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Gauss Law and Charged Fields in the Presence of a Chern-Simons Term

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Abstract

Some characteristic features of the Gauss law in 2+1 space-time dimensions and in the presence of a Chern-Simons term are studied. Physical charged fields localized in space-like cones and carrying both electric charge and magnetic flux are explicitly constructed. It is shown that these fields in general obey Abelian braid statistics. The associated statistics parameter is also derived.

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In this paper we investigate the physical charged sectors of an Abelian 2 + 1-dimensional gauge theory coupled with matter and in the presence of a Chern-Simons (CS) term. We explicitly construct observables satisfying Gauss law and determine their electric charge Q and magnetic flux Φ . Afterwards we compute the exchange phase factor among these observables and establish the dependence of the associated statistics parameter on the spectra of Q and Φ . A possible application to electron systems on the plane is also indicated.

In what follows we focus on spinor QED₃ with a CS term, but the inclusion of other matter fields can be done straightforwardly. The classical Lagrangian density reads¹

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mu\epsilon_{\mu\nu\rho}\partial^\mu A^\nu A^\rho - B\partial^\mu A_\mu + \frac{1}{2\xi}B^2 + \bar{\psi}(i\partial - e\hat{A} - m)\psi, \quad (1)$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu,$$

ξ is a non-vanishing real constant and the dimensions of the parameters μ , m and e in units of mass M are $[\mu] = [m] = M$ and $[e] = \sqrt{M}$. The density $\mathcal{L}$ is a non-degenerate functional of the *basic* fields A_μ , B , ψ and $\bar{\psi}$; as canonical conjugate pairs one can choose for example

$$A_\mu, \quad \pi^\mu \equiv F^{\mu 0} + \mu\epsilon^{0\mu\nu}A_\nu - g^{\mu 0}B; \quad (2)$$

$$\psi, \quad \pi \equiv i\psi^\dagger. \quad (3)$$

We postulate that (2,3) satisfy the standard canonical equal-time Poisson brackets (PB's), which are symmetric for the Grassmann variables ψ and $\psi^\dagger$.

The equations of motion following from $\mathcal{L}$ are

$$\partial^\nu F_{\nu\mu} + \mu\tilde{F}_\mu + \partial_\mu B = e\bar{\psi}\gamma_\mu\psi \equiv j_\mu, \quad (4)$$

$$B = \xi\partial^\mu A_\mu, \quad (5)$$

$$(i\partial^\mu - eA^\mu)\gamma_\mu\psi - m\psi = (i\partial^\mu + eA^\mu)\bar{\psi}\gamma_\mu + m\bar{\psi} = 0, \quad (6)$$

where

$$\tilde{F}_\mu \equiv \epsilon_{\mu\nu\rho}F^{\nu\rho}.$$

As is well known, eqs.(6) imply

$$\partial^\mu j_\mu = 0, \quad (7)$$

¹ We adopt the following conventions: $\text{diag } g_{\mu\nu} = (+ \dots -)$ and $\epsilon_{012} = 1$.

which reflects the invariance of $\mathcal{L}$ under global $U(1)$ transformations. The conserved charge N , associated with j_0 , is the so-called global $U(1)$ -charge. Taking the divergence of eq.(4) and using eq.(7), one gets

$$\partial^\mu \partial_\mu B = 0. \quad (8)$$

Therefore, in spite of the non-trivial dynamics described by $\mathcal{L}$, the gauge-fixing Lagrange multiplier B remains a free field. This fact permits us to compute the exact (non-equal-time) PB's of B with the other basic fields.² One finds

$$\begin{aligned} \{B(x), A_\mu(y)\} &= \partial_\mu D(x-y), \\ \{B(x), \psi(y)\} &= ieD(x-y)\psi(y), \\ \{B(x), \bar{\psi}(y)\} &= -ieD(x-y)\bar{\psi}(y), \end{aligned} \quad (9)$$

where the Pauli-Jordan commutator function in 2+1 dimensions reads

$$D(x) = \frac{1}{2\pi\sqrt{x^2}}\epsilon(x_0)\theta(x^2).$$

Consider now the configuration space $\mathcal{C}$ generated by the basic fields. According to the theory of constrained systems, the physical submanifold $\mathcal{M} \subset \mathcal{C}$ is given by the hypersurface defined by

$$B = 0. \quad (10)$$

In what follows, all equations which are valid only on $\mathcal{M}$ are called "weak", and are denoted by $\approx$. Owing to eqs.(4,10), the Maxwell equation

$$\partial^\nu F_{\nu\mu} + \mu\tilde{F}_\mu \approx j_\mu \quad (11)$$

is a weak one. Eq.(11) respects the general rule of gauge theories, being a weak equality between an identically conserved (topological) current (the left-hand side of (11)) and a dynamically conserved current, which is j_μ in this case. The novelty in the presence of a CS term is that the topological current involves two different identically conserved contributions:

² As a consequence of eq.(8), any such bracket is a solution of the Cauchy problem for $\partial^\mu \partial_\mu$ with initial conditions provided by the canonical equal-time PB's.

