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PREFACE TO VOLUME I

The 1967 CERN School at Råttvik was the sixth in a series that began in 1962. The lectures of the 1967 School started on the morning of 22 May and were attended by 84 students from 21 countries in Western Europe, Eastern Europe, the Middle East, and India. The School closed on 2 June with the traditional banquet.

The purpose of the School was to familiarize young experimental postgraduate students of experimental physics with the current theoretical and experimental situation in elementary particle studies. Eleven lecturers contributed to this end by giving a total of 34 seminars, lectures, or after-dinner talks.

This volume contains the lectures given by Prof. Jan Nilsson on "The Discrete Symmetries P, C, and T", and by Dr. Maurice Jacob on "Weak Interactions and Higher Symmetries". In the interests of speed we have photographed the typescripts given to us by the authors and used the photo-offset reproduction process to produce the four volumes of the Proceedings. All errors, corrections, illegible formulae, etc., are therefore the responsibility of the individual authors and not of the editors or the CERN Scientific Information Service, who have not proofread the texts. We trust that what has been lost in beauty of presentation will be compensated by the fact that the Proceedings will be available to the scientific community in a much shorter time than previously after the end of the School. We would be pleased to receive comments from readers on this change in our publishing policy.

Our thanks are due to the authors who have worked very hard to provide us with their manuscripts either at the School itself or a very few weeks afterwards, and to the Scientific Information Service for their careful and rapid work of publication.

Editorial Board.
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Jan Nilsson

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1. Introduction.

In recent years there has been a revival of the interest in the discrete symmetries P (space reflection), C (charge conjugation), T (time reversal) and their various combinations. There are several reasons for this turn of events. The first and most immediate motivation for a reexamination of these invariance principles is the fact that some, if not all, seem to be violated; the main difference being the level at which the violations occur.

It all started in 1956 when Lee and Yang suggested that space reflection is not a valid symmetry for weak interaction processes. Experiments in nuclear $\beta$-decay and muon decay confirmed that this suggestion was correct. It was realized at about the same time that weak interactions are not only P-violating but also C-violating. Some sort of balance in the laws of physics was restored by the assumption by Landau and others that although P and C are separately violated, the combined CP is still a symmetry operation respected by all interactions. Recent experiments on $K^0_2$ decays have cast reasonable doubts on the validity of the CP symmetry. Since the operation CPT seems to be a true invariance of any reasonable theory (the CPT theorem) a CP violation necessarily implies that also T-invariance is violated.

Before 1956 it was generally believed that all the symmetries mentioned here were absolute symmetries of nature. At present we are in the midst of drastic changes in our understanding of nature and none of the symmetries, with the possible exception of CPT, may survive this development. At the same time we have been forced to take a good look at the present formulation of the discrete symmetries with an open mind to possible redefinitions. Clearly this state of affairs places the field of invariance principles in the center of interest.

The second reason for the present surge in interest in symmetries including the discrete symmetries is the fact that we still lack a dynamical theory for elementary particle processes. It has been known for a long time that invariance arguments can be used to derive predictions, which can be subjected to experimental tests, and the deductions do not require knowledge about the detailed form of the interaction. Hence, symmetry principles can be verified or falsified at a stage when we still lack the
complete theory. In this way they provide very powerful tools to bring
order in our empirical findings with regard to particle processes while
we still search for the appropriate theory.

2. The basic formalism and invariance principles.

To establish our notations and the general conventions, to which
we shall adhere, we begin with a brief review of the basic formalism.
In this context we shall also introduce the concept of symmetry and
derive some of the immediate implications which follow from invariance
principles. For a more complete discussion of these questions we refer
to standard textbooks 1).

2.1 The Hilbert space of physical state vectors.

An experiment in physics consists of two parts: (i) the preparation
of the initial state of the physical system which is under study, and
(ii) performing measurements on the system at a later time. It is the
aim of theory to account for or "explain" the correlations one may
observe between different preparations and the observations which one
may make at a later time.

The preparation of the initial state corresponds to performing a
series of measurements on the physical system to define its properties.
After the preparation the system is represented by a state vector
\[ |\alpha (\Omega)\rangle \]
where \( \alpha \) denotes the outcome of the various measurements.
Thus each state vector is labeled by the results of measurements. The
preparation extends over a finite domain \( \Omega \) in space and time, and the
state vectors will in general depend on \( \Omega \) as implied by the notation,
which is used.

The conventional formulation of quantum theory, on which we base
our description, can be summarized in the following way. The manifold
\( \mathcal{H} \) of all possible state vectors form a linear vector space or, more
precisely, a separable Hilbert space. Each observable, that is,
measurable property \( A \) of a physical system is represented by a linear
hermitian operator \( A \) acting on \( \mathcal{H} \). The (real) eigenvalues \( a_i \) of \( A \) are
the only possible results of an exact, sharp measurement of the
property A. The maximal amount of information one may obtain for a physical system corresponds to a complete set of compatible (commuting) measurements, that is, measurements which do not perturb or interfere with each other. If the set is complete, then the system is represented by a state vector which is unique apart from an arbitrary over-all phase factor. Such a state vector is labeled by all the eigenvalues $a, a', \ldots$ of the observables $A, A', \ldots$ in the complete set. If the set of measurements is not complete, then one distinguishes between two cases: (i) Pure states; the representing state vector is a linear superposition of unique state vectors so that the phase relations between the various components are known. Clearly the case of a unique state vector is a special case of a pure state. (ii) Mixed states; the representing state vector is a statistical superposition of unique state vectors so that the phase relations between the components are unknown. The two cases are physically distinguishable by the fact that for pure states one may observe interference effects which are absent if one is dealing with mixed states.

Returning to our discussion of an experiment we note that some time after the preparation the system is once more subjected to measurements, which constitute the second phase of the experiment. The interaction between the physical system and the measuring apparatus will affect the latter and the registration of how it is affected is the outcome of the experiment. As a result of this second set of measurements one has new information about the physical system and, hence, it must be represented by a new state vector $|\beta (\Omega')\rangle$, the final state. This new state vector will depend on the space-time region $\Omega'$ in which the measurements were performed. The properties of the initial state at the time and place of the later measurements is given by the probability distribution over all possible outcomes of these measurements. The fundamental principle of quantum mechanics now asserts that this probability distribution $w (\beta, \alpha)$ is given by

$$w (\beta, \alpha) = | \langle \beta (\Omega') | \alpha (\Omega) \rangle |^2$$  \hspace{1cm} (1)
2.2 The S-operator.

In elementary particle experiments there are specific features which we shall take advantage of. We will be concerned with decay processes and scattering processes. These have in common that for times much larger than some characteristic reaction time all particles are free, i.e. noninteracting, and we can label the states with single particle labels such as the individual momenta and spins. However, we have previously recognized that the state vectors in general depend on the space-time region for the labeling measurements. For an elementary particle process the "asymptotic" regions, where the particles are free, correspond to $t \rightarrow -\infty$ for the initial state and $t \rightarrow +\infty$ for the final outgoing state. With this in mind we write the eq. (1) in the following way

$$w(\beta, \alpha) = | <\beta; \text{out} | \alpha; \text{in} > |^2$$

where "in" and "out" refers to asymptotic free particle states for $t \rightarrow -\infty$ and $t \rightarrow +\infty$ respectively.

For a free particle of mass $m$ and spin $s$ we may take as a complete set of measurements a determination of its three-momentum $\vec{p}$ and its helicity $\lambda$, but other choices are possible and in some cases more convenient. For many applications the "plane wave" states $|m, s; \vec{p}, \lambda >$ are useful, however, since in most experiments one actually measures at least the momenta of the particles. Of course, one cannot determine the momentum exactly since it is a continuous quantity, but one can determine it within an arbitrarily small but finite interval, and the corresponding state vector will then be an appropriate superposition of plane wave states. If the momentum spread is small enough the idealization of a plane wave state is often relevant and suitable to facilitate computations. The price one has to pay for this convenience is sometimes a lack of mathemtical rigor in the formalism.

With regard to two-particle states and many-particle states the most obvious extension is to form simple product states of the form

$$|\vec{p}, \vec{p'}, \lambda, \lambda' > \equiv |\vec{p}, \lambda > |\vec{p'}, \lambda' >$$

(3)
In some cases the analysis of a process is more transparent if one uses other basis vectors corresponding to a different choice of a complete set of observables. For example, it is often instructive to analyse the process in terms of angular momentum. The transformation from the basis (3) to an angular momentum basis is straightforward but involves tools which we shall not develop here. A brief discussion of these questions is given in appendix 2.

For the no-particle state, i.e. the vacuum state, and the single particle states there is no distinction between in-states and out-states since nothing can happen in those cases. For states of two or more particles a distinction is necessary, however, since our in-state may undergo scattering etc. so that

$$| \alpha; \text{in} \rangle \neq | \alpha; \text{out} \rangle$$

The set of all free particle in-states as well as the set of all free particle out-states each from a complete set of basis vectors in the physical Hilbert space $H$. Thus, the set of in-states and the set of out-states provide us with two different choices of basis vectors and, consequently, there must exist a unitary operator $S$ which connects the two sets, that is

$$| \alpha; \text{in} \rangle = S | \alpha; \text{out} \rangle \quad (4)$$

for all $\alpha$. The unitarity of the $S$-operator implies that

$$SS^\dagger = S^\dagger S = 1$$

or

$$S^\dagger = S^{-1}$$

Introducing the $S$-operator of the eq. (4) we may now write

$$| \alpha; \text{in} \rangle = \sum_\beta | \beta; \text{out} \rangle \langle \beta; \text{out} | S | \alpha; \text{out} \rangle =$$

$$= \sum_\beta | \beta; \text{out} \rangle S_{\beta \alpha}$$

$$(5)$$
with

\[ S_{\beta \alpha} = \langle \beta ; \text{out} | S | \alpha ; \text{out} \rangle \]  

(6)

Form the eq. (5) we conclude that given the in-state \( | \alpha ; \text{in} \rangle \) the matrix elements \( S_{\beta \alpha} \) represent the probability amplitudes for this in-state to appear in the channel \( | \beta ; \text{out} \rangle \) at large times. This is consistent with the eq. (2) which may now be rewritten in the following way

\[ w(\beta, \alpha) = | \langle \beta ; \text{out} | S | \alpha ; \text{out} \rangle |^2 = | S_{\beta \alpha} |^2 \]  

(7)

From this expression for the transition probability it is a trivial matter to derive the expressions for cross sections and decay rates. We omit the derivations and just quote the results\(^2\) for the following two types of processes

\[ a \rightarrow b_1 + b_2 + \ldots + b_n \]  

(8)

and

\[ a + b \rightarrow c_1 + c_2 + \ldots + c_n \]  

(9)

For the decay rate \( \lambda \) of process (8) one finds

\[ \lambda = \frac{1}{(2\pi)^{3n-4}} \cdot \frac{1}{2E_a} \sum \frac{d^3 p_1}{2E_{l_1}} \ldots \frac{d^3 p_n}{2E_{l_n}} \delta^4(p_f - p_i) | \langle f | T | i \rangle |^2 \]  

(10)

where the transition matrix element \( \langle f | T | i \rangle \) is defined by

\[ \langle f | S | i \rangle = \delta_{fi} + i (2\pi)^4 \delta^4(p_f - p_i) \cdot N \cdot \langle f | T | i \rangle \]  

(11)

and \( N \) is a normalization factor defined by

\[ N = \frac{1}{\sqrt{2VE_i}} \prod_{\text{initial particles}} \frac{1}{\sqrt{2VE_f}} \prod_{\text{final particles}} \]  

(12)
For convenience a finite normalization volume $V$ is used. The cross section $\sigma$ for the process (9) is similarly given by

$$\sigma = \frac{1}{(2\pi)^{3n-4}} \cdot \frac{1}{4E_1^b v_{in}} \cdot \sum \int \frac{d^3 p_1}{2E_1} \cdots \frac{d^3 p_n}{2E_n} \delta^4(p_f - p_i) \left| <f | T | i> \right|^2$$

where $v_{in}$ is the relative velocity of the incoming particle with respect to the target particle. If the initial particles carry spin, then one has to take the appropriate spin averages over the initial state.

We conclude this section by noting that all physical information is contained in the S-operator or, alternatively, the T-operator. A discussion of invariance principles, which is the main subject of these lectures, can then clearly be restricted to a discussion of symmetry properties of these operators.

2.3 Invariance principles.

An invariance principle or symmetry operation of a physical system is a one to one correspondence which assigns to each physically realizable state $| \alpha >$ another state $| \alpha' >$ such that all transition probabilities are preserved, that is

$$w(\beta', \alpha') = w(\beta, \alpha)$$

or

$$|<\beta' | \alpha'>|^2 = |<\beta | \alpha>|^2$$

Similarly there is a correspondence between the observables in the two alternative descriptions. It has been shown by Wigner that if the mapping $| \alpha > \rightarrow | \alpha' >$ is to satisfy the eq. (14) then it is realized by means of a unitary or an antiunitary operator. We summarize the essential features of the two alternatives.
(i) Unitary symmetry operations:

\[ | \alpha \rangle \rightarrow | \alpha' \rangle = U | \alpha \rangle \iff | U \alpha \rangle \]

\[ U^\dagger = U^{-1} \] (15)

\[ \langle \beta | U^\dagger | \alpha \rangle = \langle U \beta | \alpha \rangle = \langle \alpha | U | \beta \rangle \]

For an arbitrary operator \( \Omega \) one obtains

\[ \langle \beta | \Omega | \alpha \rangle = \langle \beta | U^{-1} U \Omega U^{-1} U | \alpha \rangle = \langle \beta | \Omega' | \alpha' \rangle \] (16)

where

\[ \Omega' = U \Omega U^{-1} \]

This is the transformation law for an operator of the theory. If the operator \( \Omega \) is invariant under the symmetry operation \( U \), then \( \Omega' = \Omega \) so that

\[ \Omega = U \Omega U^{-1} ; \quad [ \Omega, U ] = 0 \] (17)

and

\[ \langle \beta | \Omega' | \alpha' \rangle = \langle \beta | \Omega | \alpha \rangle \] (18)

The last equation implies that the matrix elements of \( \Omega \) are the same in the two descriptions.

(ii) Antiunitarity symmetry operations:

\[ | \alpha \rangle \rightarrow | \alpha' \rangle = A | \alpha \rangle \iff | A \alpha \rangle \]

\[ A^\dagger = A^{-1} \] (19)

\[ A \lambda | \alpha \rangle = \lambda^* A | \alpha \rangle \]

\[ \langle \beta | A^\dagger | \alpha \rangle = \langle A \beta | \alpha \rangle = \langle \alpha | A | \beta \rangle \]
For an arbitrary operator $\Omega$ one obtains
\[
\langle \beta | \Omega | \alpha \rangle = \langle \beta | A^{-1} \Omega A^{-1} A | \alpha \rangle = \langle \beta | (A \Omega A^{-1}) | \alpha \rangle^* = \\
= \langle \alpha^* | (A \Omega A^{-1})^\dagger | \beta \rangle = \langle \alpha^* | \Omega^* | \beta \rangle,
\]
(20)
where
\[
\Omega^* = (A \Omega A^{-1})^\dagger.
\]
This is the transformation law for $\Omega$ under an antiunitary symmetry transformation. More precisely, the eq. (20) implies that the operator $\Omega^*$ taken between the states $|\alpha^*\rangle$ and $|\beta^*\rangle$ yields the same result as the operator $\Omega$ taken between the states $|\beta\rangle$ and $|\alpha\rangle$. If $\Omega$ is invariant under the transformation, then
\[
\Omega^* = A \Omega A^{-1}
\]
(21)
which is to be compared to the eq. (17) for the unitary case.

