes

$$\exp\left[\frac{1}{2} \beta_{eff} \text{tr} U_{\square}\right] = \sum_{j=0, 1/2, 1, \dots} (2j+1) c_j(\beta_{eff}) \hat{\chi}_j(U_{\square}) \tag{25}$$

where the coefficients c_j are obtained via the orthogonality relation of the characters as

$$c_j(\beta_{eff}) = \frac{2}{\beta_{eff}} I_{2j+1}(\beta_{eff}) \tag{26}$$

with the modified Bessel functions I_{2j+1} . It is advantageous to separate off the first term and define

$$\exp\left[\frac{1}{2} \beta_{eff} \chi_{1/2}(U)\right] = c_0(\beta_{eff}) \left[1 + \sum_{j=1/2, 1, \dots} b_j(\beta_{eff}) \hat{\chi}_j(U_{\square})\right] \tag{27}$$

with

$$b_j(\beta_{eff}) = (2j+1) \frac{I_{2j+1}(\beta_{eff})}{I_1(\beta_{eff})}. \tag{28}$$

The expansion coefficients $c_0(\beta_{eff})$ and $b_j(\beta_{eff})$ depend in the dielectric theory on the dielectric fields at the position of the individual plaquettes on the lattice. We try a Landau-approximation for the dielectric fields and set all fields $\chi_{\mu}(x) \equiv \chi$. For the calculation of expectation values it is useful to discuss operators $f(\chi)$ which depend only on χ separately from operators which depend on χ and the gauge field U .

In the first case the integral over gauge fields is similar to the conventional strong coupling expansion of the partition function for $\beta_{eff}(\chi^4) = \beta\chi^4$.

$$\begin{aligned} \frac{1}{N_L} \ln Z_U(\beta_{eff}(\chi^4)) &= \frac{d-1}{2} \ln c_0(\beta_{eff}) + \frac{(d-1)(d-2)}{6} \frac{1}{16} b_{1/2}^6(\beta_{eff}) + \dots \\ &= \frac{3}{16} \beta_{eff}^2 - \frac{1}{256} \beta_{eff}^4 - \dots \end{aligned} \quad (29)$$

The expectation values for $f(\chi)$ are calculated with the effective action composed of the plaquette part eq. (29) and the rest of the action S_χ for constant χ (eq. 21) and phase space

$$S_\chi^{eff} = -\frac{\ln Z_U(\beta_{eff}(\chi^4))}{N_L} + \frac{\beta^2}{\kappa^2} (\chi^4 - 1)^2 + \frac{3}{2} \beta \chi^4 - 3 \ln \chi \quad (30)$$

$$\langle f(\chi) \rangle_{eff} = \frac{\int_0^\infty d\chi f(\chi) e^{-S_\chi^{eff} \cdot N_L}}{\int_0^\infty d\chi e^{-S_\chi^{eff} \cdot N_L}} \quad (31)$$

This expression agrees with the one of ref. [7]. For the gauge field observable $\langle 1/2 \text{tr} U_\square \rangle$, only the same plaquette $U_\square$ in the exponential contributes up to order β_{eff}^5 ,

$$\langle \frac{1}{2} \text{tr} U_\square \rangle = \langle \frac{1}{2} b_{1/2}(\beta_{eff}) \rangle_{eff} \approx \langle \frac{1}{4} \beta \chi^4 - \frac{\beta^3 \chi^{12}}{96} \rangle_{eff} \quad (32)$$

As the argument of the exponential in eq.31 is proportional to the number of links, the integral can be well approximated by the value of the integrand at the minimum of S_χ^{eff} . Let this minimum be χ_{min} , then the strong coupling expansion of the fluctuating gauge theory is identical to the same expansion in pure gauge theory with the coupling

$$\beta_{eff}(\chi_{min}^4) = \beta \cdot \chi_{min}^4. \quad (33)$$

As one can see from a comparison of the mean field results with the Monte Carlo simulations in fig. 6 the accuracy of the lowest order approximations of the gauge terms in the action reproduces the exact results for $\langle \frac{1}{2} \text{tr} \Phi_\square \rangle$ rather well in the weak and strong coupling limits. In fig. 7 we show the phase transition line obtained in the strong and weak coupling approximations. The strong coupling calculation corresponds to the line $\beta_{eff}(\chi_{min}^4) = 2$.