(i) the electric current

$$k_\mu \equiv \partial^\nu F_{\nu\mu} \quad (12)$$

with which one associates the electric charge Q ;

(ii) the current

$$l_\mu \equiv \mu \tilde{F}_\mu \quad (13)$$

whose zero component gives rise to the magnetic flux charge Φ .

The integrals of motion Q , Φ and N satisfy the Gauss law, which according to eq.(11) reads

$$Q + \Phi \approx N \quad (14)$$

The counterpart of eq.(14) in the absence of a CS term is $Q \approx N$, and we are accustomed in this case (e.g. QED₄) to identify (on-shell) the electric charge Q with the global $U(1)$ -charge N . Eq.(14) shows that in the presence of a CS term such an identification is in general no longer possible; Q differs from N when the flux charge Φ is non-vanishing.

The first question arising at this stage concerns the spectrum of Q and Φ . Clearly, since we are dealing with a gauge theory, it is meaningful to pose this question for observables, i.e. on the submanifold $\mathcal{M}$. We emphasize this point for the following reason. Already from the pioneering works of Deser et al. [1] and Hagen [2], it is known that $\mathcal{L}$ exhibits a topological mass generation. More precisely, the field A_μ gives rise to *one* physical degree of freedom which is massive.³ Therefore one may be tempted to conclude that observables in QED₃ with a CS term have vanishing electric charge Q , the reason being that Q measures the flux at space-like infinity of the exponentially decaying electric field F_{i0} . Note however that this argument applies only for observables which are localizable in bounded domains of space-time. In analogy with QED₄ (see e.g. [3,4]), the physical charged sectors of the model we are considering are generated by fields whose support cannot be localized in a bounded region of space-time but rather extends to infinity in space-like direction. Such fields are constructed explicitly in what follows and it is shown that in general they carry a non-trivial charge Q .

Since observables will play an essential role in our discussion, we recall in passing that a functional $\mathcal{O}$ of the basic fields is an observable iff⁴

$$\{B(x), \mathcal{O}(y)\} \approx 0 \quad (15)$$

³ A_μ should be distinguished from the standard massive Abelian gauge field in $2+1$ dimensions, which has *two* massive physical degrees of freedom.

⁴ The BRS approach offers an alternative but equivalent definition of $\mathcal{A}$.

On the classical level, the observables form an associative algebra $\mathcal{A}$ with respect to the pointwise multiplication of c-number functions. By means of eqs.(9) one easily verifies that $F_{\mu\nu}$, j_μ , k_μ , l_μ belong to $\mathcal{A}$, but A_μ , ψ , $\bar{\psi} \notin \mathcal{A}$. An observable $\mathcal{O}$ is local iff

$$\{\mathcal{O}(x), \mathcal{O}(y)\} = 0 \quad \text{for } (x-y)^2 < 0 \quad .$$

Moreover, $\mathcal{O}_{1,2} \in \mathcal{A}$ are called "relatively local" iff

$$\{\mathcal{O}_1(x), \mathcal{O}_2(y)\} = 0 \quad \text{for } (x-y)^2 < 0 \quad .$$

We denote by $\mathcal{A}_{\text{loc}}$ the subalgebra of local observables which are relatively local among each other. We shall see in a while that in our model $\mathcal{A}_{\text{loc}}$ is properly included in $\mathcal{A}$.

Coming back to the spectra of Q and Φ , we may first ask if there are observables carrying a non-trivial charge N and what they look like. In the case of QED₄ this question has been answered in a beautiful paper [3] by Dirac. Following [3] we define the field

$$\Psi(x, a^\mu) \equiv \psi(x) \exp \left[ie \int d^3 y a^\mu(x-y) A_\mu(y) \right] \quad , \quad (16)$$

where a^μ is a real-valued (tempered) distribution obeying

$$\partial_\mu a^\mu(x) = \delta(x) \quad . \quad (17)$$

Denote by $\bar{\Psi}$ the Dirac conjugate to Ψ . Using eqs.(9) one easily verifies that

$$\{B(x), \Psi(y; a^\mu)\} = \{B(x), \bar{\Psi}(y; a^\mu)\} = 0 \quad ,$$

i.e. $\Psi, \bar{\Psi} \in \mathcal{A}$. It is also obvious that under global $U(1)$ transformations Ψ acquires the same phase factor as ψ . This means that ψ and $\bar{\psi}$ have the same charge N , which equals $-e$ with our conventions.