After this rather formal introduction of the symmetry concept it is instructive to consider an explicit example. We choose the principle of Lorentz invariance, that is, the invariance under proper, orthochronous Lorentz transformations (Poincaré transformations). This principle asserts that the laws of physics are invariant under Poincaré transformations which relate the spacetime coordinates of the same physical event as it is observed from two different inertial frames. From the previous discussion we know that this implies that there exists a unitary operator relating the state vectors and the observables which represent the physical system in the two descriptions. The case of an antiunitary correspondence is clearly inapplicable for transformations that can be reached continuously from the identity transformation, since the identity operator is unitary. This formulation of the relativity principle is usually called the passive formulation. It is a rule which relates the descriptions of the same physical system with reference to two different coordinate frames. However, one may adopt an alternative point of view, which is known as the active formulation of the relativity principle. Instead of considering
two different observers, in relative motion or otherwise related by a Poincaré transformation, one may consider one and the same observer as he observes two physical events, which are related by the same Poincaré transformation as the two observers in the passive formulation. The active formulation implies that if $|\alpha\rangle$ is a possible state of a system, then $|\alpha'\rangle = U|\alpha\rangle$, where $U$ represents an arbitrary Poincaré transformation, is also a possible state of the system as seen by the same observer. Thus, in the active formulation the relativity principle relates the observations one observer may make on two different physical systems, which are identical apart from a relative Poincaré transformation. All the transformations of the Poincaré group are of geometrical nature and it is experimentally feasible to test the relativity principle both in the active and the passive formulation and to establish their equivalence. For other transformations such as inversions (i.e. time reversal) we can in effect only test the invariance principle formulated in the active manner, since we cannot realize the "inverted" observer. Nevertheless, in the passive manner one may always formulate hypothetical invariance principles involving such "pathological" coordinate transformation, but they will not be physically meaningful unless they represent a true invariance operation. If not they will merely be mathematical operations. Correspondingly, in the active formulation one finds that the operations, which are not symmetry operations, in some cases transform physically realizable states into states which do not occur in nature. This means that the corresponding state vectors lie outside the physical Hilbert space. Mathematically this means that the symmetry operator cannot be unitary or antiunitary since it leads out of the space. A well-known example of this phenomenon occurs in connection with the space reflection operation and the neutrino particle for which only one of the two conceivable helicity states is realized in nature. Of course, one may enlarge the space of state vectors to include also these non-physical states so that one may define the corresponding symmetry operators. However, this is, as previously stated, only a mathematical construction and we can attach no physical meaning to it.
Since we are ultimately interested in experimental tests of the invariance principles we shall consistently adopt the active formulation of all symmetry operations.

2.4 Invariance properties of the $S$-operator

The basic requirement on any symmetry operation was expressed in the eq. (14). Recalling the relation (7) we see that a symmetry principle can be expressed as an equality between the squared modulus of certain $S$-matrix elements. Thus, in the active formulation symmetry operators yield relations between the appropriate $S$-matrix elements corresponding to different experimental situations. From this we may then derive relations between different measurable quantities and these predictions can be tested in experiments. It is important to realize that no further information about the $S$-operator beyond the assumed symmetry property is needed to make such predictions. It is then clear that invariance principles provide useful and powerful tools in the study of elementary particle processes.

From the previous section we know that the invariance of the $S$-operator implies

$$S = USU^{-1}$$  \hspace{1cm} (22)

if it is a unitary operation, and

$$S^\dagger = ASA^{-1}$$  \hspace{1cm} (23)

if it is an antiunitary operation. We will encounter symmetry principles of both types. In those cases where we have an explicit expression for the $S$-operator the eqs. (22) and (23) permit us to examine invariance properties of the theory directly. This is for example the case in many field-theoretic models. Assuming that we know the hamiltonian density and that it has the form

$$H(x) = H_0(x) + H'(x)$$  \hspace{1cm} (24)
where \( H_0(x) \) and \( H'(x) \) refer to the free part and the interaction part respectively, standard methods yield the following perturbation expression for the S-operator

\[
S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n \ T \{ H'(x_1) \ldots H'(x_n) \} \tag{25}
\]

where \( T \) is the conventional time-ordering symbol. From the eq. (25) it is clear that in this case invariance of the S-operator follows from the following conditions

\[
H'(x) = U H'(x) U^{-1} \tag{26}
\]

if it is a unitary symmetry operator, and

\[
H'(x) = A H'(x) A^{-1} \tag{27}
\]

if the symmetry operation is represented by an antiunitary operator. In the eq. (27) we have used the fact that \( H'(x) \) is hermitian.

To any unitary operator \( U_1 \) one may define a hermitian operator \( H_1 \) by the relation

\[
U_1 = \exp [ i H_1 ] \tag{28}
\]

For continuous symmetry groups the operators \( H_1 \) are closely related to the infinitesimal generators of the group. Since the operators \( H_1 \) are hermitian they may represent observables and their eigenvalues may be used as quantum numbers, which are conserved provided the operators \( U_1 \) represent invariance operations. To see this consider the process \( | \alpha \rangle \rightarrow | \beta \rangle \). We assume

(i) that \( H_1 \) is related to a symmetry operation \( U_1 \) by the eq. (28), that is \( [ H_1, S ] = 0 \)

(ii) \( | \alpha \rangle \) and \( | \beta \rangle \) are eigenvectors of \( H_1 \) corresponding to the eigenvalues \( h_\alpha \) and \( h_\beta \) respectively.
From this we obtain

$$h_{\alpha} < \beta | S | \alpha \rangle = < \beta | s H_1 | \alpha \rangle = < \beta | S \alpha \rangle = h_{\beta} < \beta | S | \alpha \rangle \quad (29)$$

or

$$(h_{\alpha} - h_{\beta}) < \beta | S | \alpha \rangle = 0 \quad (30)$$

and hence

$$< \beta | S | \alpha \rangle \neq 0 \Rightarrow h_{\alpha} = h_{\beta} \quad (31)$$

This result implies that the initial and the final states are eigenstates of $H_1$, corresponding to the same eigenvalue; the quantum number $h$ is conserved. Invariance principles may in this way give rise to conservation laws. We reach the same result for hamiltonian theories by considering the Heisenberg equation for an observable $H_1$, which does not depend on the time coordinate explicitly. In that case the Heisenberg equation reads

$$\frac{dH_1}{dt} = i \quad [H, H_1] \quad (32)$$

where $H$ denotes the hamiltonian of the system. If $H_1$ is conserved, i.e., its time derivative vanishes, then $H_1$ and $H$ commute and we may deduce the conservation law for $h_1$.

If we apply these results to the case of Poincaré invariance for the $S$-operator we find that

$$[p_\mu, S] = 0 \quad (33)$$

$$[j_{\mu \nu}, S] = 0$$

where $p_\mu$ and $j_{\mu \nu}$ are the hermitian generators of the Poincaré group. It follows from the first relation above that the matrix elements of $S$ between states of definite values of $p_\mu$ must have
the following diagonal form

\[ \langle p'_\mu, a' | S | p_\mu, a \rangle = \delta^4 (p'_\mu - p_\mu) S (p_\mu, a', a) \quad (34) \]

where \( a' \) and \( a \) are the additional quantum numbers which are used to specify the states. If the state vectors are so normalized that the S-matrix elements are relativistic invariants, then the function \( S(p_\mu, a', a) \) must also be an invariant function constructed out of the variables \( p_\mu, a' \) and \( a \). The four-dimensional \( \delta \)-function in eq. (34) is precisely the \( \delta \)-function extracted in the definition of the T-matrix in eq. (11). In that definition a normalization factor \( N \) was taken out to make the remainder a relativistically invariant function for spinless particles. For particles with spin the spin average has the same property.

2.5 Final state interactions

So far we have not taken full advantage of the unitarity relation for the S-operator. In this section we shall exploit this property and deduce some important relations for the T-matrix elements. These relations will turn out to be very useful in the later discussions of time reversal and CPT invariance. They shed light on a complication called final state interactions which plague some particle processes as we shall see later.

Consider a process whereby an initial state \( | \alpha \rangle \) goes over into a final state \( | \beta \rangle \). We have given the probability \( w(\beta, \alpha) \) for this to occur in the eq. (7)

\[ w(\beta, \alpha) = | \langle \beta | S | \alpha \rangle |^2 = \langle \alpha | S^\dagger | \beta \rangle \langle \beta | S | \alpha \rangle \quad (35) \]

Conservation of probability implies that the total probability that the state \( | \alpha \rangle \) at the end is found in some channel \( | \beta \rangle \)
must be unity, that is
\[
\Sigma_{\beta} w(\beta, \alpha) = 1
\]  \hspace{1cm} (36)

or
\[
\Sigma_{\beta} \langle \alpha | S^\dagger | \beta \rangle = \langle \beta | S | \alpha \rangle = \langle \alpha | S^\dagger S | \alpha \rangle = 1
\]  \hspace{1cm} (37)

If this is to hold for any state |\alpha\rangle then it follows that
\[
S^\dagger S = 1
\]  \hspace{1cm} (38)

or
\[
S^\dagger = S^{-1}
\]  \hspace{1cm} (39)

and this is the unitarity condition for the S-operator.

To rewrite the unitarity condition (38) in terms of the T-operator we put
\[
S = 1 + \mathbf{i} T
\]  \hspace{1cm} (40)

This definition of the transition operator differs from the previous one of eq. (11) by a trivial normalization factor, which is irrelevant for the present discussion. Inserting (40) into (38) we find
\[
i (T^\dagger - T) = T^\dagger T
\]  \hspace{1cm} (41)

If we take the matrix element of this operator relation between the states |\alpha\rangle and |\beta\rangle we obtain
\[
i \left[ \langle \beta | T^\dagger | \alpha \rangle - \langle \beta | T | \alpha \rangle \right] = \langle \beta | T^\dagger T | \alpha \rangle
\]  \hspace{1cm} (42)

Introducing a complete set of intermediate states on the right hand side yields
\[
i \left[ \langle \beta | T^\dagger | \alpha \rangle - \langle \beta | T | \alpha \rangle \right] = \sum_{\gamma} \langle \beta | T^\dagger | \gamma \rangle \langle \gamma | T | \alpha \rangle
\]  \hspace{1cm} (43)
\[ i [ T^*_{\alpha \beta} - T_{\beta \alpha} ] = \Sigma_{\gamma} T^*_{\gamma \beta} T_{\gamma \alpha} \tag{44} \]

Of course, the summation on the right hand side is subjected to restrictions imposed by conservation laws and other symmetry conditions. In special cases there are no possible intermediate states \( | \gamma \rangle \), and we deduce from (44)

\[ < \alpha | T | \beta >^* = < \beta | T | \alpha > \tag{45} \]

In other cases the right hand side of the eq. (44) is very small because possible intermediate states can only be reached by means of weak or electromagnetic interactions. In those cases it is sometimes possible to evaluate the corrections to the approximate relation (45).

The physical content of these results is most easily described by considering a specific process where the right hand side of the eq. (44) cannot be neglected. Such a process is the normal decay of a \( \Lambda \)-particle

\[ \Lambda \rightarrow p + \pi^- \]

If we neglect all intermediate states which are possible, then the process is described by the diagram (a) of figure 1. However, the proton and the \( \pi \)-meson in the final state may rescatter, a strong process described by the diagram (b) of figure 1. This intermediate state will give a non-negligible contribution to the sum in the eq. (43). Due to the rescattering the momenta of the proton and the \( \pi \)-meson will change. Of course, there are other intermediate states which contribute to this process. Collectively these effects are called final state interactions for obvious reasons. As mentioned before these effects will be of importance for the considerations related to time reversal invariance and we shall then make use of the results of this section.
3. Space reflection and parity.

3.1 Definition and action on state vectors.

The space reflection transformation is a classical concept defined by the coordinate transformation

\[(x_o, \bar{x}) \rightarrow (x'_o, \bar{x}') = (x_o, -\bar{x})\]  \hspace{1cm} (46)

It corresponds to a description of the same physical event in a right-handed respectively a lefthanded coordinate frame. In quantum theory the transformation is implemented by a unitary operator \(P\). Since the transformation is of classical origin we know its action on various observables. For example, it must satisfy the following conditions

\[P x^\mu \left( P^{-1} \right)^\mu = \varepsilon (\mu) x^\mu\]

\[P P^\mu \left( P^{-1} \right)^\mu = \varepsilon (\mu) P^\mu\]  \hspace{1cm} (47)

\[P J_{\mu\nu} \left( P^{-1} \right)^{\mu\nu} = \varepsilon (\mu) \varepsilon (\nu) J_{\mu\nu}\]
where $x^\mu$ is the coordinate operator, $P^\mu$ the momentum operator and $J_{\mu\nu}^\rho$ the angular momentum operator (more precisely, $J_{\mu\nu}^\rho$ for $\mu, \nu \neq 0$ is an angular momentum operator while $J_{\mu\nu}^0 = -J_{\nu\mu}^0$ is a generator of a pure Lorentz transformation). We have further introduced the symbol $\varepsilon(\mu)$ which is defined as $\varepsilon(0) = 1$ and $\varepsilon(k) = -1$ for $k = 1, 2, 3$. In adopting the active formulation we must recall that the space reflection invariance is not generally valid (see further discussion below). As a consequence, $P$ acting on a physical state vector may in some cases lead out of the physical Hilbert space. However, only the weak interactions seem to violate $P$ invariance and for many important processes the influence of the weak interactions is negligible. One may then neglect their contributions to the $S$-matrix element, which will then satisfy the requirements imposed by $P$ invariance.

For later applications we need to know the action of $P$ on state vectors. From the equation (47) it is seen that it reverses the sign of linear momenta but leaves angular momenta (spins) unchanged. If one uses a set of basis vectors labeled by the individual three-momenta $\vec{p}$ and the helicities $\lambda$ (spin component along the direction of motion) then one finds the following transformation law for a single particle state vector

$$P | \vec{p}, \lambda > = \eta_p \exp(-i\pi s) | -\vec{p}, -\lambda >$$

(48)

where $\eta_p$ is the intrinsic parity of the state and $s$ is its invariant spin. For a precise definition of the state vector $| \vec{p}, \lambda >$ we refer to the appendix 2. In appendix 2 we have collected the definitions of all the different sets of basis vectors which will be used in this series of lectures. Also some of their main properties are summarized there.

It follows from the eq. (48) that $P^2$ acting on any state vector will map it onto itself and, hence, $P^2$ must be a multiple of the identity and one may choose the undetermined phase factor in such a way that $P$ is hermitian and unitary, that is

$$P = P^\dagger = P^{-1}$$

(49)
With this choice of phase the eigenvalues of $P$ are restricted to $\pm 1$. We further note that in the ordinary coordinate space a space reflection $P$ followed by a rotation $R(2n\pi)$ around an arbitrary axis yields the same result as $P$ alone, provided $n$ is an integer. Thus if

$$P'(n) = P \circ R(2n\pi); \quad n \text{ integer}$$

then either one of $P$ and $P'$ may serve as the space reflection transformation. Consider now an eigenstate $|\eta\rangle$ of $P$ for which

$$P |\eta\rangle = |\eta\rangle,$$

If $|\eta\rangle$ is a state of integer spin one finds

$$P'(n) |\eta\rangle = |\eta\rangle.$$ 

Similarly, for a state of half-integer spin one obtains

$$P'(n) |\eta\rangle = (-1)^n |\eta\rangle,$$

since the half-integer spin representations of the rotation group are double-valued. As a consequence of this one may only define relative parities for fermions. In passing we finally note that one could also have identified $P^2$ with a rotation $R(2n\pi)$. For $n$ odd one would then obtain the eigenvalues $\pm 1$ for $P^2$ with the minus sign referring only to states of half-integer spin. If this is done then the physical space reflection operator may have the eigenvalues $\pm 1$ and $\pm 1$. We mention this since attempts have been made to exploit this additional freedom in the definition of the space reflection transformation $^3$. Since nothing of interest has come out of these attempts there is at present no reason to consider this possibility further, however.

Returning to the transformation properties of the various sets of basis vectors we list the ones we shall make use of. For single particle states we have

$$P |\bar{p}, \lambda\rangle = \eta_p \exp(-i \pi s) |\bar{p}, \lambda\rangle,$$

$$P |j, m, p, \lambda\rangle = \eta_p (-1)^{j-s} |j, m, p, \lambda\rangle.$$ (50)
Similarly for two-particle states one finds

\[ P | \vec{P} = 0; J, M, p; \lambda_1, \lambda_2 > = \]

\[ = \eta_p(1) \eta_p(2)(-1)^J s_1 - s_2 | \vec{P} = 0; J, M, p; -\lambda_1, -\lambda_2 > \]  \hspace{1cm} (51)

\[ P | \vec{P} = 0; J, M, p; l, \sigma > = \eta_p(1) \eta_p(2)(-1)^l | \vec{P} = 0; J, M, p; l, \sigma > \]

In this last set of basis vectors one has first coupled the two individual spins \( \vec{s}_1 \) and \( \vec{s}_2 \) to a resultant spin \( \vec{\sigma} \), which is then coupled to the orbital angular momentum \( \vec{l} \) to yield a total angular momentum \( (J, \omega) \). It is seen from the eq. (51) that these vectors are particularly useful for space reflection considerations since they are eigenvectors of \( P \). Finally, in some applications we shall also encounter three-particle states. We shall then always make use of basis vectors in which the relative orbital angular momentum \( \vec{l} \) of particle 1 and 2 is coupled to the orbital angular momentum \( \vec{L} \) of particle 3 (with reference to the center of mass of the particles 1 and 2) to a total orbital angular momentum. Similarly the spins \( \vec{s}_1 \) and \( \vec{s}_2 \) are coupled to \( \vec{\sigma} \) and this in

\[ \text{Figure 2: The coupling of orbital angular momenta in a three-particle state.} \]

turn is coupled to \( \vec{s}_3 \). Finally the total orbital angular momentum and the total spin \( \vec{\Sigma} \) are coupled to yield a total angular momentum \( (J, M) \). The corresponding basis vectors are denoted \( | \vec{P} = 0; J, M; l, \sigma; L, \Sigma > \)
They are eigenvectors of $P$ as indicated by the transformation law

$$ P | P = 0; J, M; l, \sigma; L, \Sigma > = $$(52)

$$ \eta_p (1) \eta_p (2) \eta_p (3) (-1)^{1+L} | \bar{P} = 0; J, M; l, \sigma; L, \Sigma > $$

From the transformation rules for the state vectors it is a straightforward task to deduce the transformation rules which creation and destruction operators must obey. From this one may then obtain the transformation rules for the field operators of a second-quantized theory. Although we shall not make use of field-theoretic concepts we list the transformation rules of the fields $\varphi(x), \psi(x)$ and $A_\mu(x)$, describing particles of spin $0, \frac{1}{2}$ and 1 respectively. For spin 1 we only consider the case of a massless field (the electromagnetic field).

$$ P \varphi (x_0, \vec{x}) P^{-1} = \eta_p \varphi (x_0, -\vec{x}) $$

$$ P \psi (x_0, \vec{x}) P^{-1} = \eta_p \gamma_0 \psi (x_0, -\vec{x}) $$

$$ P \bar{\psi} (x_0, \vec{x}) P^{-1} = \bar{\eta}_p \bar{\psi} (x_0, -\vec{x}) \gamma_0 $$

$$ P A_\mu (x_0, \vec{x}) P^{-1} = \epsilon (\mu) \eta_p A_\mu (x_0, -\vec{x}) $$

(53)

A spin-0 field for which $\eta_p = +1$ is called a scalar field. If $\eta_p = -1$ the field is said to be a pseudoscalar field.