5 Discussion of the results

In various papers [6, 7, 11] on color dielectric theory an action of the form given in Section 2 was proposed as an effective theory to interpolate between the Wilson theory for weak coupling and the strong coupling approximation. We think our investigation has shown that the action with fluctuating coupling has certain desirable properties but also a phase structure which may cause problems in renormalization

group calculations. Among the pleasant features of the Wilson theory with fluctuating coupling is the explicit parametrization of the “order” parameter χ , which distinguishes the ordered state $\chi = 1$ from the disordered vacuum $\chi \ll 1$. In the Monte Carlo renormalization group (MCRG) [7] the magnitude of χ on the coarse lattice of size $2a$ depends on the energy density of the field strength tensor on the small lattice of size a

$$1 - \chi^2 \propto \frac{1}{2g^2} \text{tr} F_{\mu\nu} F_{\mu\nu}. \quad (34)$$

The build-up of a nonperturbative vacuum produces a χ value smaller than unity. The mean field calculation in weak coupling (cf. Appendix A) $\beta \rightarrow \infty$ shows that due to the phase space factor $\chi^3 d\chi$ and the entropy the minimum of χ^4 is taken at $\chi_{min}^4 \approx 1 - \frac{3}{8} \frac{\kappa^2}{\beta^2} + 0\left(\frac{1}{\beta^4}\right)$ different from unity. A MCRG-calculation must reproduce $\langle \chi^4 \rangle$ and the correct value of $\langle \frac{1}{2} \text{tr} \Phi_{\square} \rangle$ with two constants, therefore the χ -fluctuations may not be well fitted. In the asymptotic freedom regime it may be better to include an extra term $\Delta S = -\frac{3}{4} \ln \chi^4$ in the action, such that the $\chi_{min}^4 \approx 1 + 0\left(\frac{1}{\beta^4}\right)$ independent of (β, κ) to a higher order in $1/\beta$.

The main result of our work, however, concerns the phase structure of the Wilson theory with fluctuating coupling. When the effective coupling $\beta_{eff} = \beta \langle \chi \rangle^4$ approaches $\beta \langle \chi \rangle^4 \approx 2$ and $\kappa^2 > 5$, there are strong indications for a first order phase transition. It is quite usual that multiparameter plaquette actions have 1st order transitions like e.g. the Wilson theory with the plaquette energy as a sum of plaquettes in the fundamental and adjoint representations. This phase transition is disturbing for numerical work in the MCRG. The coupling constants of the effective theory have a tendency to swing to either the weak or strong coupling side of the two phases. The resulting effective operator expectation values then may not match well with the operators of the blocked theory. The presented simulations show no phase transition for $\kappa^2 \leq 5$. This finding allows to connect the strong coupling domain of color dielectric theory to the weak coupling region without crossing the phase transition line. For a blockspin with small enough fluctuations one may successfully block Wilson theory to an effective strong coupling dielectric theory (c.f. fig. 8). Moreover, it may be that a continuum color dielectric theory with a color neutral “glueball” degree of freedom may be defined at the endpoint of the phase transition line. This result would then be the desired dilaton action for an effective hadron theory [5].

Appendix A

In this appendix we represent an analytical calculation of the dielectric field strength $A = \langle \chi^2 \rangle$. The mean field free energy at the selfconsistency point (eqs. (19,20)) is given as:

$$\begin{aligned} \frac{F_{mf}}{N_t} = & - \ln \int_0^\infty \chi^3 d\chi \int DU e^{+6\beta M^3 \chi (\frac{1}{2} tr U) - \frac{\theta^2}{\kappa^2} (\chi^4 - 1)^2 - \chi^2 \cdot 2\lambda' A} \\ & + \frac{9}{2} \beta \cdot M^4 - \lambda' A^2. \end{aligned} \quad (35)$$

The gauge field can be integrated out and one obtains an effective action depending on χ alone.

$$\begin{aligned} I = & - \ln \int d\chi e^{\ln \left[\frac{2}{\beta_{eff}(\chi)} I_1(\beta_{eff}(\chi)) \right] - \frac{\theta^2}{\kappa^2} (\chi^4 - 1)^2 - 2\lambda' \chi^2 A + 3 \ln \chi} \\ = & - \ln \int d\chi e^{-S_h(\chi)}, \end{aligned} \quad (36)$$

with $\beta_{eff}(\chi) = 6\beta M^3 \chi$. The extremum of the integrand is defined by $\frac{\partial S_h}{\partial \chi} \big|_{\chi=\sqrt{A}} = 0$

$$\begin{aligned} \frac{3}{\sqrt{A}} - 4\lambda' A^{3/2} - \frac{2\beta'^2}{\kappa^2} (A^2 - 1) 4A^{3/2} + \frac{I_2(\beta_{eff}(\sqrt{A}))}{I_1(\beta_{eff}(\sqrt{A}))} 6\beta M^3 = 0 \\ 3 - 4\lambda' A^2 - \frac{8\beta'^2}{\kappa^2} (A^2 - 1) A^2 + \frac{I_2(\beta_{eff}\sqrt{A})}{I_1(\beta_{eff}\sqrt{A})} = 0. \end{aligned} \quad (37)$$

This equation (37) replaces (20). For the other self-consistency equation (19) we approximate the χ -integration by the value of the integrand at $\chi = \sqrt{A}$ and get

$$M = \langle \frac{1}{2} tr \Phi \rangle = \sqrt{A} \langle \frac{1}{2} tr U \rangle = \sqrt{A} \frac{I_2(\beta_0(A))}{I_1(\beta_0(A))} \quad (38)$$

with $\beta_0(A) = 6\beta M_0 A^2$ and $M_0 \equiv M/\sqrt{A}$.