The physical meaning of the exponential factor in eq.(16) is clear; it describes the "photon" cloud surrounding any charged particle. The possible "photon" distributions in the cloud correspond to various solutions of eq.(17). A quite general family of such solutions is given in momentum space by

$$\hat{a}^\mu(p) = \int d^3 n [ig^{\mu\nu} \varrho(n) + \varepsilon^{\mu\nu\rho} p_\rho \sigma(n)] n_\nu \frac{1}{np + i\varepsilon} \quad , \quad (18)$$

where the densities ϱ and σ are real-valued and ϱ is normalized according to

$$\int d^3 n \varrho(n) = 1 \quad . \quad (19)$$

If one requires in addition Lorentz covariance, then ϱ and σ are actually functions of the variable n^2 . Concerning the physical interpretation of a^μ it is perhaps useful to recall that the four-dimensional counterpart of eq.(18) with $\sigma(n) = 0$ can be successfully used (see e.g. [5]) in describing scattering in QED₄. In this case the phase factor in eq.(16) automatically reproduces the summation over soft photons adopted in the traditional approach [6] to the S -matrix.

In coordinate space one has for (18) the following integral representation

$$a^\mu(x) = \int ds \int d^3n e^{-is} \theta(s) [g^{\mu\nu} \varrho(n) + \varepsilon^{\mu\nu\rho} \sigma(n) \partial^\rho] n_\nu \delta(x - sn) . \quad (20)$$

Questions related to the time-ordering of operator products of Ψ and $\bar{\Psi}$ in the quantum case suggest the consideration of densities ϱ and σ with space-like support. In particular one may take space-like cones as supports of ϱ and σ . The most popular example is of course

$$\varrho(n) = \delta(n + v) , \quad \sigma(n) = 0 ,$$

with fixed space-like vector v . In this case the support of the field (16) is a string,⁵ starting in x and going to infinity in the direction of v . The fact that in the above examples the support of Ψ is not localizable in a bounded region of space-time is actually a general feature [7]; one has $\Psi \in \mathcal{A}$ but $\Psi \notin \mathcal{A}_{\text{loc}}$.

Our main goal now is to compute the charges Q and Φ carried by Ψ and $\bar{\Psi}$, as well as the PB's among them. In general this is a complicated problem, but for a^μ with support on a space-like surface, one can use the canonical PB's and give the final answer explicitly. Consider for example the plane $x_0 = 0$ and take

$$a^0(x) \equiv \frac{1-\lambda}{2\mu} \delta(x_0) \varepsilon^{jk} \partial_j g_k(x) , \quad (21)$$

$$a^i(x) \equiv \delta(x_0) \left[\frac{1+\lambda}{2} f^i(x) + \frac{1-\lambda}{2} g^i(x) \right] - \frac{1-\lambda}{2\mu} \delta'(x_0) \varepsilon^{ik} g_k(x) ,$$

where f^i and g^i are real-valued distributions, λ is a real parameter and $\varepsilon^{ij} \equiv \varepsilon^{0ij}$. The condition (17), imposed for any λ , leads to

$$\partial_i f^i(x) = \partial_i g^i(x) = \delta(x) . \quad (22)$$

⁵ This object, which is usually called a Mandelstam string, was known to Dirac in the early fifties. Dirac calls it a Faraday line of force, and attempts to explain in [3] the electric charge quantization in QED₄ by means of such lines.

Inserting (21) in eq.(16) one gets

$$\Psi(x; \zeta) = \psi(x) \exp \left\{ \frac{ie}{2} \int d^2y \left[(1+\lambda) f^i(x-y) A_i(x_0, y) - (1-\lambda) g^i(x-y) \varepsilon_{ij} \frac{\pi^j(x_0, y)}{\mu} \right] \right\} , \quad (23)$$

where ζ is a shorthand notation for the set of variables $\{\lambda, f^i, g^i\}$. The field $\Psi(x; \zeta)$ arises naturally also in the Hamiltonian approach and for a particular choice of the data ζ has been previously considered in [8-10]. Our construction is more general and covers the relevant case when $\Psi(x; \zeta)$ is localized in a space-like cone (string). We also draw reader's attention to the following technical point. The discussion below completely avoids the use of the angle function $\Omega(x)$ adopted in [8-10]. Consequently, we do not need to assume the identity [9]

$$\varepsilon^{ij} \partial_i \partial_j \Omega(x) = 2\pi \delta(x) ,$$

whose validity and interpretation (for instance in terms of distributions over a suitable test-function space) are rather obscure.