In passing we note that off hand there is no compelling reason to choose the intrinsic parity $\eta_p$ of the eq. (48) to be the same for a particle and its antiparticle. In fact, for fermions and antifermions one must choose the relative intrinsic parity to be odd in order that the field equations should remain invariant. Hence, for a fermion - antifermion state one has

$$ \eta_p (1) \eta_p (2) = -1 $$

(54)
3.2 Some applications of the space reflection symmetry.

Under the assumption that the S-operator is space reflection invariant we have from the eq. (22)

$$\langle \beta; \text{out} \mid S \mid \alpha; \text{out} \rangle = \langle P\beta; \text{out} \mid S \mid P\alpha; \text{out} \rangle$$

(55)

where $|P\alpha\rangle$ refers to the space-reflected state. This relation states that the matrix element and hence the transition probability for the process $|\alpha\rangle \rightarrow |\beta\rangle$ is the same as that of the process $|P\alpha\rangle \rightarrow |P\beta\rangle$.

From this one may in many important cases derive selection rules or restrictions on the possible form of the matrix elements. We will demonstrate this in a few examples below.

(i) The $\theta - \tau$ puzzle.

The original suggestion of Lee and Yang that the weak interactions do not respect space reflection invariance was made to resolve the so-called $\theta - \tau$ puzzle. One had observed what one believed to be two different particles $\theta$ and $\tau$. They had the same mass and were distinguished solely by their different decay modes

$$\theta^+ \rightarrow \pi^+ + \pi^0$$

$$\tau^+ \rightarrow \pi^+ + \pi^+ + \pi^-$$

If parity is a good quantum number in these decays, then we will demonstrate below that the relative intrinsic parity of the two particles must be odd. Now we know that parity is violated in these decays and that the two particles in fact are the same (the K meson). To establish the difficulty which one was faced with before the parity violation was discovered let us assume that both particles have spin zero (the argument can easily be generalized to higher spins).

For the $\theta$-particle decay the total angular momentum $l$ of the final state is zero due to angular momentum conservation. Thus the parity of the final state is

$$\eta = \eta(\pi^+)(\pi^0)(-1)^0 = +1$$
since the $\pi$-mesons are pseudoscalar particles. Parity conservation then requires that $\eta_\Theta = +1$.

In the $\tau$ decay we denote the relative orbital angular momentum of the $2\pi^+$ system by $l$ and the orbital angular momentum of the $\pi^-$ relative to the center of mass of the $2\pi^+$ system by $L$. If the spin of the $\tau$ is zero, then angular momentum conservation requires $l + L = 0$ or $l = L$. The total parity of the final state is then $(-1)^3 (-1)^1 + L = -1$, and conservation of parity requires that $\eta_\tau = -1$. Thus the $\Theta$ and the $\tau$ seem to have opposite parity. The complete discussion of this problem is found in reference 4).

(ii) The $\beta^-$-decay of polarized Co$^{60}$.

The first actual verification of Lee and Yang's hypothesis that parity is not conserved in weak processes was obtained from the study of the decay $^5$)

$$\text{Co}^{60} \rightarrow \text{Ni}^{60} + e^- + \nu_e$$

The Co$^{60}$ nucleus has the spin-parity assignment $5^+$ in its ground state and it decays into an excited state of Ni$^{60}$ with $4^+$. Since Co$^{60}$ has nonvanishing spin one may polarize a sample of Co$^{60}$ nuclei. In this way one defines a direction in space. Consider now the two experimental configurations of figure 3. Clearly the two configurations are related by a space reflection transformation in the active

![Diagram](image)

**Figure 3**: Decay configuration in the $\beta^-$-decay of polarized Co$^{60}$ nuclei.
sense. From the eq. (55) we know that if the theory is invariant under P, then the transition probability for the two decays must be the same, a prediction one can easily test experimentally. Thus, one should investigate if in a sample of polarized Co$^{60}$ nuclei the number of electrons emitted in the forward direction with regard to $\langle \vec{q}_N \rangle$ is the same as the number emitted in the backward direction. In the experiment by Wu et al. 5) one found that the electrons were preferentially emitted in the backward direction from which one then could conclude that P in fact is violated.

We may express the result in a slightly different way. In an experiment, where one observes (i) $\langle \vec{q}_N \rangle$ and (ii) the momentum $\vec{p}_e$ of the emitted electron but no other quantities, the most general form of the transition rate consistent with rotational invariance is given by

$$\frac{d\lambda}{d(\cos \theta)} = \Lambda (1 + \alpha \langle \vec{q}_N \rangle \cdot \vec{p}_e)$$

where $\theta$ is the angle between the two vectors $\langle \vec{q}_N \rangle$ and $\vec{p}_e$.

Since the second term changes sign under a space reflection it is clear that P is a valid symmetry only if $\alpha = 0$. Experimentally one found $\alpha^{\exp} \approx -v/c$, where $v$ is the electron velocity.

We note in passing that an observed asymmetry is evidence for P-violation. On the other hand the absence of a P-violating effect may always be accidental and cannot be taken as conclusive evidence for P conservation.

(ii) Space reflection and the process $\nu_\mu + n \rightarrow \mu^- + p$.

In many important applications invariance arguments are used to restrict the possible form of matrix elements. We shall demonstrate how the method works by considering the process $\nu_\mu + n \rightarrow \mu^- + p$ in relation to the various discrete transformations. The process is
due to the weak interactions and it has been investigated in connection with the discovery that there exist two kinds of neutrinos, namely $\nu_\mu$ and $\nu_e$.

Two configurations related by a space reflection are depicted in figure 4. Since parity is not conserved by the weak interaction we cannot deduce an equality between the two matrix elements.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Two configuration of the process $\nu_\mu + n \rightarrow \mu^- + p$ related by a space reflection.}
\end{figure}

Instead we shall demonstrate the difference between (a) and (b) within the conventional V-A theory of the weak interactions. This is a field-theoretic model with an interaction described by the hamiltonian density

$$H'(x) = \frac{G}{\sqrt{2}} \cdot j_\mu^\dagger(x) j_\mu^\mu(x)$$

where $G \approx 10^{-5}/m_p^2$ is the weak interaction coupling constant and the current $j_\mu$ may be decomposed into

$$j_\mu(x) = j_\mu^1(x) + j_\mu^h(x)$$

corresponding to a leptonic part and a hadronic part (involving strongly interacting particles $\equiv$ hadrons). This separation is motivated by the fact that leptons do not interact strongly. For that reason the matrix element of the leptonic current has a very simple structure.

If we treat the problem to lowest order in the perturbation expansion we obtain for the process (a)
\[ \langle p, \mu^- | S | n, \nu_\mu^- \rangle = -i \langle p, \mu^- | \int d^4x \ H'(x) | n, \nu_\mu^- \rangle = \]

\[ = -\frac{iG}{\sqrt{2}} \delta^4 \delta \langle p | J^\dagger_\mu (0) | n > 1^\mu \]

with

\[ 1^\mu \equiv \langle \mu^- | J^\mu (0) | \nu^- \rangle = \bar{u} (\overline{p}; \overline{s}) \gamma^\mu (1 + \gamma_5) u(p, s) \]  

(60)

In (60) we have omitted some trivial normalization factors which are irrelevant for the present discussion. For the hadronic matrix element \[ \langle p | J^\dagger_\mu (0) | n > \] we cannot neglect the structure effects due to the strong interactions. However, we know that the matrix element shall transform as a four-vector under proper Lorentz transformations. Using this fact and the properties of the \( \gamma \) -matrices and the spinors \( u(p, s) \) one can easily show that the most general form is given by

\[ h^\mu_\mu \equiv \langle p | J^\dagger_\mu (0) | n > = \]

\[ = \bar{u} (\overline{k}; \overline{w}) \{ F_1 (q^2) \gamma^\mu_\mu + i F_2 (q^2) \sigma_{\mu\nu} q^\nu + F_3 (q^2) q^\mu + \]

\[ + [ G_1 (q^2) \gamma^\mu_\mu + i G_2 (q^2) \sigma_{\mu\nu} q^\nu + G_3 (q^2) q^\mu ] \gamma_5 \} u(\overline{k}, \overline{w}) \]

(61)

where \( F_1 (q^2) \) and \( G_i (q^2) \) are unknown formfactors depending on the variable \( q^2 \), where \( q = k' - k \) is the momentum transfer. These formfactors are complex functions and the factors i have been inserted for convenience. We shall see later that with the conventions of the eq.(61) time reversal invariance will permit us to choose the form factors to be real functions. The corresponding expressions \( l^\mu_\mu (p) \) and \( k^\mu_\mu (p) \) for the process are similarly given by

\[ l^\mu_\mu (p) = \bar{u} (\overline{-p}; \overline{s'}) \gamma^\mu (1 + \gamma_5) u(-\overline{p}, \overline{s}) \]

(62)
and
\[ h_{\mu} (\bar{P}) = \bar{u}(q^2) \gamma_{\mu} \{ P_1(q^2) \gamma_{\mu} + i \varepsilon(\nu) P_2(q^2) \sigma_{\mu\nu} q^\nu \] + \varepsilon(\mu) P_3(q^2) q_{\mu} + \] + \{ G_1(q^2) \gamma_{\mu} + i \varepsilon(\nu) G_2(q^2) \sigma_{\mu\nu} q^\nu + \varepsilon(\mu) G_3(q^2) q_{\mu} \gamma_5 \} u(-\bar{k}, \bar{w}) \] \tag{63}

The factors \( \varepsilon(\mu) \) take care of the change in sign for the space components of the momentum transfer under the space reflection. To compare the matrix elements for the two processes we make use of the following relations
\[ u(-\bar{k}, \bar{w}) = \gamma \circ u(k, w) \] \tag{64}
\[ \bar{u}(-\bar{k}, \bar{w}) = \bar{u}(k, w) \gamma \circ \]

Inserting this in the eqs. (62) and (63) we obtain
\[ 1_{\mu}(\bar{P}) = \bar{u}(\bar{P}', \bar{s}') \gamma \circ \gamma_{\mu} (1 + \gamma_5) \gamma \circ u(\bar{P}, \bar{s}) = \] \[ = \varepsilon(\mu) \bar{u}(\bar{P}', \bar{s}') \gamma_{\mu} (1 - \gamma_5) u(\bar{P}, \bar{s}) \] \tag{65}

and
\[ h_{\mu}(\bar{P}) = \varepsilon(\mu) u(k, \bar{w}) \{ P_1(q^2) \gamma_{\mu} + i P_2(q^2) \sigma_{\mu\nu} q^\nu + P_3(q^2) q_{\mu} - \] \[ - \{ G_1(q^2) \gamma_{\mu} + i G_2(q^2) \sigma_{\mu\nu} q^\nu + G_3(q^2) q_{\mu} \} \gamma_5 \} u(k, \bar{w}) \] \tag{66}

In forming the product \( h_{\mu}(\bar{P}) l_{\mu}(\bar{P}) \) we see that the factors \( \varepsilon(\mu) \) cancel out. Comparison between \( l_{\mu}(\bar{P}) \) and between \( h_{\mu}(\bar{P}) \) and \( h_{\mu}(\bar{P}) \) shows that all terms containing \( \gamma_5 \) have changed sign. Since an over-all change of sign for the S-matrix element does not affect the transition probability we find that the cross sections for the
where the final state is reached via the resonant intermediate state \( \text{Li}^6 \). If the intermediate state is unstable under the break-up process then there is a partial width \( \Gamma_{d_a} \) of the corresponding level, and this width enters in the cross section for the process (70) as a measure of the probability that the initial state may reach the intermediate state. A small width \( \Gamma_{da} \) clearly corresponds to a small chance for the process to occur. Experimentally one has determined an upper limit for the width of 0.2 ev from which one may conclude that \( |F|^2 < 10^{-7} \).

(2) Circular polarization in nuclear \( \gamma \)-transitions.

If parity is strictly conserved, then the \( \gamma \)-rays emitted following the bombardment of an unpolarized target by unpolarized particles can show no circular polarization. Since circular polarization is a correlation of the type \( (\bar{\sigma} \cdot \bar{k}) \) it is a result of an interference between parity conserving and parity non-conserving parts in the interaction. It is then proportional to the amplitude \( P \) introduced in the previous example. Several experiments of this general type have been reported. We shall only mention one of these with implications for the presence of weak parity violating forces between the nucleons in nuclei. In the experiment \(^8\) one measured the \( \gamma \)-transitions of 482 kev in \( \text{Ta}^{181} \). The transition is between a 5/2\(^+\) state and a 7/2\(^+\) state and it may occur as a M1 or E2 emission if parity is conserved. If parity is not conserved there may be admixtures of 5/2\(^-\) and 7/2\(^-\) states and also the E1 matrix element may be nonvanishing. A circular polarization may then be the result of interference between the E1 and M1 matrix elements. One observed a circular polarization \( P = - (2.0 \pm 0.4) \times 10^{-4} \) from which one concluded that there are parity non-conserving nuclear forces. Expressed in terms of the amplitude \( F \) one concludes from this experiment that \( |F| \approx 10^{-6} \). On the basis of the V-A theory with a current x current interaction one expects an effect of about this size, and the result was interpreted as a verification of this prediction. However, more recent experiments show no effect and the result of reference \(^8\) is at present in doubt \(^9\).

(3) Angular correlation of \( \gamma \)-emission from polarized nuclei.

Unless parity conservation breaks down the emission of particles from an isolated nuclear level cannot display odd powers of \( \cos \theta \),
where $\theta$ is the angle between the initial polarization direction and the momentum of the emitted particle. An experiment of this type is the decay of polarized Cd$^{114*}$ through $\gamma$-emission \(^{10})\ns + \text{Cd}^{113} \rightarrow \text{Cd}^{114*} \rightarrow \text{Cd}^{114} + \gamma \tag{71}

By using polarized neutrons they obtain polarized Cd$^{114*}$. Spin and parity assignments are $1^+$ and $0^+$ for Cd$^{114*}$ respectively Cd$^{114}$ so that it must be a $M1$ emission if parity is conserved. If parity is not conserved also $E1$ radiation is possible and there will be a term proportional to $\cos^2 \theta$ in the angular correlation. From the experiment one obtained $E1/M1 = (4 \pm 8) \times 10^{-4}$ corresponding to $|F|^2 < 2 \times 10^{-9}$, which indicates a very small admixture of parity violating contributions.

(4) The electric dipole moment of the neutron.

A very stringent but less unambiguous test of $P$ invariance for the strong interactions is the measurement of the electric dipole moment of the neutron. At present the upper limit is set by

$$\frac{\mu_n}{e} < (0.1 \pm 2.4) \times 10^{-20} \text{ cm}$$

but more accurate measurements are in progress. Space reflection invariance requires it to vanish. However, also from time reversal invariance one arrives at the same conclusion. Therefore, the result cannot be attributed to $P$ invariance with any certainty.

(ii) Parity conservation in electromagnetic processes.

All the experiments quoted in connection with the strong interactions clearly have bearing also on the electromagnetic interactions although the accuracy in this case is rather poor. Much more accurate evidence for parity conservation in electromagnetic interactions is obtained from optical spectroscopy of atomic levels. The evidence in the literature suggest that impurities of parities in the energy levels of atoms or in their electromagnetic transitions is less than about $10^{-7}$ in intensity, that is, the parity-violating amplitude must be less than $10^{-3}$ of the parity conserving amplitude.

(iii) Parity conservation and the weak interactions.