This equation always has a trivial solution $M_0 = 0$. In the weak coupling region $\beta A^2 \gg 1$ there is an additional solution which one can determine approximately by using the series expansion of the Bessel functions. One finds a solution for the self-consistency equation for $M_0 = \langle \frac{1}{2} tr U \rangle = 1 - \epsilon$ by determining ϵ as a power series in $(1/\beta A^2)$:

$$M_0 = 1 - \left(\frac{1}{4\beta A^2} + \frac{17}{96\beta^2 A^4} + \frac{241}{1152\beta^3 A^6} + \dots \right). \quad (39)$$

The resulting expression for $M = \sqrt{A} M_0$ is inserted into (37), thus determining A as a function of β and κ^2 . The mean field free energies for $M_0 \neq 0$ and $M_0 = 0$ are

compared for large β with the help of the formulas (35-39). The energy ϵ and the entropy σ are given with the density matrix ρ in mean field approximation

$$\rho = e^{-S_h} / \text{tr } e^{-S_h} \quad (40)$$

$$\begin{aligned} \epsilon &= \text{tr } \rho S = \text{tr}(e^{-S_h} S) / (\text{tr } e^{-S_h}) = \\ &= +\frac{3}{2}\beta A^2(1 - M_0^4) + \frac{\beta^2}{\kappa^2}(A^2 - 1)^2 - \frac{3}{2}\ln A; \end{aligned} \quad (41)$$

$$\begin{aligned} \sigma &= -\text{tr}(\rho \ln \rho) = \text{tr}(S_h e^{-S_h}) / (\text{tr } e^{-S_h}) + \ln \text{tr } e^{-S_h} = \\ &= -6\beta A^2 M_0^4 + \ln \left(\frac{2}{\beta_{eff}(\sqrt{A})} I_1(\beta_{eff}\sqrt{A}) \right). \end{aligned} \quad (42)$$

In eq. (41,42) the integral (36) has been approximated by the saddle point. The mean field free energy becomes:

$$\frac{F_{mf}}{N_e} = \epsilon - \sigma. \quad (43)$$

The limit $\beta \rightarrow \infty$ corresponds to the ordered state. In this limit the free energy has the form

$$\frac{F_{mf}}{N_e} = \frac{\beta^2}{\kappa^2}(A^2 - 1)^2 + \frac{3}{4}\ln A^2. \quad (44)$$

The minimum of F_{mf} gives with $A^2 = \chi^4$:

$$\langle \chi^4 \rangle \approx 1 - \frac{3}{8} \frac{\kappa^2}{\beta^2}. \quad (45)$$

This means $\chi = 1$ corresponds to the most ordered state and $\chi \ll 1$ signals disorder. Note that the entropy may be negative at β finite, but it is always increasing from $\beta \rightarrow \infty$ towards finite β .

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Figure captions

Figure 1: The energy E_κ and the heat capacity c_κ as a function of β . For the heat capacity a characteristic error is drawn.

Figure 2: Equilibration of the plaquette expectation value at the critical point as a function of the number of Monte Carlo steps. Please note the different scales for the x-axis. For $\kappa^2 = 10$ some measurements have been skipped, therefore the curves are interrupted.

Figure 3: Hysteresis curves for various β and κ^2 . Each β was equilibrated during 16 updating steps and then changed by $\Delta\beta = 0.03$.

Figure 4: Solutions $M \neq 0$ of the mean field equations. The drawn curves give the solutions of the equation $M = \langle \frac{1}{2} \text{tr } \Phi \rangle_h$, and the dashed curves the solutions of the equation $A = \langle \chi^2 \rangle_h$.

Figure 5: The local minimum (dash dotted line) and maximum (dashed line) of the free energy. The drawn line corresponds to $M = 0$.

Figure 6: Comparison of the Monte Carlo results (crosses) with the strong coupling solution and mean field solutions (drawn lines).

Figure 7: The drawn curve shows the mean field calculation of the critical line $\beta = \beta(\kappa)$, the dashed line the strong coupling approximation. The crosses are obtained from the Monte Carlo simulations.

Figure 8: Schematic phase structure of color dielectric theory in the (β, κ) plane. The arrow indicates a possible renormalization group trajectory.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8