In the rest of the paper we shall be mainly concerned with the field $\bar{\Psi}(x; \zeta)$. A straightforward computation using the canonical PB's gives

$$i\{Q, \Psi(x; \zeta)\} = -\lambda e \Psi(x; \zeta) , \quad (24)$$

$$i\{\Phi, \bar{\Psi}(x; \zeta)\} = (\lambda - 1) e \bar{\Psi}(x; \zeta) , \quad (25)$$

which imply the values $-\lambda e$ and $(1-\lambda)e$ for the charges Q and Φ carried by Ψ . These values respect the Gauss law (14) because, as already mentioned, the global $U(1)$ -charge N of Ψ is $-e$. At this point the role of the parameter λ is clear; it describes the partition of the global $U(1)$ -charge in electric charge and magnetic flux. For $0 < \lambda < 1$ the electric charge of Ψ is a fraction of $-e$. We shall see below that this feature of Ψ persists also in the quantized theory, where only the sector $\lambda = 0$ has been explored till now.

One can construct in terms of $\Psi(x; \zeta)$ some composite fields with interesting characteristics. We shall consider for example

$$\chi(x; \zeta_1, \zeta_2) \equiv \bar{\Psi}(x; \zeta_1) \Psi(x; \zeta_2) . \quad (26)$$

The remarkable property of χ is that it has vanishing global $U(1)$ -charge, but nevertheless carries a non-trivial electric charge and magnetic flux; indeed

$$i\{Q, \chi(x; \zeta_1, \zeta_2)\} = (\lambda_2 - \lambda_1) e \chi(x; \zeta_1, \zeta_2) , \quad (27)$$

$$i\{\Phi, \chi(x; \zeta_1, \zeta_2)\} = (\lambda_1 - \lambda_2) e\chi(x; \zeta_1, \zeta_2) . \quad (28)$$

Before discussing the quantization of $\mathcal{L}$, it is instructive to compute the PB's among the components of Ψ . One finds

$$i\{\Psi_\alpha(x; \zeta), \Psi_\beta(y; \zeta')\} \Big|_{z_0=y_0} = -i\vartheta \tau(x-y; \zeta) \Psi_\beta(x_0, y; \zeta) \Psi_\alpha(x_0, x; \zeta) , \quad (29)$$

where

$$\tau(x-y; \zeta) \equiv (\lambda^2 - 1) \int d^2 z \epsilon_{ij} [f^i(x-z) g^j(y-z) - f^i(y-z) g^j(x-z)] \quad (30)$$

and ϑ is the dimensionless parameter

$$\vartheta \equiv \frac{e^2}{4\mu} . \quad (31)$$

From eq.(30) it is obvious that

$$\tau(x-y; \zeta) = -\tau(y-x; \zeta) . \quad (32)$$

Our discussion below shows that eq.(29) is actually a first signal for the exotic statistics of the quantized field Ψ .

After these preliminary considerations on a classical level, we turn to canonical quantization of $\mathcal{L}$. Let us summarize the known features of the perturbative expansion. First of all the model defined by eq.(1) is superrenormalizable. The degree of divergence of a Feynman graph γ is

$$\delta_\gamma = 3 - \frac{1}{2} B_\gamma - F_\gamma - \frac{1}{2} V_\gamma , \quad (33)$$

where B_γ , F_γ and V_γ denote the number of external boson lines, external fermion lines and vertices of γ . It follows from eq.(33) that *primitive* divergences may occur only in graphs with $V_\gamma \leq 6$. Among them one can neglect the graphs with closed fermion loops involving an odd number of vertices. Such graphs are in general non-trivial but, as a consequence of the invariance of $\mathcal{L}$ under charge conjugation, their total contribution to any proper Green's function vanishes (Furry's theorem). Therefore, discarding the vacuum diagrams ($B_\gamma = F_\gamma = 0$), the only graphs which *potentially* contain a primitive divergence are:

- (i) the one-loop self-energy graphs $\Sigma^{(1)}(p)$ and $\Pi_{\mu\nu}^{(1)}(p)$;

- (ii) the familiar two diagrams $\Pi_{\mu\nu}^{(2a)}(p)$ and $\Pi_{\mu\nu}^{(2b)}(p)$ contributing to the two-loop self-energy $\Pi_{\mu\nu}^{(2)}(p)$.