In many of the applications of space reflection invariance we used examples of weak processes, and it was found that in fact they
violate this invariance principle strongly. In the conventional V-A theory the parity violation is maximal as implied by the fact that vector and axial vector parts are of equal strength. This theory seems to account for all known weak processes quite accurately and parity violating effects have been established in a wide variety of reactions.

4. Charge conjugation

The classical theory of electromagnetism can be derived from (i) Coulomb’s law for the force between charges can (ii) the theory of special relativity. In Coulomb’s law there is full symmetry between positive and negative charges; equal charges repel each other and unequal attract each other. Since the theory of special relativity only concerns space-time concepts, which are detached from the concept of charge, it is clear that the classical theory of electromagnetism also exhibits a basic symmetry between the two kinds of charges.

Later when the relativistic quantum theory was developed it was found that to each particle there must exist an antiparticle with the same space-time properties as the original one. Electromagnetism was incorporated in analogy with the classical theory and consistency then required that a particle and its antiparticle must have opposite charges. In this way one was led to consider the symmetry of charge more appropriately as a particle - antiparticle symmetry. For historical reasons one retained then name charge conjugation symmetry, although particle-antiparticle symmetry is a more accurate name for it now.

We may remark already at this point that invariance under charge conjugation C most probably holds for strong and electromagnetic interactions (see section 4.3 below), while the weak interactions violate it. This situation thus equals the situation for P invariance previously discussed and it seems that the appropriate particle - antiparticle transformation is the combined operation CP. To this
we shall return in section 5.

4.1 Definition and action on state vectors.

It has already been stated that particles and antiparticles have the same space-time properties. This requires that the operator representing the transformation must commute with all the generators of space-time transformations, that is, the Poincaré group extended to include also the inversions (P and T). On the other hand, charge, baryon number and lepton number have opposite sign for particle and antiparticle, and the corresponding operators in Hilbert space must then anticommute with the charge conjugation operator C. Thus we may classify the various operators with regard to their commutation rules with C; operators of the first class commute with C and operators of the second class anticommute with C.

We now define the action of the charge conjugation operator on a single particle state $|\alpha; \bar{k}, \lambda>$ by the relation

$$|\alpha; \bar{k}, \lambda> \rightarrow |\alpha'; \bar{k}', \lambda'> = C |\alpha; \bar{k}, \lambda> = \eta_C |\alpha; \bar{k}, \lambda>$$ (72)

where $\alpha$ denotes all the labels corresponding to observables of the second class (charge, etc) and $\eta_C$ is a phase factor (in order that C be a unitary operator). Applying a second charge conjugation operator one is led back to the original state and by choosing the phase factor $\eta_C$ suitably one may identify $C^2$ with the identity operator. With this choice the eigenvalues of C are $\pm 1$. Clearly, eigenstates of C can those states be, for which all charges $\alpha$ are zero. For example, only those states which have $Q = 0$, $B = 0$ and $L = 0$ can be eigenstates of C. These conditions express the obvious fact that only if the particle is identical with its antiparticle can the corresponding single particle state be an eigenstate of C. In the examples we shall encounter there are three particles which satisfy these conditions; the photon, the $\pi^0$ meson and the $\eta$ meson. They are of particular importance in this context because only in the decays of these particles do we have selection rules due to C-invariance.
To determine the charge parity \( \eta_c \) for the photon we recall that the whole concept of charge conjugation originated from the electromagnetic interaction which exhibited invariance under the transformation. The interaction is due to the coupling of the electromagnetic current to the electromagnetic field. Under a C-transformation the current changes sign and, hence, also the electromagnetic field must change sign in order that the interaction be invariant. Since the photon is the elementary quantum of the electromagnetic field we conclude

\[
C | \gamma > = - | \gamma >
\]  

(73)

Since space-time properties are not affected by the C-transformation we may extend eq. (73) to an arbitrary n-photon state

\[
C | n \gamma > = (-1)^n | n \gamma >
\]  

(74)

Similar considerations for the \( \pi^0 \) meson and the \( \eta \) meson yield

\[
C | \pi^0 > = | \pi^0 >
\]  

(75)

\[
C | \eta > = | \eta >
\]  

(76)

and correspondingly for the n-particle states involving neutral mesons.

We next proceed to the two-particle states such as \( e^+ e^- \), \( p \bar{p} \) etc. since these may be eigensates of C, and we shall here only consider such states. Then the so-called \textit{generalized Pauli principle} applies. To specify the content of this principle we regard a particle and its antiparticle as two different states of the same particle. The two states differ by opposite sign for all the quantum numbers (charges) corresponding to observables of the second class. The generalized Pauli principle now asserts that any fermion - antifermion state is antisymmetric under the transformation which exchanges the two particles, while a boson - antiboson state is symmetric under the same transformation. In field theory this principle expresses the fact that creation
and destruction operator satisfy (i) commutation relations for bosons and (ii) anticommutation relations for fermions. If we denote the spin and coordinate exchange operator by \( A(\sigma, r) \), then we may write the principle in the following way

\[
C A(\sigma, r) |\alpha, \bar{\alpha} > = (-1)^{2s} |\alpha, \bar{\alpha} >
\]  

(77)

Here \( \alpha \) and \( \bar{\alpha} \) denote any particle respectively its antiparticle of spin \( s \). Thus any particle-antiparticle state which is an eigenstate of \( A(\sigma, s) \) is simultaneously an eigenstate of \( C \) by the eq. (77).

We have previously introduced the two-particle basis vectors

\[
|\vec{p}=0; J, M, p; l, \sigma >
\]

which are eigenstates of the parity operator. These states are also eigenvectors of \( A(\sigma, r) \) and a closer examination yields

\[
A(\sigma, r) |\vec{p}=0; J, M, p; l, \sigma > = (-1)^{1+\sigma-2s} |\vec{p}=0; J, M, p; l, \sigma >
\]

(78)

and hence

\[
C |\vec{p}=0; J, M, p; l, \sigma > = (-1)^{1+\sigma} |\vec{p}=0; J, M, p; l, \sigma >
\]

(79)

This result will be most important for applications later. The relation (79) clearly holds even if one does not restrict oneself to states with vanishing total momentum.

Once the transformation rules for state vectors are given we may deduce the corresponding rules for field operators. Just as in the case of the space reflection transformation we just quote the results for a fields of spin 0, \( \frac{1}{2} \) and 1.

\[
C \varphi(x) C^{-1} = \eta \varphi^T(x)
\]

\[
C \psi(x) C^{-1} = \eta \varphi^T(x)
\]

\[
C \bar{\varphi}(x) C^{-1} = -\eta \bar{\varphi}^T(x) \varphi^{-1}
\]

\[
C A_{\mu}(x) C^{-1} = -\eta A_{\mu}(x)
\]

(80)
where $\mathcal{C}$ is a 4x4 matrix, whose properties are given in the eq. (92) below.

### 4.2 Some applications of charge conjugation invariance.

Invariance of the $S$-operator under charge conjugation implies

$$
\langle \beta | S | \alpha \rangle = \langle C \beta | S | C \alpha \rangle 
$$

(81)

If the states $|\alpha\rangle$ and $|\beta\rangle$ are eigenstates of $C$ with the eigenvalues $\eta_\alpha$ and $\eta_\beta$, then the eq. (81) may be written

$$
\langle \beta | S | \alpha \rangle = \eta_\alpha \eta_\beta \langle \beta | S | \alpha \rangle
$$

(82)

so that only if $\eta_\alpha \eta_\beta = +1$ do we obtain a nonvanishing matrix element. Under these circumstances $C$-invariance provides us with a selection rule. The more general case of eq. (81) always leads to relations between matrix elements imposing restrictions on the possible form for these matrix elements. We shall see below how these remarks apply in a few cases.

**(i) The decay $\pi^0 \rightarrow n \gamma$**

The more restrictive relation (82) applies only if the two states $|\alpha\rangle$ and $|\beta\rangle$ are eigenstates of $C$. We have previously seen that a necessary but not always sufficient condition for a state to be an eigenstate of $C$ is that it is neutral ($Q = 0$) and we have found that both $\pi^0$ states and photon states in fact are eigenstates of $C$. From the eqs. (74) and (75) we know that $\eta_c(\pi^0) = +1$ and $\eta_c(\gamma) = -1$ and hence

$$
\langle n\gamma | S | \pi^0 \rangle = (-1)^n \langle n\gamma | S | \pi^0 \rangle
$$

(83)
For $<n\gamma|\bar{\pi}^0|\neq 0$ we then obtain $n = 0 \mod 2$, that is, a $\pi^0$ can only decay into an even number of photons if the interaction is $C$-invariant. Clearly the same conclusion holds for the decay of the $\eta$ meson into photons.

(ii) The decay of positronium into photons

Positronium is a bound state of $(e^+e^-)$. Provided positronium is in a definite $(1, \sigma)$ state it will be an eigenstate of $C$ with the eigenvalue $(-1)^{1+\sigma}$. This follows directly from the eq. (79). For the decay

$$(e^+e^-)_{1,\sigma} \rightarrow n\gamma$$

we obtain the following selection rule

$$(-1)^{1+\sigma} = (-1)^n$$

or $1 + \sigma = n \mod 2$. From this selection rule we find (conventional spectroscopic notation is used for the positronium states)

$$^1S_0 \rightarrow 2\gamma; \ \ l=\sigma = 0; \ n=2$$

$$^3S_1 \rightarrow 3\gamma; \ \ l=\sigma = 0; \ n=3$$

$$^3S_1 \rightarrow 3\gamma; \ \ l=\sigma = 1; \ n=2$$

$$^3S_1 \rightarrow 3\gamma; \ \ l=\sigma = 1; \ n=3$$

etc.

The $p\bar{p}$ annihilation into $\pi^0$'s is, of course, completely analogous to the positronium decay and one obtains equivalent selection rules.

(iii) The decay $\eta \rightarrow \pi^+ + \pi^- + \pi^0$

This decay is of electromagnetic nature, since it does not conserve $G$-parity, a concept we shall not be concerned with, however.
The decay has been proposed as a suitable process to test C-invariance for the electromagnetic interactions. As pointed out by Bernstein et al. 11) many previous tests of C-invariance with regard to the electromagnetic interactions are ambiguous.

If the interaction responsible for the $\eta$ decay is C-invariant, then we obtain from the eq. (81)

$$\langle \pi^+ (k_+), \pi^- (k_-), \pi^0 (k) | S | \eta \rangle = \eta_c \langle \pi^- (k_+), \pi^+ (k_-), \pi^0 (k) | S | \eta \rangle$$

(87)

where $\eta_c$ is a phase factor which, however, is irrelevant for the present discussion. This relation implies that in $\eta$ decay the probability to find a $\pi^+$ with momentum $k_+$ and a $\pi^-$ with momentum $k_-$ is the same as finding a $\pi^+$ with momentum $k_-$ and a $\pi^-$ with momentum $k_+$. This in turn implies that the energy spectra for $\pi^+$ and $\pi^-$ in $\eta$ decay should be the same (the energy spectrum of $\pi^+$ is obtained by integrating the decay rate over all angles and the energies of the $\pi^-$ and the $\pi^0$ etc.). It also implies that the charged $\pi$ meson with the largest energy in $\eta$ decays should be a $\pi^+$ as often as a $\pi^-$. An unbalance in this respect clearly violates C invariance. The experimental results seem to favour no C-violation for this process but further clarification would be desirable (see below).

(iv) Charge conjugation and the process $\psi_{\mu} + n \rightarrow \mu^- + p$

We have previously studied this process in section 3.2 in connection with the space reflection transformation. In this section we will discuss it with regard to the charge conjugation transformation and as before we work within the framework of the V-A theory.

In figure 5 the diagrams for the process $\psi_{\mu} + n \rightarrow \mu^- + p$ and the charge conjugate process $\bar{\psi}_{\mu} + \bar{n} \rightarrow \mu^+ + \bar{p}$ are given. Since the C-transformation does not affect the space-time properties of a state there is in this case no change in the momenta and the spins. The matrix element of the process (a) is given by the eqs. (59), (60) and (61). To compute the matrix element of process (b) it must be realized
Figure 5: Two processes (a) $\nu_\mu + n \to \mu^+ + p$ and (b) $\bar{\nu}_\mu + \bar{n} \to \mu^+ + \bar{p}$ which are related by a charge conjugation transformation.

that although the V-A Hamiltonian $H_1(x)$ of the eq. (57) is hermitian each term by itself does not have this property, that is, the currents of the eq. (58) are not hermitian. The relevant part of the S-operator for the process (b) is in fact the hermitian conjugate of the part describing the process (a)\(^{12}\), and the matrix element is given by

$$< \bar{p}, \mu^+ | S | \bar{n}, \nu_\mu > = - \frac{ig}{\sqrt{2}} (2\pi)^4 \delta^4 (p_1 - p_2) < \bar{p} | J_{\mu} (c) | \bar{n} > \gamma_1^{\nu} (c)$$

where

$$1_\nu (c) \equiv \bar{\nu} (\bar{p}, s) \gamma_\mu (1 + \gamma_5) \nu (p', s')$$

and

$$h_{\mu} (c) \equiv < \bar{p} | J_{\mu} (0) | \bar{n} > = < \bar{n} | J^{\dagger}_{\mu} (0) | \bar{p} >$$

$$= \{ \bar{\nu} (\bar{p}, \bar{w}) \} \left[ F_1 (q^2) \gamma_\mu - iF_2 (q^2) \sigma_{\mu \nu} q^\nu - iF_3 (q^2) q_\mu \right] +$$

$$+ G_1 (q^2) \gamma_\mu \gamma_5 - iG_2 (q^2) \sigma_{\mu \nu} q^\nu \gamma_5 - G_3 (q^2) q_\mu \gamma_5 \bar{\nu} (\bar{p}, \bar{w}) \}^*$$

This last relation is most easily derived if one makes explicit use of an effective current operator in terms of field operators\(^{13}\).

To compare the matrix elements of the two processes we make use of the following properties of the Dirac spinors
\[ v(\vec{p}, \vec{s}) = \mathcal{C} \bar{u}^T(\vec{p}, \vec{s}) \]

\[ \bar{v}(\vec{p}, \vec{s}) = -u^T(\vec{p}, \vec{s}) \mathcal{C}^{-1} \]  \hspace{1cm} (91)

with the matrix \( \mathcal{C} \) defined by

\[ \mathcal{C}^{-1} \gamma^\mu \mathcal{C} = -\gamma^\mu \]
\[ \mathcal{C}^{-1} \gamma^5 \mathcal{C} = \gamma^5 \]
\[ \mathcal{C}^\dagger = \mathcal{C}^{-1} \]
\[ \mathcal{C}^T = -\mathcal{C} \]  \hspace{1cm} (92)

The superscript \( T \) means transposition. Inserting (91) into \( l_\mu^\mu(C) \) we obtain

\[ l_\mu^\mu(C) = -u^T(\vec{p}, \vec{s}) \mathcal{C}^{-1} \gamma^\mu (1+\gamma^5) \mathcal{C} \bar{u}^T(\vec{p}; \vec{s}') \]
\[ = -\bar{u}(\vec{p}; \vec{s}') \left\{ \mathcal{C}^{-1} \gamma^\mu (1+\gamma^5) \mathcal{C} \right\}^T u(\vec{p}, \vec{s}) = \]
\[ = \bar{u}(\vec{p}; \vec{s}') \left\{ \gamma^\mu (1+\gamma^5) \right\}^T u(\vec{p}, \vec{s}) \]
\[ = \bar{u}(\vec{p}; \vec{s}') \left\{ \gamma^\mu (1-\gamma^5) \right\} u(\vec{p}, \vec{s}) \]  \hspace{1cm} (93)

and similarly for \( h_\mu^\mu(C) \)

\[ h_\mu^\mu(C) = \bar{u}(\vec{k}; \vec{w}') \left\{ F_1^*(q^2) \gamma^\mu + iF_2^*(q^2)\sigma_{\mu\nu}q^\nu + F_3^*(q^2)q^\mu - \right\} \]
\[ - \left[ G_1^*(q^2) \gamma^\mu + iG_2^*(q^2)\sigma_{\mu\nu}q^\nu + G_3^*(q^2)q^\mu \right] \gamma^5 \}
\[ u(\vec{k}, \vec{w}) \]  \hspace{1cm} (94)

With the symbolic notations of the eq. (67) we may write this result in the following way

\[ h_\mu^\mu(C) l_\mu^\mu(C) = \varphi(F_1^*, G_1^*; 1 - \gamma^5) \]  \hspace{1cm} (95)

Once more we find that it is the simultaneous presence of terms with \( \gamma^5 \) and without \( \gamma^5 \) in both \( l_\mu^\mu \) and \( h_\mu^\mu \) that makes it impossible to achieve C-invariance. Any quantity which depends on the interference between the two kind of terms will change sign when we pass from the process(a) to the process(b).
4.3 Tests of charge conjugation invariance

We proceed to discuss some of the more stringent tests of C-invariance for the three types of interactions.

(i) C-invariance in strong interaction processes.

There is so far no evidence for C-violation in strong interaction processes. The most accurate test seems to be annihilation experiments of \( \bar{p}p \) at rest. It is believed that the annihilation takes place from an S-state (\( l = 0 \)). From the eq. (79) it then follows that the initial state is an eigenstate of C with \( \eta_c = +1 \) for \( \sigma = 0 \) (antiparallel spins) and \( \eta_c = -1 \) for \( \sigma = 1 \) (parallel spins). Thus, from angular momentum conservation we conclude that the \( \bar{p}p \) state with \( \eta_c = +1 \) will go over into an over-all S-state and the state with \( \eta_c = -1 \) into a P-state. Therefore, if one performs the angular integrations over the final states there will be no interference between the contributions from the two possible initial \( \bar{p}p \) states, and the momentum distribution for the annihilation products must be invariant under interchange of positive and negative particles. In the experimental test of this prediction one has examined for example the following annihilation channels

\[
\begin{align*}
\bar{p}p & \rightarrow \pi^+ + \pi^- + \pi^0 \\
& \rightarrow \pi^+ + \pi^+ + \pi^- + \pi^- \\
& \rightarrow \pi^+ + \pi^- + \pi^0 \\
& \rightarrow K^+ + \pi^- + K^0 \\
& \rightarrow K^- + \pi^+ + K^0 \\
& \rightarrow K^+ + \pi^- + \pi^0 + K^0 \\
& \rightarrow K^- + \pi^+ + \pi^0 + K^0
\end{align*}
\]
In the $K$-channels one compared energy and momentum distributions of the $K^+$ and the $K^-$ etc. If $\alpha$ is the ratio of the C violating amplitude to the C conserving amplitude, then one obtained
\[
|\alpha_\pi| < 0.01 \quad \quad |\alpha_K| < 0.03
\]

In passing we note that in the absence of final state interactions the same result follows from CPT invariance. However, the final state interactions are of strong nature and a C-violation is thus expected to give observable effects. This remark touches upon a very important question, namely, a specific prediction may follow from several alternative invariance principles and it may be impossible to interpret a verification of the prediction as evidence for any one of them.

(ii) C-invariance in electromagnetic processes.

In many cases of electromagnetic processes one or several of the participating particles also interact strongly and tests of C-invariance in such cases obviously have bearing also on the question of C-invariance for the strong interactions although indirectly.

(1) The decay $\pi^0 \rightarrow 3\gamma$

We have previously found that this decay is strictly forbidden if C-invariance holds. The dominant decay mode of the $\pi^0$ is clearly $\pi^0 \rightarrow 2\gamma$ and an upper limit on the branching ratio of the two processes has been determined\(^{16}\)
\[
B \equiv \frac{\lambda(\pi^0 \rightarrow 3\gamma)}{\lambda(\pi^0 \rightarrow 2\gamma)} < 5 \times 10^{-6}
\]

(2) The decay $\eta \rightarrow \pi^+ + \pi^- + \pi^0$.

Denote by $N_+$ the number of $\eta$ decays where $E(\pi^+) > E(\pi^-)$ and by $N_-$ the number of events with $E(\pi^-) > E(\pi^+)$. We have previously shown that C-invariance implies that $N_+ = N_-$. One has determined the asymmetry
parameter $A$, where

$$A = \frac{N_+ - N_-}{N_+ + N_-}$$

in a number of experiments \(^{17}\) and the results are listed below.

<table>
<thead>
<tr>
<th></th>
<th>Number of decays</th>
<th>$A$ per cent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baltay et al.</td>
<td>1351</td>
<td>7.2 ± 2.8</td>
</tr>
<tr>
<td>Rutherford - Saclay</td>
<td>795</td>
<td>-6 ± 4</td>
</tr>
<tr>
<td>CERN</td>
<td>10665</td>
<td>0.3 ± 1.0</td>
</tr>
</tbody>
</table>

It is seen that the Cern result rests on the best statistics. In view of the fundamental character of this experiment it is certainly desirable to investigate the matter further.

(3) The decay $\eta \rightarrow \pi^+ + \pi^- + \gamma$

The previous discussion of $\eta \rightarrow 3\pi$ applies equally well to this process, and one may look for an asymmetry in the energy distribution for the two charged $\pi$ mesons as evidence for a C-violation. If we define the asymmetry parameter $A$ in the same way as before it is found \(^{18}\) that

$$A = (1.5 \pm 2.5) \times 10^{-2}$$

which is consistent with C-conservation for this electromagnetic process.

(4) The decay $\eta \rightarrow \pi^0 + e^+ + e^-$.  

This is clearly an electromagnetic process and to lowest order in the electromagnetic coupling it is described by the diagram of figure 6. The relevant part of the transition matrix element is
Figure 6: The decay $\eta \rightarrow \pi^0 + e^+ + e^-$

$\langle \pi^0 | J^\text{EM}_\mu (0) | \eta^0 \rangle$ where $J^\text{EM}_\mu (0)$ is the electromagnetic current operator. We have previously given the transformation properties of the two state vectors, which appear in this matrix element, and of the current operator. With those relations in mind we find

$$
\langle \pi^0 | J^\text{EM}_\mu (0) | \eta^0 \rangle = \langle \pi^0 | C^{-1} C J^\text{EM}_\mu (0) C^{-1} C | \eta^0 \rangle = \\
= \langle \pi^0 | C J^\text{EM}_\mu (0) C^{-1} | \eta^0 \rangle = - \langle \pi^0 | J^\text{EM}_\mu (0) | \eta^0 \rangle
$$

and we conclude that this matrix element must vanish unless $C$ is violated. Of course, if the electromagnetic current operator has a component which is even under $C$ \textsuperscript{11} then (i) $C$ invariance is violated by the electromagnetic interactions and (ii) the decay $\eta \rightarrow \pi^0 + e^+ + e^-$ is no longer forbidden to this order in the coupling constant. Experimentally \textsuperscript{19} one has determined the branching ratio for this decay with the following result

$$
B \equiv \frac{\lambda (\eta \rightarrow \pi^0 + e^+ + e^-)}{\lambda (\eta \rightarrow \pi^0 + \pi^+ + \pi^-)} < 0.0045
$$

which supports the assumption of $C$-invariance.

It should be noted that this process is allowed by higher order
terms in the perturbation expansion even if C-invariance is an exact
symmetry. For example, the two-photon exchange diagram does not vanish due
to C-invariance. Therefore, one expects that the process may occur but
it must be substantially suppressed if C-invariance holds.

(iii) C-invariance of the weak interactions.

The weak interactions violate C-invariance strongly. There are many
verifications of this and in the conventional V-A theory the violation is
maximal. We will only mention one experiment which clearly demonstrates the
C-violation in weak processes.

Consider the processes $\pi^+ \rightarrow \mu^+ + \nu_\mu$ and $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$. They are clearly
related by a C-transformation. It is experimentally established that in the
first one of these decays the $\mu^+$ is emitted fully polarized with the
spin antiparallel to its momentum. Since a C-transformation does not affect
space-time properties (such as the polarization) C-invariance implies that
the $\mu^-$ in the second decay should also come out with its spin antiparallel
to its momentum. It is found that it instead is fully polarized with
its spin parallel to its momentum in violation of the prediction based
on C-invariance, but in full agreement with the V-A theory.

5. The CP transformation

In the previous discussion the P and C transformations were treated
separately. Reviewing the present experimental situation it was concluded
that with regard to the strong and the electromagnetic interactions we
have no evidence for violations of separate P and C invariance. For the
weak interactions non-invariance under the separate P and C transformations
is firmly established. In fact, in almost all weak processes which have been
observed so far the P and C violating effects are maximal or nearly maximal.
The few exceptions to this general rule are all results of accidental can-
cellations. In view of these observations the question immediately arises
whether the weak interactions are invariant under the combined operation
CP, and we shall devote this chapter to that question. For obvious rea-
sons we restrict the discussion to weak processes.
Since we have already given the transformation properties for state vectors under separate P and C transformations it is a trivial matter to derive the corresponding rules for the operation CP, and we proceed directly to a discussion of some relevant applications.

5.1 Some applications of the CP symmetry

From the eq. (22) we obtain the following condition on the S-matrix elements provided the S-operator is invariant under the CP transformation.

\[ \langle \beta | S | \alpha \rangle = \langle \text{CP} \beta | S | \text{CP} \alpha \rangle \]

(96)

If we are dealing with states \(| \alpha \rangle \) and \(| \beta \rangle \) which are eigenstates of CP, the the eq. (96) leads to a selection rule. Since the charge conjugation is involved it is clear that the eigenstates of CP must have \( Q = B = Y = 0 \), and this restricts the class of processes for which CP invariance imposes absolute selection rules. On the other hand it should be borne in mind that absolute selection rules generally provide us with the best opportunities to test invariance principles to a high degree of accuracy. This will be evident in the case of CP invariance below.

(i) The decays \( K^0 \rightarrow 2\pi \) and \( K^0 \rightarrow 3\pi \).

For the present discussion we assume that invariance under the CP transformation is a rigorous symmetry law respected by all interactions. Under this assumption we shall derive absolute selection rules which can be and actually have been subjected to experimental tests.

Neutral K mesons are obtained by associated production for which the strong (electromagnetic) interactions are responsible. These interactions have in common that they conserve the additive quantum number of hypercharge Y or alternatively the strangeness S, where \( Y = B + S \). The implication of associated production is, that the neutral K-mesons obtained in this way at the moment of creation represent eigenstates of hypercharge. There are two such states denoted \( K^0 \) and \( \bar{K}^0 \) for which \( Y = +1 \) and \( Y = -1 \) respectively. Obviously Y is an observable of the second class with regard to C and neither \( K^0 \) nor \( \bar{K}^0 \) by itself can be an eigenstate of C or CP.
However, with our previous phase convention for the space reflection transformation we obtain for a single particle state of a neutral K meson at rest

\[ P \left| k^0 \right> = - \left| k^0 \right> \]
\[ P \left| \bar{k}^0 \right> = - \left| \bar{k}^0 \right> \]  \hspace{1cm} (97)

and we may choose the phases for the C-transformation such that

\[ C \left| k^0 \right> = - \left| \bar{k}^0 \right> \]
\[ C \left| \bar{k}^0 \right> = - \left| k^0 \right> \]  \hspace{1cm} (98)

and hence 20)

\[ CP \left| k^0 \right> = \left| \bar{k}^0 \right> \]
\[ CP \left| \bar{k}^0 \right> = \left| k^0 \right> \]  \hspace{1cm} (99)

On the basis of the eq. (99) we may now define the following eigenstates of CP

\[ \left| k_1^0 \right> = \frac{1}{\sqrt{2}} \left( \left| k^0 \right> + \left| \bar{k}^0 \right> \right) \]
\[ \left| k_2^0 \right> = \frac{-1}{\sqrt{2}} \left( \left| k^0 \right> - \left| \bar{k}^0 \right> \right) \]  \hspace{1cm} (100)

corresponding to the eigenvalues +1 respectively -1. Of course, the states \( \left| k^0 \right> \) and \( \left| \bar{k}^0 \right> \) may just as well be regarded as coherent superpositions of \( \left| k_1^0 \right> \) and \( \left| k_2^0 \right> \). From (100) one obtains

\[ \left| k^0 \right> = \frac{1}{\sqrt{2}} \left( \left| k_1^0 \right> + i \left| k_2^0 \right> \right) \]
\[ \left| \bar{k}^0 \right> = \frac{1}{\sqrt{2}} \left( \left| k_1^0 \right> - i \left| k_2^0 \right> \right) \]
We have previously stated that the strong and the electromagnetic interactions conserve hypercharge. As a consequence there will be no transitions of the type $|K^0> \rightarrow |\bar{K}^0>$ due to these interactions. However, the decays of the neutral K mesons are all due primarily to the weak interactions and these do not conserve hypercharge. This means that due to weak interactions the transitions $|K^0> \rightarrow |\bar{K}^0>$ occur and although we create pure states $|K^0>$ and $|\bar{K}^0>$ these will develop into coherent mixtures of the type

$$|\varphi> = \alpha(t) |K^0> + \beta(t) |\bar{K}^0>$$

For the discussion of the decays of neutral K mesons the states $|K^0>$ and $|\bar{K}^0>$ are clearly not appropriate since they do not permit us to identify from which state the decay occurred. Thus the states $|K^0>$ and $|\bar{K}^0>$ will not exhibit exponential decay laws and they will not be characterized by a unique life-time. If we instead consider the states $|K_1^0>$ and $|K_2^0>$ and if CP is a rigorously valid symmetry, then we know that the transitions $|K_1^0> \rightarrow |K_2^0>$ are rigorously forbidden and a $|K_1^0>$ state respectively a $|K_2^0>$ state will retain its identity until the moment of decay. These states will then decay exponentially with unique life-times. One concludes, that in this case the decay of $K^0$ will be characterized by two life-times corresponding to the two components $|K_1^0>$ and $|K_2^0>$ and similarly for $\bar{K}^0$. These conclusions are verified by observation and it turns out that

$$\tau(K_1^0) = (0.87 \pm 0.01) \times 10^{-10} \text{ sec.}$$

$$\tau(K_2^0) = (5.68 \pm 0.26) \times 10^{-8} \text{ sec.}$$

that is, the life-times differ by almost three orders of magnitude.

Beyond this, the decay modes of the states $|K_1^0>$ and $|K_2^0>$ are different and we shall in this context restrict our discussion to the decay modes with two or three $\pi$ mesons in the final state.

Consider first the decay of a neutral K meson into two $\pi$ mesons; $\pi^+ \pi^-$ or $\pi^0 \pi^0$. In the rest system of the K meson $J = 0$ for the initial
state, and angular momentum conservation then implies that \( J = l = 0 \)
also in the final state. From the eqs. (51) and (79) we conclude that
since \( l = 0 \) we have
\[
\text{CP} | \pi^+, \pi^- > = | \pi^+, \pi^- > \\
\text{CP} | 2 \pi^0 > = | 2 \pi^0 >
\]
(103)

To the extent that CP is a good quantum number we then find that the
decay
\[
K^0_2 \rightarrow 2 \pi
\]
is forbidden, while
\[
K^0_1 \rightarrow 2 \pi
\]
can occur.

For the decay into three \( \pi \) mesons we conclude that \( J = 0 \) implies
\( L = 1 \). The parity of any such state is odd and hence
\[
\text{CP} = -1 \quad \text{for } 3 \pi^0 \\
\text{CP} = (-1)^{l+1} \quad \text{for } \pi^+ + \pi^- + \pi^0
\]
where \( l \) is the relative orbital angular momentum of the \( \pi^+ \pi^- \) pair. We
conclude that
\[
K^0_1 \rightarrow 3 \pi^0
\]
is forbidden, while
\[
K^0_2 \rightarrow 3 \pi^0
\]
is allowed. Similarly
\[
K^0_1 \rightarrow \pi^+ + \pi^- + \pi^0
\]
is allowed if \( l = 1, 3, 5, \ldots \), and
\[
K^0_2 \rightarrow \pi^+ + \pi^- + \pi^0
\]
if \( l = 0, 2, 4, \ldots \), while for other \( l \)-values these decays are forbidden
by CP invariance.
We shall not pursue this subject further at this point but we shall return to it in section 5.2 since these decays have played a very important role in recent investigations regarding CP invariance.

(ii) The decays $\pi^+ \rightarrow \mu^+ \nu_\mu$ and $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$

In section 4.3 we examined the experimental results with regard to the muon polarization for the decays $\pi^+ \rightarrow \mu^+ \nu_\mu$ and $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$. Experimentally it is known that the muon is fully polarized in both cases but the $\mu^+$ has the helicity (= spin component along the momentum vector) -1 while the helicity of the $\mu^-$ in the second process is +1. This was found to violate C invariance, since the C transformation relates the matrix elements for the two processes provided the helicity of the $\mu^+$ is the same as that of the $\mu^-$. However, the added space reflection in the CP transformation changes the sign of the helicity and there is no contradiction between the observations and CP invariance. This is demonstrated more in detail in figure 7.

![Diagram](attachment:image.png)

**Figure 7**: Decay configuration for the process $\pi^+ \rightarrow \mu^+ \nu_\mu$ (a), the space reflected configuration (b) and the CP transformed configuration (c).

(iii) The CP transformation and the process $\nu_\mu + n \rightarrow \mu^- + p$.

We have previously considered this process in relation to the P and the C transformations. As before, we write the relevant S-matrix element for this process in the following symbolic way.
\[ h_\mu^\mu = (F_i, G_i; \gamma_5) \] (104)

then we have found the following relations for the P and the C transformed processes

\[ h_\mu^\mu (P) l_\mu^\mu (P) = \varphi(F_i, G_i; - \gamma_5) \] (105)

and

\[ h_\mu^\mu (C) l_\mu^\mu (C) = \varphi(F_i^*, G_i^*; - \gamma_5) \] (106)

For the combined operation CP we then obtain

\[ h_\mu^\mu (CP) l_\mu^\mu (CP) = \varphi(F_i^*, G_i^*; \gamma_5) \] (107)

and we conclude that CP invariance can hold only if all the form factors \(F_i\) and \(G_i\) \((i = 1, 2, 3)\) are real. This prediction is verified by experiments \(^{21}\) although the accuracy is not sufficient to rule out small CP violating effects since a small but non-vanishing relative phase factor for the form factors is very difficult to detect. We shall return to these questions later in the discussion of time reversal invariance.