In order to investigate these graphs one needs a regularization. It can be shown that in any regularization which preserves gauge invariance the graphs mentioned in point (i) are actually finite.⁶ Separately the two diagrams in point (ii) are in general divergent, but their sum $\Pi_{\mu\nu}^{(2)}(p)$ is finite.

Summarising, all proper Green's functions⁷ of the model are finite, although some Feynman graphs need to be regularized. One should keep in mind that different regularizations, preserving gauge invariance, give in general different finite results for the proper Green's functions. As is well known (see e.g. [11]), all these ambiguities consist of local (contact) terms and can be parametrized by the following *finite* renormalizations:

$$A_{\text{ren}}^\mu \equiv z_3^{-1/2} A^\mu , \quad \psi_{\text{ren}} \equiv z_2^{-1/2} \psi , \quad B_{\text{ren}} \equiv z_3^{1/2} B , \quad (34)$$

$$e_{\text{ren}} \equiv z_3^{1/2} e , \quad \xi_{\text{ren}} \equiv z_3 \xi , \quad \mu_{\text{ren}} \equiv z_3 z \mu , \quad m_{\text{ren}} \equiv m - \delta m . \quad (35)$$

In perturbation theory z , z_2 , z_3 and δm are power series in the bare coupling constant. The factor z in μ_{ren} accounts for possible finite renormalization of the parameter ϑ , namely

$$\vartheta_{\text{ren}} \equiv \frac{e_{\text{ren}}^2}{4\mu_{\text{ren}}} = \frac{e}{4\mu} z^{-1} = z^{-1} \vartheta . \quad (36)$$

Let us mention that there is some controversy in the literature concerning the relevance of the bare parameters e , μ and m and the finite renormalizations (35,36) in the quantized theory. In this respect we recall [11] that the dependence of the renormalized (quantum) parameters on the bare (classical) ones is not at all universal; it depends on both the regularization procedure and the renormalization prescription, and for this reason has no physical meaning. This fact creates no problems of course, because the quantized theory is *entirely* expressed in terms of the renormalized parameters.

The quantum field Ψ_{ren} is defined according to the general rules for constructing composite operators [12]. One has

$$\Psi_{\text{ren}}(x; a^\mu) \equiv Z_2^{-1/2} \psi_{\text{ren}}(x) \exp \left[i e_{\text{ren}} \int d^3 y a_\mu(x-y) A_{\text{ren}}^\mu(y) \right] =$$

⁶ Other regularizations lead in general to the presence of divergent local terms which break down gauge invariance, but are harmless because of the existence of corresponding local counterterms restoring the symmetry.

⁷ For off-mass-shell external momenta.

$$= (Z_2 z_2)^{-1/2} \psi(x) \exp \left[i e \int d^3 y a_\mu(x-y) A^\mu(y) \right] . \quad (37)$$

The factor Z_2 can be computed⁸ for example in perturbation theory; it accounts for the possible *infinite* renormalization needed in the construction of the composite operator and in general depends on a^μ . We choose a^μ as given by eq.(21) and denote by $\tilde{\Psi}_{\text{ren}}(x; \zeta)$ the resulting composite field. The composite field $\chi_{\text{ren}}(x; \zeta_1, \zeta_2)$ can be constructed analogously. In what follows we adopt exclusively renormalized fields and parameters and for notational convenience omit the index "ren".

We concentrate now on the charge content and the statistics of the quantum field $\Psi(x; \zeta)$. Using the canonical commutation (anticommutation) relations for the renormalized basic fields one finds

$$[Q, \Psi(x; \zeta)] = -\lambda e \Psi(x; \zeta) , \quad (38)$$

$$[\Phi, \Psi(x; \zeta)] = (\lambda - 1) e \Psi(x; \zeta) . \quad (39)$$

Analogously one derives the exchange relation

$$\Psi_\alpha(x_0, x; \zeta) \Psi_\beta(x_0, y; \zeta) = R(x - y; \zeta) \Psi_\beta(x_0, y; \zeta) \Psi_\alpha(x_0, x; \zeta) , \quad (40)$$

where $x \neq y$ and

$$R(x - y; \zeta) \equiv \exp[i\partial\tau(x - y; \zeta) + i\pi] . \quad (41)$$

Eqs.(38,39) confirm the expected values of the quantum charges. By means of eq.(40) one can compute the equal-time anticommutator of $\Psi(x; \zeta)$. One obtains

$$\{\Psi_\alpha(x; \zeta), \Psi_\beta(y; \zeta)\} \Big|_{x_0=y_0} = [R(x - y; \zeta) + 1] \Psi_\beta(x_0, y; \zeta) \Psi_\alpha(x_0, x; \zeta) . \quad (42)$$

It is instructive to compare now eqs.(29) and (42). Expanding $R(x - y; \zeta)$ in the r.h.s. of eq.(42) in powers of ∂ , we see that the first-order contribution reproduces exactly the r.h.s. of eq.(29). In this sense $R(x - y; \zeta)$ is the quantization of $\tau(x - y; \zeta)$.