5.2 Experimental tests of CP invariance in weak processes.

After the discovery of charge conjugation and parity violations in weak processes the possibility of symmetry under CP was suggested \(^{22}\) as a remedy for the apparent asymmetry between world and anti-world. The CP symmetry is closely related to the symmetry under time reversal through the CPT theorem (cf. section 7) and many of the early tests refer to T invariance rather than to CP invariance directly.

We have seen in the previous section that the implications of CP symmetry fall into two categories; (i) absolute selection rules, and (ii) reality properties for form factor. Predictions of the second kind are mostly derived from T invariance and we shall discuss them in section 6. They
all have in common that small deviations from $T$ ($\sim CP$) invariance
are very difficult to detect and this is the reason why no CP violating
effects were discovered until one made use of an absolute selection
rule of the type (i) above. We shall restrict the discussion of this
section to the fundamental experiment by Christenson et al. 23) in
which one found a small but nonvanishing branching ratio for the
decay $K_2^0 \to \pi^+ + \pi^-$. We have previously seen that this decay is abso-
lutely forbidden if CP is conserved. Since CP and $T$ invariance have
been very dear to theorists there have been a large number of proposals
made to account for the observations and still retain CP invariance.
Many of them have been ruled out on experimental grounds already and
at the present time the simplest explanation is that in fact CP is vi-
olated, and we shall take this attitude. With CP violated the relevant
decay states are not the $K_1^0$ and $K_2^0$ states previously introduced but
some other linear combinations of $K_1^0$ and $K_2^0$. These new states are usually
denoted $K_L^0$ and $K_S^0$ refering to the longliving and the shortliving com-
ponents. Since the CP violation is small (see below) one finds $K_S^0 \approx K_1^0$
and $K_L^0 \approx K_2^0$. A measure for the violation is given by the parameter
$\eta_{+-}$ and $\eta_{00}$ which are defined in the following way

$$\eta_{+-} = \frac{A(K_L^0 \to \pi^+ + \pi^-)}{A(K_S^0 \to \pi^+ + \pi^-)} \quad (108)$$

$$\eta_{00} = \frac{A(K_L^0 \to \pi^0 + \pi^0)}{A(K_S^0 \to \pi^0 + \pi^0)} \quad (109)$$

where $A$ stands for the transition amplitude. Experimentally one has
found 24)

$$| \eta_{+-} | = (1.94 \pm 0.09) \times 10^{-3}$$

$$\arg \eta_{+-} = (84 \pm 17)^0$$

$$| \eta_{00} | = (4.9 \pm 0.5) \times 10^{-3}$$
We shall not pursue this subject further but refer to the extensive discussion of the neutral K mesons in reference 26).

6. Time reversal.

6.1 Definition and action on state vectors.

The time reversal transformation is defined as the coordinate transformation

\[ (x_0, \bar{x}) \rightarrow (x'_0, \bar{x}') = (-x_0, \bar{x}) \]  \hspace{1cm} (110)

on the physical space-time. In quantum theory it is implemented by an operator \( T \). Since the basic transformation involves classical concepts we immediately know its action on various observables related to space and time. It must satisfy the following conditions

\[ T x^\mu T^{-1} = -\varepsilon (\mu) x^\mu \]
\[ T p^\mu T^{-1} = \varepsilon (\mu) p^\mu \]
\[ T j_{\mu\nu} T^{-1} = -\varepsilon (\mu)\varepsilon (\nu) j_{\mu\nu} \]  \hspace{1cm} (111)

provided that \( x^\mu, p^\mu \) and \( j_{\mu\nu} \) are chosen to be hermitian operators. It can now easily be shown that the relations (111) are consistent with the commutation rules for the Poincaré algebra only if \( T \) is an anti-unitary operator. To see this we recall the commutation rule for an angular momentum operator \( J_{k\ell} \) and \( P_k \). In our metric it reads

\[ [J_{k\ell}, P_k] = i P_\ell \]  \hspace{1cm} (112)

Since both \( J_{k\ell} \) and \( P_k \) changes sign under time reversal the left hand side is unchanged, while the right hand side would change sign if \( T \) is
a linear operator. However, if $T$ is antilinear, then there is an extra change of sign on the right hand side due to the factor $i$ (cf. the eq. (19)). Thus, to preserve the basic commutation rules $T$ must be chosen to be an antiunitary operator.

Since $T$ is an antiunitary operator it cannot represent an observable. However, the operator $T^2$ is a linear operator. It is further clear that performing two consecutive time reversal transformations restores the original situation and hence $T^2$ must commute with all observables. It is, therefore, a multiple of the unit operator

$$T^2 = \lambda I$$  \hspace{1cm} (113)

or

$$T = \lambda T^{-1} = \lambda T^\dagger$$  \hspace{1cm} (114)

By hermitian conjugation of the eq. (114) we find

$$T^\dagger = \lambda^* T$$  \hspace{1cm} (115)

which inserted in (114) yields $|\lambda|^2 = 1$, that is, $\lambda$ is a phase factor. If we multiply by $T$ from left and right in the eq. (114) and further make use of the antilinearity of $T$, then we obtain

$$\lambda^* I = T^2 = \lambda I$$  \hspace{1cm} (116)

and thus $\lambda = \pm 1$. Since $T^2$ commutes with the $S$-operator it follows that its eigenvalue $\lambda$ is a constant of motion; a state with $\lambda = +1$ can never develop into a state with $\lambda = -1$ if it is isolated. There is a profound difference between this result for $T^2$ and the previous result in section 3.1 that for the space reflection operator $P$ a suitable phase convention will yield $P^2 = I$. This latter result does not yield a selection rule. The selection rule which one obtains from $T^2$ is called a superselection rule \cite{26} to mark its distinction from ordinary selection rules. A superselection rule occurs whenever there is an observable (like $T^2$) which is strictly conserved and which commutes with all other observables so that any physical state necessarily is
an eigenstate of the operator. In contrast, an ordinary selection rule is obtained if there is a strictly conserved observable, which does not commute with all other observables. Physically superselection rules respond to the fact that there are physical quantities which always have definite values for any realizable physical state, and these values remain constant as long as the system is not subjected to external interactions. Mathematically the existence of a superselection rule leads to a decomposition of the Hilbert space into incoherent subspaces. In the case of $T^2$ one obtains

$$ H = H_+ \otimes H_- $$

where $H_+$ is spanned by vectors with $\lambda = 1$ and $H_-$ by vectors with $\lambda = -1$. It is easily seen that no observables can have non-vanishing matrix elements between states which belong to different incoherent subspaces, since $T^2$ commutes with all observables. Thus, if $\Omega$ is an arbitrary observable then one finds

$$ \langle \lambda = +1 | \Omega | \lambda = -1 \rangle = \langle + | (T^2)^+ T^2 \Omega | - \rangle = $$

$$ = \langle + | (T^2)^+ \Omega T^2 | - \rangle = - \langle + | \Omega | - \rangle $$

from which it follows that the matrix element must vanish. What has been stated here for $T^2$ clearly holds for any operator which yields a superselection rule. The result of (117) may be phrased differently; there is no way to determine the relative phase between state vectors which belong to different incoherent subspaces. We shall not pursue the subject further, but note that the notion of an observable sometimes is used incorrectly for quantities which are not measurable and, therefore, not observable. For example in order to assign a charge parity to the photon we made reference to the form of the electromagnetic interaction and concluded that the photon has odd charge parity. This is just a useful but not unique convention and, hence, the
charge parity is not measurable. As a consequence, C does not represent an observable. This is an important observation since the electric charge operator usually is assumed to yield a superselection rule but the charge operator does not commute with C as it would be required to do if C was an observable. Also, C clearly has non-vanishing matrix elements between states of different charge.

We proceed to discuss the action of T on various one- and two-particle state vectors. Consider first the plane wave state \( |\vec{p}, \lambda > \equiv |\psi, \theta, p; \lambda > \).

With the same phase convention as before one finds

\[
T |\psi, \theta, p; \lambda > = \exp[-i \pi \lambda] |\psi + \pi, \pi - \theta, p; \lambda > \tag{118}
\]

corresponding to a change in sign for the three-momentum but no change in the helicity. Similarly for the angular momentum states one easily obtains

\[
T |j, m, p, \lambda > = (-1)^{j-m} |j, -m, p, \lambda > \tag{119}
\]

For the two-particle state vectors, which we have previously considered in section 3.1, we just quote the following results

\[
T |\vec{F} = 0; J, M, p; \lambda_1, \lambda_2 > = (-1)^{J-M} |\vec{F} = 0; J, -M, p; \lambda_1, \lambda_2 > \tag{120}
\]

and

\[
T |\vec{F} = 0; J, M, p; l, \sigma > = (-1)^{J-M} |\vec{F} = 0; J, -M, p; l, \sigma > \tag{121}
\]

In the same fashion one may then construct many-particle states and investigate their properties under time reversal.

From the eqs. (118) - (121) it is immediately seen that \( T^2 \) will have the eigenvalue +1 for integer \( J \) and -1 for half-integer \( J \) so that the corresponding state vectors belong to \( H_+ \) for integer and to \( H_- \) for half-integer \( J \). One can show that for the general case the eigenvalues of \( T^2 \) are given by
\[ T^2 = (-1)^{2J} \]  \hspace{1cm} (122)

in accord with the special cases considered above.

From these transformations laws for the state vectors one may deduce the following transformation properties for the fields describing particles of spin 0, 1/2 and 1 respectively

\[
\begin{align*}
\varphi(x_o, \overline{x}) \rightarrow \varphi'(\ensuremath{-}x_o, \overline{x}) & \equiv \left[ T \varphi(-x_o, \overline{x}) T^{-1} \right]^\dagger = \varphi^+(x_o, \overline{x}) \\
\psi(x_o, \overline{x}) \rightarrow \psi'(\ensuremath{-}x_o, \overline{x}) & \equiv \left[ T \psi(-x_o, \overline{x}) T^{-1} \right]^\dagger = \overline{\psi}(x_o, \overline{x}) \\
\overline{\psi}(x_o, \overline{x}) \rightarrow \overline{\psi}'(\ensuremath{-}x_o, \overline{x}) & \equiv \left[ T \overline{\psi}(-x_o, \overline{x}) T^{-1} \right]^\dagger = T^{-1} \overline{\psi}(x_o, \overline{x}) \\
A^\dagger_{\mu}(x_o, \overline{x}) \rightarrow A_{\mu}^\dagger(\ensuremath{-}x_o, \overline{x}) & \equiv \left[ T A_{\mu}(-x_o, \overline{x}) T^{-1} \right]^\dagger = \epsilon(\mu)A_{\mu}(x_o, \overline{x})
\end{align*}
\]  \hspace{1cm} (123)

where \( T \) is a 4x4 matrix, whose properties are specified in the appendix 1.

6.2 Some applications of time reversal invariance.

From the eq. (23) it follows that invariance of the S-operator under the time reversal transformation implies the following relation

\[ \langle \beta | S | \alpha \rangle = \langle T \alpha | S | T \beta \rangle \]  \hspace{1cm} (124)

where \( | T \alpha \rangle \equiv T | \alpha \rangle \). We have previously seen that the action of \( T \) on a vector \( | \alpha \rangle \) implies a change in sign for three-momenta while the helicities remain unchanged. The explicit form of \( | T \alpha \rangle \) beyond this depends on the choice of basis vectors. We may express the eq. (124) in terms of the transition operator \( t \) \(^{27}\) and it then reads

\[ \langle \beta | t | \alpha \rangle = \langle T \beta | t^\dagger T \alpha \rangle^* = \langle T \alpha | t | T \beta \rangle \]  \hspace{1cm} (125)

In section 2.5 we have shown, that if final state interactions are negligible, then the eq. (45) holds and one obtains from (125)
\[ \langle \beta | t | a \rangle = \langle T \beta | t | Ta \rangle^* \]  

(126)

This eq. (126) implies equality for the transition amplitudes when all three-momenta are reversed while helicities are unchanged corresponding to a reversal of spins. In the applications below we will demonstrate what results one may obtain from (125) and (126) when they are applicable.

(i) Reciprocity relations.

Consider the following two processes

\[ a + b \rightarrow c + d \]
\[ c + d \rightarrow a + b \]

at the same center of mass energy \( W \). They are related by an exchange of the initial and the final states. For convenience we choose to describe them in their respective center of mass frames and we denote the cross sections by \( \sigma \) and \( \sigma' \) respectively. Since we work in the center of mass frame we need only give \( \vec{p}_a \) and \( \vec{p}_c \) to specify the kinematics. Alternatively we may give the center of mass total energy \( W = E_a + E_b = E_c + E_d \) and the scattering angles for, say, the particle \( c \). From the eq. (13) we may now derive the differential cross section for the processes. For the first process we have

\[
\frac{d \sigma}{d \omega_c} = \frac{1}{(2\pi)^2} \frac{1}{4E_a E_b v_{in}} \int \frac{dE_c}{2} \frac{p_c}{2} \int \frac{d^3 p_d}{2E_d} \delta^3 (p_c + p_d) \times
\]
\[
x \delta (E_a + E_b - E_c - E_d) \ | \vec{c}, \lambda_c; \vec{p}_d, \lambda_d \ | t | \vec{p}_a, \lambda_a; \vec{p}_b, \lambda_b \rangle^2
\]

(127)

where \( d\omega_c \) is the solid angle element within which particle \( c \) emerges.

For \( v_{in} = |\vec{v}_a - \vec{v}_b| \) we obtain

\[
v_{in} = v_a + v_b = p_a \cdot \frac{E_a + E_b}{E_a \cdot E_b} = p_a \frac{W}{E_a \cdot E_b}
\]

(128)

The integration in \( p_d \) can be performed by means of the three-dimensional \( \delta \) -function. This fixes \( p_d \) and, therefore, \( E_d \) may be expressed as a
function of $E_c$. This must be taken into account when the remaining integration is carried out by means of the remaining $b$-function. If this is done correctly one finds

$$\frac{d\sigma}{d\omega_c} = \frac{1}{(8\pi)^2} \cdot \frac{p_c}{p_a} \cdot |\langle \bar{p}_c, \lambda_c; -\bar{p}_c, \lambda_d | t | \bar{p}_a, \lambda_a; -\bar{p}_a, \lambda_b \rangle|^2 \quad (129)$$

Similarly for the second process one obtains (cf. figures 8a and 8b)

$$\frac{d\sigma'}{d\omega_a} = \frac{1}{(8\pi)^2} \cdot \frac{p_a}{p_c} \cdot |\langle \bar{p}_a, -\lambda_a; \bar{p}_a, -\lambda_b | t | \bar{p}_c, -\lambda_c; \bar{p}_c, -\lambda_d \rangle|^2 \quad (130)$$

We have chosen a configuration in (130) which is the reverse of the first process as demonstrated in figure 8a and 8b. If invariance under time reversal is assumed we may apply the eq. (125) to obtain

$$|\langle -\bar{p}_a, -\lambda_a; \bar{p}_a, -\lambda_b | t | -\bar{p}_c, -\lambda_c; \bar{p}_c, -\lambda_d \rangle|^2 =$$

$$= |\langle \bar{p}_c, -\lambda_c; -\bar{p}_c, -\lambda_d | t | \bar{p}_a, -\lambda_a; -\bar{p}_a, -\lambda_b \rangle|^2 \quad (131)$$

The right hand side of the eq. (131) is almost the same as the matrix element which occurs in the eq. (129). The difference is demonstrated in figure 8c. It corresponds to a change of the spin directions. This difference vanishes if we sum over possible final helicities and average over the initial helicities. In this way we obtain the following equality from the eqs. (129) - (131)

$$\frac{d\sigma(W)}{d\omega_a} = \frac{p_c}{p_a}^2 \cdot \frac{(2S_c+1)(2S_d+1)}{(2S_a+1)(2S_b+1)} \cdot \frac{d\sigma'(W)}{d\omega_c} \quad (132)$$

which is our final result. Relations of the type (132) are known as reciprocity relations.
Figure 8: Momentum and spin orientations for a process (a), the reverse process (b) and the process obtained from (b) by a time reversal transformation (c).

Of course, if we instead of considering the reversed state (b) had considered states with the opposite helicities, then a time reversal transformation had taken us back to (a). Thus, from time reversal invariance one may deduce relations between a process and the reversed process with all spins reversed.

Reciprocity relations have been used to test time reversal invariance (cf. section 6.3). One has also used a relation of the type (132) to determine the spin of the charged π meson from the reactions 28

\[ p + p \rightarrow \pi^+ + d \]

(ii) The decay \( K^+ \rightarrow \pi^0 + \mu^+ + \nu^\mu \)

This decay has attracted attention recently in connection with possible tests of time reversal invariance for the weak interactions. We shall return to this aspect of the decay below, but for the time
being we assume invariance under the time reversal transformation and note that no strong final state interactions can occur and hence one may presumably neglect the final state interactions. This makes relation (126) applicable and we shall make use of it. First, however, a few remarks about the structure of the relevant S-matrix element.

Since it is a weak process it is described by the hamiltonian (57). If the weak interactions are treated to first order in the perturbation expansion, then we may write the relevant S-matrix element

\[
\langle \pi^0, \mu^+, \nu\mu | S | K^+ \rangle = -\frac{4G}{\sqrt{2}} (2\pi)^4 \delta^4(p_T - p) x < \pi^0 | J_\mu (0) | K^+ \rangle l^\mu
\]

As before, we have omitted some trivial normalization factors and

\[
l^\mu \equiv \bar{u}_\nu (q_i w) \gamma_\mu (1 + \gamma_5) v_\mu (q_i w')
\]

From Lorentz invariance we may deduce the following form for the hadronic matrix element

\[
L_\mu \equiv \langle \pi^0 | J_\mu (0) | K^+ \rangle = f_+(p_K + p_\pi)_\mu + f_-(p_K - p_\pi)_\mu
\]

where \( f_+ \) and \( f_- \) are unknown complex-valued functions of the only relativistic invariant one may form out of \( p_K \) and \( p_\pi \), namely \((p_K^\mu p_\pi^\mu)\). To apply the result of eq. (126) we must consider the same S operator sandwiched between the time reversed states, that is, with the sign for three-momenta and spins reversed. More precisely, from the eqs. (126) and (133) we obtain

\[
L_\mu l^\mu = [\bar{L}_\mu (T) l^\mu (T)]^*
\]

where the star denotes complex conjugation. The factors \( \bar{L}_\mu (T) \) and \( l^\mu (T) \) are defined by

\[
\bar{L}_\mu (T) \equiv < T \pi^0 | J_\mu (0) | T (K^+) > \equiv \epsilon (\mu) \{ f_+(p_K^+ p_\pi) + f_-(p_K - p_\pi) \}_{\mu} \]

\[
(137)
\]

\[
(134)
\]
and

$$l_\mu(T) = \bar{u}_\nu(-\bar{q}, -\bar{w}) \gamma_\mu (1 + \gamma_5) \mu(-\bar{q}', -\bar{w}')$$  \hspace{1cm} (138)$$

The last equation may be rewritten in the following way

$$l_\mu^*(T) = \varepsilon(\mu) \bar{u}_\nu(\bar{q}, \bar{w}) \gamma_\mu (1 + \gamma_5) \mu(\bar{q}', \bar{w}')$$  \hspace{1cm} (139)$$

or

$$l_\mu^*(T) = \varepsilon(\mu) l_\mu$$  \hspace{1cm} (140)$$

In the scalar product $[L_\mu(T) l_\mu^*(T)]^*$ the factors $\varepsilon(\mu)$ cancel out and from (136) it then follows that the form factors $f_+$ and $f_-$ must be real functions. We have omitted some over-all phase-factors which are unmeasurable, however, and hence irrelevant. A more careful statement of the condition placed upon the form factors in order to satisfy time reversal invariance is, therefore, that $f_+$ and $f_-$ have the same phase which we may put equal to zero without changing the content of the theory.

For processes like the one just considered where final state interactions are negligible the simple argument with vectors, previously used in connection with space reflection invariance, again holds. For example, in a $K_{\mu 3}^-$ decay one may determine (i) the three-momentum of the $\pi$ meson (ii) the three-momentum of the $\mu^+$ and (iii) the polarization of the $\mu^+$. One may then determine whether there in a sample of $K_{\mu 3}^-$ decays exists a correlation of the type $\alpha \cdot \langle \delta \pi^\mu \rangle \cdot \langle \sigma \rangle$. If $\alpha \neq 0$ one observes a transverse polarization of the muon, that is, a polarization perpendicular to the plane defined by $\delta \pi^\mu$ and $\delta \mu$. Since a term of this type changes sign under time reversal any value of $\alpha$ different from zero is evidence for a violation of time reversal invariance. A more detailed calculation based on the expression (133) for the matrix element shows that $\alpha$ is proportional to $\text{Im}(f_+^* f_-)$, which vanishes if $f_+$ and $f_-$ have the same phase. Thus $\alpha = 0$ implies that the form factors must have the same phase in agreement with our previous findings. Experimentally there is so far no evidence for a transverse polarization of the muon but the experimental errors are still relatively large 29).
If final state interactions are non-negligible (strong or electromagnetic) then the simple argument with vectors fails since the momenta and the spins of the particles in the final state may change as a result of rescattering. By the same token, if the corresponding scattering phase shifts are small for some reason, then the corrections to the vector argument are small. To demonstrate the role played by the final state interactions we shall next consider the process $\Sigma^- \rightarrow n + \pi^-$. We shall not deal with a T-violating effect since the corresponding correlations are rather complicated and hence the computational work is more extensive. Instead we choose a P-violating correlation where final state interactions are important and T-invariance imposes restrictions.

(iii) The decay $\Sigma^- \rightarrow n + \pi^-$

We have chosen this process since no charge exchange scattering can occur in the final state and we do not have to invoke isospin in the discussion. An analogous process is $\Lambda \rightarrow p + \pi^-$. In that case there are contributions from two isospin channels and the analysis will be somewhat more complicated.

The $\Sigma^-$ hyperons are produced by associated production. In general the $\Sigma$ particles have a non-vanishing polarization due to final state interactions in the production process. For simplicity we consider the case of fully polarized particles whereby we may omit a discussion of the production process and we can focus our attention on the decay. In the rest system of the $\Sigma$ the decay configuration is given by figure 9.

![Figure 9. Decay configuration for a fully polarized $\Sigma^-$ decaying into a $\pi^-$ and a neutron.](image)

Figure 9. Decay configuration for a fully polarized $\Sigma^-$ decaying into a $\pi^-$ and a neutron.
Angular momentum conservation requires that \( J = M = \frac{1}{2} \) in the final state (the z-axis along the spin vector of the particle). If we anticipate a P-violation in this weak decay, then the \( \eta \pi^- \) system may be in an S-state or in a P-state. The separation with regard to orbital angular momentum is convenient in this analysis. We have (cf. appendix 2)

Initial state: \( |j, m, p, \lambda > = |\frac{1}{2}, \frac{1}{2}, \frac{1}{2} > \)

Final state: \( |\bar{F} = 0; \varphi, \theta, p; \lambda_1, \lambda_2 >= |\bar{F} = 0; \varphi, \theta, p; \pm \frac{1}{2}, 0 > \)

corresponding to the two possible helicities for the final neutron.

The decay rate \( \lambda \) is given by (cf. the eq. (10))

\[
\frac{d \lambda}{d(\cos \theta)} = \text{const.} \times \sum_{s=\pm \frac{1}{2}} |<0; \varphi, \theta, p; s, 0|t|\frac{1}{2}, \frac{1}{2}, \frac{1}{2} >|^2 (141)
\]

The two possible final states can be expressed in terms of an orbital angular momentum basis \( |\bar{F} = 0; j, m, p; l, \sigma > \) by the relations

\[
|\bar{F} = 0; \varphi, \theta, p; \pm \frac{1}{2}, 0 > = -\frac{1}{\sqrt{4\pi}} \sin \theta \exp(-i\theta) |0; \frac{1}{2}, \frac{1}{2}, p; 1, \frac{1}{2} > (142)
\]

so that

\[
\frac{d \lambda}{d(\cos \theta)} = \text{const.} \times \{ |A_S + A_P \cos \theta|^2 + |A_P \sin \theta|^2 \} (143)
\]

with

\[
A_S = <0; \frac{1}{2}, \frac{1}{2}, p; 0, \frac{1}{2}|t|\frac{1}{2}, \frac{1}{2}, \frac{1}{2} >
\]

\[
A_P = -\frac{1}{\sqrt{4\pi}} <0; \frac{1}{2}, \frac{1}{2}, p; 1, \frac{1}{2}|t|\frac{1}{2}, \frac{1}{2}, \frac{1}{2} > (144)
\]

corresponding to the S and P wave amplitudes. Finally, absorbing a factor \([|A_S|^2 + |A_P|^2]\) in the constant we obtain
\[
\frac{d \lambda}{d(\cos \theta)} = \text{const.} \times (1 + \alpha \cos \theta)
\]  \hspace{1cm} (145)

with

\[
\alpha = \frac{2\text{Re}(A_S A_P^*)}{|A_S|^2 + |A_P|^2}
\]  \hspace{1cm} (146)

We next proceed to show that with time reversal invariance the phase angles for the two complex amplitudes \(A_S\) and \(A_P\) are given by the S and P wave scattering phase shifts for \(\pi^n\) scattering. In the absence of final state interactions both amplitudes are real. Treating the S and P waves separately we employ the following simplified notation

\[
< n, \pi; L | t | \Sigma > \equiv < 0; \frac{1}{2}, \frac{1}{2}; p; L, \frac{1}{2} | t^\dagger | \frac{1}{2}, \frac{1}{2} >
\]  \hspace{1cm} (147)

Assuming time reversal invariance we obtain from the eq. (125)

\[
< n, \pi; L | t | \Sigma > = < T(n, \pi; L) | t^\dagger | T(\Sigma) >^*
\]  \hspace{1cm} (148)

From (119) and (121) we find

\[
< T(n, \pi; L) | t^\dagger | T(\Sigma) > = < 0; \frac{1}{2}, -\frac{1}{2}; p; L, \frac{1}{2} | t^\dagger | \frac{1}{2}, -\frac{1}{2} >
\]  \hspace{1cm} (149)

and further using rotational invariance we conclude

\[
< T(n, \pi; L) | t^\dagger | T(\Sigma) > = < n, \pi; L | t^\dagger | \Sigma >
\]  \hspace{1cm} (150)

so that the eq. (148) reads

\[
< n, \pi; L | t | \Sigma > = < n, \pi; L | t^\dagger | \Sigma >^*
\]  \hspace{1cm} (151)

If final state interactions were negligible we would use the eq. (126) to deduce that the matrix elements in (151) are real. In this case, however, we must employ the complete expression (42) and we obtain
\[ < n, \pi; L | t^+ | \Sigma > ^* = < n, \pi; L | t | \Sigma > ^* + \]

\[ + i < n, \pi; L | t^+ | t | \Sigma > ^* \]

Thus, from (151) and (152) we arrive at

\[ < n, \pi; L | t | \Sigma > = < n, \pi; L | t | \Sigma > ^* + \]

\[ + i \Sigma < n, \pi; L | t^+ | \gamma > ^* < \gamma | t | \Sigma > ^* \gamma \]

where a complete set of physical states \( | \gamma > \) has been introduced.

Considering for a moment only the last term in (153) it is clear that each factor is a matrix element for a physical process (note that \( t \) is the full T-operator which in principle describes all physical processes).

We only have to consider those intermediate states \( | \gamma > \) for which the order of the last term in (153) is the same as the other two terms. This is for example the case if \( | \gamma > = | n, \pi; L > \) in which case the first factor corresponds to \( n \pi \) scattering in the appropriate channel \( L \).

This is a strong process, while the second factor in this case describes \( \Sigma^- \) decay into \( (n\pi^-) \). A closer study immediately reveals that at the relevant energy, which is below the inelastic threshold, all other possible intermediate states involve additional electromagnetic and/or weak processes and hence we can neglect those terms. In this approximation we obtain

\[ < n, \pi; L | t | \Sigma > = < n, \pi; L | t | \Sigma > ^* + \]

\[ + i < n, \pi; L | t^+ | n, \pi; L > < n, \pi; L | t | \Sigma > ^* \]

\[ (154) \]

It follows from parity conservation for the strong interactions that the part of the S-operator which describes \( n \pi \) scattering is diagonal in the orbital angular momentum basis which we have chosen. More precisely one has

\[ S | n, \pi; L > = \exp (2i \delta_L) | n, \pi; L > \]

\[ (155) \]
where $\delta_L$ is the scattering phase shift in the channel $L$. From this we obtain

$$t|n, \pi; L\rangle = -i \left[ \exp(2i\delta_L) - 1 \right] |n, \pi; L\rangle \quad (156)$$

since $S = 1 + i t$. Inserting this into (154) we find

$$< n, \pi; L | t | \Sigma > = \exp(2i \delta_L) < n, \pi; L | t | \Sigma > ^* \quad (157)$$

and thus

$$\text{arg} \{ < n, \pi; L | t | \Sigma > \} = \delta_L \quad (158)$$

which is the result claimed before. With this result we can now write the asymmetry parameter $\alpha$ of the eq. (146) in its final form

$$\alpha = \frac{2 |A_s|^2 - |A_p|^2}{|A_s|^2 + |A_p|^2} \cos (\delta_s - \delta_p) \quad (159)$$

and the presence of the cosine factor is here a result of final state interactions. Clearly $\delta_s$ and $\delta_p$ refer to the phase shifts taken at a center of mass energy equal to the energy release in the $\Sigma$ decay.

(iv) Time reversal and the process $\nu^- + n \to \mu^- + p$

We return to this process which serves as our test ground for invariance arguments. The two configurations corresponding to the process $\nu^- + n \to \mu^- + p$ and the time-reversed process are given in figure 10. We have given the explicit form of the matrix element 10 a in section 3.2, and we proceed directly to the matrix element 10 b. It is given by

$$< T(\nu^-_\mu, n) | S | T(\mu^-_\mu, p) > = -\frac{iG}{\sqrt{2}} (2\pi)^4 \delta^4(p_f - p_i) h_{\mu}^T(T) l^H(T) \quad (160)$$
Figure 10: Two processes (a) $\nu_{\mu}^- + n \rightarrow \mu^- + p$ and (b) $\mu^- + p \rightarrow \nu_{\mu}^- + n$ related by a time reversal transformation.

where

$$h_{\mu}^\dagger (T) \equiv < T(n) | J^\dagger_{\mu} (0) | T(p) > = < T(p) | J_{\mu} (0) | T(n) >^*$$  \hspace{1cm} (161)

and

$$1^\mu (T) \equiv < T(\nu_{\mu}^-) | J^\dagger_{\mu} (0) | T(\mu^-) > = < J(\mu^-) | J_{\mu} (0) | T(\nu_{\mu}^-) >^*$$  \hspace{1cm} (162)

In (161) and (162) we have taken notice of the fact that the hermitian conjugate parts of the currents contribute to the time reversed process since initial and final state are exchanged. We next express the matrix elements in terms of elementary spinors

$$h_{\mu}^\dagger (T) = \bar{u}(-\vec{k},-\vec{w}) \{ F_1(q^2) \gamma_{\mu} + i\epsilon(v)F_2(q^2)\sigma_{\mu\nu} \frac{\gamma}{2} +$$

$$+ \epsilon(\mu)F_3(q^2)q_{\mu} + [G_1(q^2)\gamma_{\mu} + i\epsilon(v)G_2(q^2)\sigma_{\mu\nu} \frac{\gamma}{2} +$$

$$+ \epsilon(\mu)G_3(q^2)q_{\mu}] \gamma_5 \frac{\gamma}{2} u(-\vec{k},-\vec{w})$$  \hspace{1cm} (163)
To rewrite this matrix element we make use of the following relations

\[ u(-\bar{k},-\bar{w}) = T \ u^{\mu}(\bar{k},\bar{w}) \]

\[ \bar{u}(-\bar{k},-\bar{w}) = u^{\dagger}(\bar{k},\bar{w})T^{-1} \]  \hspace{1cm} (164)

where the 4 x 4 matrix \( T \) has the following properties

\[ T^{-1} \gamma_{\mu} T = \epsilon(\mu) \gamma_{\mu} \]  \hspace{1cm} (165)

Introducing (164) into (163) one finds

\[ h_{\mu}(T) = \epsilon(\mu) \ u(\bar{k},\bar{w}) \{ P_{1}(q^{2}) \gamma_{\mu} + iP_{2}(q^{2}) \sigma_{\mu\nu} q^{\nu} + P_{3}(q^{2}) q_{\mu} + \sigma_{1}(q^{2}) \gamma_{\mu} + i\gamma_{2}(q^{2}) \sigma_{\mu\nu} q^{\nu} + \gamma_{3}(q^{2}) q_{\mu} \} \ u(\bar{k},\bar{w}) \]  \hspace{1cm} (166)

and similarly

\[ l^{\mu}(T) = \epsilon(\mu) \ u(\bar{p},\bar{s}) \gamma^{\mu} ( 1 + \gamma_{5} ) \ u(\bar{p},\bar{s}) \]  \hspace{1cm} (167)

If we write

\[ h_{\mu} l^{\mu} \equiv \varphi(\bar{p}_{1}, G_{1}; \gamma_{5}) \]

then we obtain from (166) and (167)

\[ h_{\mu}(T) l^{\mu}(T) = \varphi(\bar{p}_{1}, G_{1}; \gamma_{5}) \]  \hspace{1cm} (168)

\( T \) invariance implies that

\[ h_{\mu} l^{\mu} = h_{\mu}(T) l^{\mu}(T) \]  \hspace{1cm} (169)

and hence all form factors must be real (have the same phase). We had anticipated this result when we introduced the factors \( i \) in the definitions of the form factors in section 3.