Let us describe now some fundamental properties of $R(x - y; \zeta)$. Combining eqs.(32,41) one has

$$R(x - y; \zeta) R(y - x; \zeta) = 1 , \quad (43)$$

but in general

$$R(x - y; \zeta) \neq R(y - x; \zeta) . \quad (44)$$

⁸ A suitable regularization is the point-splitting [12].

Eqs.(30,40,41,44) imply that in general $\Psi(x; \zeta)$ obeys an *anyonic* statistics [13] which depends on the charge content of $\Psi(x; \zeta)$ via the parameter λ .

It is useful at this point to compute $R(x - y; \zeta)$ explicitly in some concrete cases. Recalling that $f^i(x)$ and $g^i(x)$ satisfy eqs.(22), we take

$$\vec{f}^i(p) = \vec{g}^i(p) = i \int_0^{2\pi} d\varphi \varrho(\varphi) \frac{1}{np + i\epsilon} , \quad (45)$$

where

$$\int_0^{2\pi} d\varphi \varrho(\varphi) = 1 , \quad n = (\cos \varphi, \sin \varphi) . \quad (46)$$

For simplicity we have assumed in eq.(45) that the density ϱ depends only on the polar angle φ but not on $|n|$. A standard computation gives ($x \neq y$)

$$r(x - y; \zeta) = (1 - \lambda^2) \int_0^{2\pi} d\varphi \varrho(\varphi) \varepsilon[\sin(\varphi - \gamma)] . \quad (47)$$

Here $\varepsilon(s) \equiv \theta(s) - \theta(-s)$ and γ is the polar angle of the vector $x - y$. Consider now the case

$$\text{supp } \varrho(\varphi) = [\alpha_1, \alpha_2] , \quad 0 \leq \alpha_1 < \alpha_2 \leq \frac{\pi}{2} .$$

With this choice the support of the field $\Psi(x; \zeta)$ for fixed x_0 is the wedge $W_x \subset \mathbb{R}^2$ with apex at the point x and angle $\alpha_2 - \alpha_1$. Using the normalization condition (46) and eq.(47) one gets

$$\tau(x - y; \zeta) = \begin{cases} (1 - \lambda^2) \left[1 - 2 \int_{\alpha_1}^{\gamma} d\varphi \varrho(\varphi) \right] , & \text{if } \alpha_1 \leq \gamma \leq \alpha_2 ; \\ (\lambda^2 - 1) , & \text{if } \alpha_2 < \gamma < \alpha_1 + \pi ; \\ (\lambda^2 - 1) \left[1 - 2 \int_{\alpha_1}^{\gamma - \pi} d\varphi \varrho(\varphi) \right] , & \text{if } \alpha_1 + \pi \leq \gamma \leq \alpha_2 + \pi ; \\ (1 - \lambda^2) , & \text{if } \alpha_2 + \pi < \gamma < \alpha_1 + 2\pi . \end{cases} \quad (48)$$

From eqs.(48) it follows that $\tau(x - y; \zeta)$ depends continuously on γ . For y outside the wedge W_x and its shadow $\tilde{W}_x$, see Fig.1, $\tau(x - y; \zeta)$ is independent of the density ϱ . Note the role played by *both* the wedges W_x and $\tilde{W}_x$ as intermediate regions where $\tau(x - y; \zeta)$ continuously varies between $(\lambda^2 - 1)$ and $(1 - \lambda^2)$. Accordingly, the exchange phase factor $R(x - y; \zeta)$ is a continuous function of the polar angle γ .

The above analysis can be extended also to the singular case when the wedge W degenerates into a string S . Being the sharpest possible localization of $\Psi(x; \zeta)$, this situation is of some interest. In order to study it, one can use any weakly convergent sequence of densities $\{\varrho_\nu(\varphi) : \nu = 1, \dots\}$, such that for $\nu \rightarrow \infty$

$$\varrho_\nu(\varphi) \longrightarrow \delta(\varphi - \alpha) , \quad 0 \leq \alpha < 2\pi , \quad (49)$$

α specifying the direction of S . Now, instead of the variable ζ , we shall simply use the angle α . The function $\tau(\mathbf{x} - \mathbf{y}; \alpha)$, which is easily computed by inserting eq.(49) in (47), is no longer continuous at the points $\gamma = \alpha$ and $\gamma = \alpha + \pi$. These are precisely the values of γ for which the strings S_x and S_y have a non-trivial overlap. The effect of this discontinuity will be discussed later on.

So, let us consider first the fields $\Psi(\mathbf{x}; \alpha)$ and $\Psi(\mathbf{y}; \alpha)$ in generic points, i. e. for $\mathbf{x}$ and $\mathbf{y}$ such that $S_x \cap S_y = \emptyset$. In this case the exchange phase factor reads

$$R(\mathbf{x} - \mathbf{y}; \alpha) = \begin{cases} R_+(\mathbf{x} - \mathbf{y}; \alpha) = \exp[i\vartheta(\lambda^2 - 1) + i\pi], & \text{if } \alpha < \gamma < \alpha + \pi; \\ R_-(\mathbf{x} - \mathbf{y}; \alpha) = \exp[-i\vartheta(\lambda^2 - 1) + i\pi], & \text{if } \alpha + \pi < \gamma < \alpha + 2\pi. \end{cases} \quad (50)$$

The occurrence of two phases $R_{\pm}$ in eq.(50) is not surprising and reflects the fact [14] that string-like objects in 2+1 dimensions in general obey braid statistics [15]. Indeed, the exchange properties of the fields $\{\Psi(\mathbf{z}_0, \mathbf{x}_i; \alpha) : i = 1, \dots, n\}$ at generic points are described by the following one-dimensional representation of the braid group B_n :

$$\sigma_i \mapsto R_+(\mathbf{x}_i - \mathbf{x}_{i+1}; \alpha), \quad \sigma_i^{-1} \mapsto R_-(\mathbf{x}_i - \mathbf{x}_{i+1}; \alpha), \quad (51)$$

$\{\sigma_i : i = 1, \dots, n-1\}$ being the generators of B_n .

The following considerations give an intuitive picture of what is going on. If $\alpha < \gamma < \alpha + \pi$, see Fig.2, in order to exchange $\mathbf{x}$ and $\mathbf{y}$ by rotating $\mathbf{y}$ around $\mathbf{x}$ without intersecting S_x , one should perform an anticlockwise π -rotation. If $\alpha + \pi < \gamma < \alpha + 2\pi$, see Fig.3, one needs instead a clockwise π -rotation. This picture is even more suggestive if one imagines the time evolution of the above rotations; the world lines associated with $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ give rise to the well-known geometric description via strings in $\mathbf{R}^3$ of the generators σ_i and σ_i^{-1} respectively.

The representation (51) is completely characterized by the so-called statistical phase factor Θ , which is defined (modulo 2) by [16]

$$\exp[i\Theta\pi] = R_+(\mathbf{x} - \mathbf{y}; \alpha). \quad (52)$$

From eqs.(50,52) one gets

$$\Theta = \frac{\vartheta}{\pi}(\lambda^2 - 1) + (2k + 1), \quad k \in \mathbb{Z}. \quad (53)$$

The Abelian braid statistics obeyed by $\Psi(\mathbf{x}; \alpha)$ is known as Θ -statistics, while the relative particle excitations are called anyons [13]. The associated wave function (in the

case $\alpha = 0$) has the form [16,17]

$$F(z_1, \dots, z_n; \bar{z}_1, \dots, \bar{z}_n) = \prod_{i < j}^n (z_i - z_j)^{\Theta} u(z_1, \dots, z_n; \bar{z}_1, \dots, \bar{z}_n), \quad (54)$$

where $z_j \equiv x_j^1 + ix_j^2$ and u is a single-valued function which is completely symmetric under particle permutation. In general F is not single-valued; this is a manifestation of the above-mentioned discontinuity in the exchange phase factor at non-generic points.

Summarising, we have shown that the charges Q and Φ as well as the statistics parameter Θ of the field $\Psi(\mathbf{x}; \alpha)$ are all parametrized by the single real parameter λ . Eliminating λ one obtains the relation

$$\Theta = -\frac{1}{4\pi\mu}(Q + N)\Phi + 2k. \quad (55)$$

The following special cases are of particular interest

$$\Theta = 2m, \quad m \in \mathbb{Z}, \quad (56)$$

$$\Theta = 2n + 1, \quad n \in \mathbb{Z}, \quad (57)$$

$$\Theta = \frac{p}{q}, \quad p, q \in \mathbb{Z}, \quad p \text{ and } q \text{ coprime}. \quad (58)$$

Eqs.(56-58) correspond to Bose-, Fermi- and p/q -statistics respectively. The parameter λ in these cases is quantized in terms of the parameter ϑ of the theory. One has

$$\lambda(m)^2 = (2m + 1)\frac{\pi}{\vartheta} + 1, \quad \lambda(n)^2 = 2n\frac{\pi}{\vartheta} + 1, \quad \lambda(p/q)^2 = \left(\frac{p}{q} - 1\right)\frac{\pi}{\vartheta} + 1. \quad (59)$$

Eqs.(59) imply in turn the quantization of Q and Φ .

The above analysis can be repeated for the composite field $\chi(\mathbf{x}; \zeta_1, \zeta_2)$ as well. Choosing $\zeta_1 = \{\lambda_1, f^i, f^i\}$ and $\zeta_2 = \{\lambda_2, f^i, f^i\}$ with $f^i(\mathbf{x})$ defined by eqs.(45,49) one gets

$$N = 0, \quad Q = (\lambda_2 - \lambda_1)e, \quad \Phi = (\lambda_1 - \lambda_2)e, \quad (60)$$

$$\Theta = \frac{\vartheta}{\pi}(\lambda_1 - \lambda_2)^2 + 2k, \quad k \in \mathbb{Z}. \quad (61)$$

The following interesting property of $\chi(\mathbf{x}; \zeta_1, \zeta_2)$ emerges at this stage. Suppose one takes as constituents two bosonic fields $\Psi(\mathbf{x}; \zeta_1)$ and $\Psi(\mathbf{x}; \zeta_2)$ which, according to the first of eqs.(59), correspond to $m_1 \neq m_2$. Then $\Psi(\mathbf{x}; \zeta_1)$ and $\Psi(\mathbf{x}; \zeta_2)$ have different charge

content and it follows from eq.(61) that in general the composite operator $\chi(z; \zeta_1, \zeta_2)$ obeys anyonic statistics. The possibility to form an anyon by combining two (distinct) bosons has been discovered recently [18] in the general framework of braid statistics. The above example provides an explicit quantum field theory realization of the phenomenon.

Finally, combining eqs.(60,61) one gets

$$\Theta = -\frac{1}{4\pi\mu} Q\Phi + 2k \quad (62)$$

This relation has been heuristically derived in [13] by using an Aharonov-Bohm type interaction. In the above context eq.(62) holds on the level of second-quantized theory.

We conclude our considerations of the charged sectors of QED₃ with a CS term with some comments concerning possible applications. Many authors (see e.g. [19] and references therein) have pointed out that this model may have some relevance for the theoretical explanation of the fractional quantum Hall effect (FQHE). A first encouraging indication in this direction is the celebrated Laughlin wave function, which has the form (54). Recently Fröhlich and Kerler [20] exhibited a scaling argument showing that the CS action is important for understanding some universal features of the FQHE. In this respect our investigation above shows that QED₃ with a CS term naturally provides a unified description in terms of the field Ψ of both fundamental fractions $1/(2n+1)$ and their hierarchical generalizations [21]. Indeed, as far as the statistics is concerned, all the carriers giving rise to the fundamental and hierarchy fractions can be associated with Ψ for $\lambda = \lambda(n)$ and $\lambda = \lambda(p/q)$ respectively (see eqs.(57,58)). The relative charge content is then determined by substituting these values of λ in eqs.(38,39). If one accepts this theoretical framework, then the challenging question arises how dynamics acts to select among all (kinematically) possible values of λ the values $\lambda(n)$ and $\lambda(p/q)$ for peculiar integers p and q . Obviously, to answer this question one needs a more detailed microscopic description of the phenomenon. Such a description will further clarify to what extent the CS model is essential for the FQHE.

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FIGURE CAPTIONS

- FIG. 1: The wedges W_x and W_y supporting the fields $\Psi(x_0, x; \zeta)$ and $\Psi(x_0, y; \zeta)$.
- FIG. 2: Exchanging $\Psi(x_0, x; \alpha)$ and $\Psi(x_0, y; \alpha)$ when $\alpha < \gamma < \alpha + \pi$.
- FIG. 3: Exchanging $\Psi(x_0, x; \alpha)$ and $\Psi(x_0, y; \alpha)$ when $\alpha + \pi < \gamma < \alpha + 2\pi$.

FIG. 1

FIG.2

FIG.3