It is of interest to note that the restrictions imposed on the matrix element (160) by time reversal invariance are exactly the same as those previously found to follow from CP invariance (cf. the eq. (107)). This is an indication of the close relationship between CP and \( T \), a
subject we shall return to in section 7.

6.3 The present status of time reversal invariance in strong, electromagnetic and weak processes.

In the last couple of years considerable attention has been devoted to the question of T invariance in various processes, and we shall briefly review the most important findings which we classify according to the type of interaction which is responsible for the process.

(i) Time reversal invariance in strong interaction processes.

The most stringent tests make use of reciprocity relations applied to nuclear reactions. For example, one has considered the processes $^{24}\text{Mg} (\alpha, p) ^{27}\text{Al}$ and $^{24}\text{Mg} (d, p) ^{25}\text{Mg}$ and one has found no disagreement with reciprocity. The upper limit on the T non-invariant amplitude obtained from these experiments is less than $1-2 \times 10^{-3}$ times the T conserving amplitude.

Another less accurate test of T invariance in strong processes refers to a polarization - asymmetry equality in nucleon-nucleon scattering. If a beam of unpolarized protons hits an unpolarized target of protons one will find that the scattered protons are partially polarized along the normal to the scattering plane, which is defined by the momenta $\vec{p}_i$ and $\vec{p}_f$ of the incident and the scattered proton respectively (cf. figure 11). Since this corresponds to a correlation of the type $<\vec{\sigma}> \cdot (\vec{p}_i \times \vec{p}_f)$ it is clear that if T invariance holds, then this polarization is the result of final state interactions. Furthermore, there can be no polarization in the scattering plane unless space reflection invariance is violated. We denote by $P(\phi)$ the polarization of a proton which has been scattered an angle $\phi$. Next consider the scattering of fully polarized protons on a target of unpolarized protons. In general there will be an azimuthal distribution of the form

$$N(\phi) = \text{const.} \times [1 + \alpha(\phi) \hat{\mathbf{w}} \cdot \hat{\mathbf{n}}]$$ (170)
where $\hat{w}$ is a unit vector along the polarization vector of the incident protons and $\hat{n} = (\vec{p}_i \times \vec{p}_f)/|\vec{p}_i \times \vec{p}_f|$ The eq. (170) implies a left-right asymmetry corresponding to whether $\hat{w}$ and $\hat{n}$ are parallel or antiparallel. It can be shown that time reversal invariance implies that

$$P(\theta) = \alpha(\theta)$$ (171)

![Diagram](image)

**Figure 11:** A configuration in scattering of a polarized proton on an unpolarized proton target.

Experimentally one has investigated these effects for pp scattering at an energy of 210 MeV and for $\theta = 30^\circ$ in the center of mass frame \(^{31}\). One found

$$P(\theta) - \alpha(\theta) = -0.014 \pm 0.014$$

with $P(\theta)$ of the order of 25 %/o. This places the upper limit on the $T$ non-conserving amplitude at a few percent of the $T$ conserving amplitude. Much more extensive measurements to test the relation (171) at higher energies are in progress.

(ii) Time reversal invariance in electromagnetic processes.

The test of $T$ invariance in electromagnetic processes are few and often ambiguous with regard to the interpretation. We shall return to
some of these questions in section 8.

(1) The electric dipole moment of the neutron.

We have previously noted that a nonvanishing electric dipole moment for the neutron violates both T and P invariance. The observed upper limit of the dipole moment sets the upper limit for the P or/and T violating amplitude at $10^{-6} - 10^{-7}$ times invariant amplitude.

(2) Quantum electrodynamics.

Beyond this one may also view the excellent agreement between theory and experiment in quantum electrodynamics as evidence for T invariance since this invariance is explicitly present in the theory. Some caution in using this argument is necessary, however, as pointed out by Lee and others [11]. Many of the consequences of T invariance follow also from the hermiticity of the electromagnetic current operator and the fact that it is conserved.

(3) The decay $\Sigma^0 \rightarrow \Lambda^0 + e^+ + e^-$

The electromagnetic decay

$\Sigma^0 \rightarrow \Lambda^0 + e^+ + e^-$

has recently been investigated in the search for T violations among electromagnetic processes [32]. It is of particular interest since the photon is virtual and hence, current conservation does not disguise a T violation (cf. section 6). One has looked for a polarization of the $\Lambda$ hyperon perpendicular to the decay plane defined by the vectors $\mathbf{p}_\Lambda$ and $\mathbf{q} = (\mathbf{p}_e^+ + \mathbf{p}_e^-)$ that is, a correlation term

$$\langle \mathbf{q} \cdot (\mathbf{p} \times \mathbf{q}) \rangle$$

A net polarization of the $\Lambda$ can be detected as an asymmetry in the decay of the $\Lambda$ hyperon

$$\Lambda \rightarrow p + \pi^-$$
Experimentally this asymmetry was found to be

\[ \langle \cos \Theta \rangle_{\text{exp}} = 0.048 \pm 0.026 \]

where \( \cos \Theta = \hat{p}_p \cdot \hat{N} \), \( \vec{N} = \vec{p}_A \times \vec{q} \). The result indicates a small polarization for the \( A \) but it seems that the result is somewhat uncertain.

(4) Reciprocity relations etc.

Experiments are under way for testing \( T \) invariance in the form of reciprocity for the processes

\[ \gamma + d \rightarrow p + n \]

which clearly are of electromagnetic nature. Furthermore, one is looking for a correlation of the type \( \langle \sigma \rangle \cdot (\vec{p}_1 \times \vec{p}_f) \) in the scattering of electrons on polarized nuclei. Since the final state interactions are of electromagnetic character one expects such a correlation also in the absence of \( T \) violation, but then it should be of the order of \( \alpha \) (= fine structure constant) and hence quite small.

(iii) Time reversal invariance in weak processes.

There is a fair number of experimental tests of \( T \) invariance for weak processes. They all have in common that they determine relative phase angles which are restricted by \( T \) invariance. In general the accuracy of such experiments is rather poor and no stringent tests are then possible. Therefore, it is not surprising that one has found no direct evidence for \( T \) violations so far. There is, of course, indirect evidence for a small \( T \) violation from the observation of the decay \( K^0 \rightarrow 2 \pi \)

This decay is strictly forbidden by CP-conservation and, hence, its occurrence implies a CP violation. From the CPT theorem (cf. next section) we then deduce that also \( T \) must be violated in this weak decay. We have previously discussed the \( K^0 \rightarrow 2 \pi \) decay and in this section we restrict ourselves to direct tests of \( T \) invariance.

(1) \( T \) violating effects in the \( \beta \)-decay of the neutron.

A test of \( T \) invariance is provided by a measurement of the electron-neutrino angular correlation in the decay of polarized neutrons.
\[ n \rightarrow p + e^- + \bar{\nu}_e \]

The observed correlation is of the form \( \langle \bar{\sigma}_n \rangle \cdot (\bar{p}_e \times \bar{p}_\nu) \). Invoking energy-momentum conservation one may rewrite this correlation in the following way (the neutrons decay at rest)

\[
\langle \bar{\sigma}_n \rangle \cdot (\bar{p}_e \times \bar{p}_\nu) = -\bar{p}_{\text{recoil}} \cdot (\langle \bar{\sigma}_n \rangle \times \bar{p}_e) \tag{172}
\]

where \( \bar{p}_{\text{recoil}} \) is the three-momentum of the recoiling proton. Thus, the presence of a term like (172) is established if one finds a non-vanishing up-down asymmetry for recoiling protons with regard to the plane defined by \( \langle \bar{\sigma}_n \rangle \) and \( \bar{p}_e \). In terms of the form factors \( F_i \) and \( G_i \), previously introduced for the matrix element of the hadron current, the dominant contribution to this correlation is proportional to \( \text{Im} [ F_i(0)\sigma_i(0)] \). We have already established that \( T \) invariance requires the form factors to be real and hence this term vanishes if time reversal is a good symmetry. The experimental result implies that the phase angle difference between these two form factors is less than \( 8^\circ \) (mod. \( \pi \)) \(^{23}\).

(2) The transverse polarization in \( K^- \) decay.

This is a weak, strangeness-changing decay for which we have already discussed the implications of time reversal invariance. Experimentally one looks for a net polarization of the emitted muon perpendicular to the plane defined by \( \bar{p}_\pi \) and \( \bar{p}_\mu \), that is, the relevant correlation is \( \langle \bar{\sigma}_\mu \rangle \cdot (\bar{p}_\pi \times \bar{p}_\mu) \). Since there are no strong or electromagnetic (to order \( \alpha \)) final state interactions such a correlation is forbidden if \( T \) invariance holds. Experimentally one has determined the transverse polarization \( P_\perp \) of the muon to be \( ^{34} \)

\[ P_\perp = 0.007 \pm 0.016 \]

which is consistent with \( T \) invariance.

(3) The proton polarization in the decay of polarized \( \Lambda \) -particles.

In the decay

\[ \Lambda \rightarrow p + \pi^- \]
of polarized $\Lambda$ hyperons one has measured the polarization of the emitted proton. It can be shown \cite{35} that the polarization of the proton is given by

$$< \bar{\sigma}_p > = \text{const.} \times \{ - (\alpha - \bar{p}_p \cdot \bar{P}) \bar{p}_p + \beta \bar{p}_p \times \bar{P} + \gamma \left[ (\bar{p}_p \times \bar{P}) \times \bar{p}_p \right] \} \quad (173)$$

where the parameters $\alpha, \beta$ and $\gamma$ are given in terms of $S$ and $P$ wave transition amplitudes (cf. the discussion of $\Sigma \rightarrow n + \pi^-$ in section 6.2), and $\bar{P}$ is the polarization of the $\Lambda$ hyperon. Clearly the polarization perpendicular to the plane defined by $\bar{p}_p$ and $\bar{P}$ is a correlation which is non-invariant under $T$ and hence $\beta = 0$ if (i) $T$ invariance holds and (ii) one can neglect final state interactions. In the presence of final state interactions this term will be proportional to $\sin(\delta_s - \delta_p)$ and one may determine the phase shifts $(\delta_s - \delta_p)$ for $p\pi^-$ scattering at the center of mass energy of 37 MeV. Experimentally one finds from the decay \cite{36}

$$[\delta_s - \delta_p] = (7 \pm 8)^o$$

which should be compared to the result obtained from $p\pi^-$ scattering

$$[\delta_s - \delta_p] = (6.5 \pm 0.5)^o$$
7. The CPT transformation.

Since we have treated the three discrete transformations C, P and T it is a straightforward task to investigate the physical implications of invariance under the combined operation CPT. For two reasons we shall pay special attention to this transformation; (i) It turns out that the CPT transformation by itself is of much more fundamental character than any one of its constituents and (ii) it imposes less stringent conditions on the theory than separate invariance under C, P and T and in that way it is a sort of minimal program to have CPT invariance.

The fundamental importance of the CPT transformation is a consequence of the so-called CPT theorem, which we shall briefly state omitting a proof since it goes beyond the scope of these lectures. The main content of the theorem is that any reasonable theory is CPT invariant. In the next subsection we shall qualify this statement making it more precise. It is clear, however, that the depth of the theorem makes it particularly important to test the implications one may derive from CPT invariance, and we shall discuss those aspects in some detail. In view of the very weak assumptions on which the CPT theorem is based it is not surprising that one finds less stringent conditions on a theory by invoking CPT invariance than, say, from separate C, P or T invariance. In fact our previous results with regard to the process $\nu_\mu + n \rightarrow \mu^- + p$ illustrates this statement, since it is immediately seen that CPT invariance imposes no restrictions whatsoever on the form of the matrix element. We shall later see that this is to be expected since we have assumed Lorentz invariance in deriving the most general form for the matrix element, and Lorentz invariance is the major ingredient of the CPT theorem.

7.1 The CPT theorem.

So far in these lectures we have made no reference to local field
theory although local field theory in many ways provides the most natural frame for a description of elementary particles. The CPT theorem emerges in local field theory from very general assumptions. It essentially asserts that any local Lagrangian field theory is invariant under the combined operation CPT, taken in any order whatever, provided the theory is invariant under proper orthochronous Lorentz transformations. For a proof or a more detailed discussion of this remarkable theorem we refer to \[37\].

Since the assumptions underlying the CPT theorem are so weak it is generally assumed that CPT invariance is an absolute symmetry principle. If that is the case, and there is so far no indication to the contrary, then, say, tests of CP invariance have immediate implications on the question of T invariance. This was briefly mentioned in connection with the \( K \rightarrow 2\pi \) decay. It was shown that the existence of this decay mode violates CP invariance and we stated that as a consequence of CPT invariance it is at the same time evidence for T non-invariance. In the same fashion the CPT invariance offers opportunities to test C, P or T invariance indirectly in cases where direct tests are inaccurate or beyond present technique.

7.2 The present status of CPT invariance.

Since the CPT transformation involves a time reversal operation it is clearly represented by an antiunitary operator for which we shall use the notation \( \Theta \). Its action on state vectors can be read off from their transformation properties under separate C, P and T transformation. For the moment it suffices to note that a single particle state of momentum \( \vec{p} \) and helicity \( \lambda \) under CPT transforms into an antiparticle state of momentum \( \vec{p} \) and helicity \( -\lambda \), that is, the spin direction is reversed. Furthermore, an in-state is turned into an out-state.

Consider the process \( |i; \text{in}\rangle \rightarrow |f; \text{out}\rangle \) where \( i \) and \( f \) stand to the arbitrary initial and final states characterized by the momenta and the helicities of the individual particles. The corresponding states with
the particles replaced by their antiparticles will be denoted \( |\tilde{1}; \text{in} \rangle \) and \( |\tilde{f}; \text{out} \rangle \). Finally we denote by \( |\tilde{1}; \text{in} \rangle \) and \( |\tilde{f}; \text{out} \rangle \) the original states with the opposite signs for the helicities. Next consider a hermitian operator \( \Omega \) which is assumed to be CPT invariant, that is

\[
\Theta \Omega \Theta = \Omega \tag{174}
\]

For the matrix elements of \( \Omega \) one then finds the following relation

\[
<\tilde{f}; \text{out} | \Omega | \tilde{1}; \text{in} > = <\tilde{f}; \text{out} | \Theta^{-1} \Theta \Theta^{-1} \Theta | \tilde{1}; \text{in} > = <\tilde{f}; \text{in} | \Omega | \tilde{1}; \text{out} > \tag{175}
\]

We may further introduce the S-operator to rewrite the eq. (175) in the following way

\[
<\tilde{f}; \text{out} | \Omega | \tilde{1}; \text{in} > = <\tilde{f}; \text{out} | S^{-1} \Omega S | \tilde{1}; \text{in} > \tag{176}
\]

If the states \( |\tilde{f}; \text{out} \rangle \) and \( |\tilde{1}; \text{in} \rangle \) are chosen to be eigenstates of the total Hamiltonian and hence of the S-operator, then we obtain

\[
<\tilde{f}; \text{out} | \Omega | \tilde{1}; \text{in} > = \exp \left[ 2i(\delta_{\tilde{f}} + \delta_{\tilde{1}}) \right] <\tilde{f}; \text{out} | \Omega | \tilde{1}; \text{in} > \tag{177}
\]

which relates the matrix elements of the hermitian operator \( \Omega \) between particle states and the corresponding antiparticle states respectively. From this very important relation we may deduce a number of predictions which may be subjected to experimental tests.

(i) **Equality for the mass of a particle and its antiparticle.**

We choose the states \( |\tilde{1}; \text{in} \rangle \) and \( |\tilde{f}; \text{out} \rangle \) to represent a single particle at rest and, hence, there is no distinction between in and out labels in this case. If we further consider
the case with $\mathcal{A} = \mathcal{H}$, where $\mathcal{H}$ is the total Hamiltonian (the energy operator), then the eq. (177) reads

$$\tilde{m} \delta_{\tilde{1}\tilde{f}} = m^* \delta_{\tilde{i}\tilde{f}}$$

where $\tilde{m}$ and $m$ represent the rest masses of the antiparticle respectively the particle. Since the energy operator is hermitian and $m$ represents a diagonal matrix element of $\mathcal{H}$ it follows that $m$ is real and we conclude that

$$\tilde{m} = m$$

or the mass of a particle equals the mass of its antiparticle. The same result clearly follows also from the more restrictive assumption of $C$-invariance. Since $C$-invariance is not generally respected it is an important observation that the very weak assumption of CPT invariance is sufficient.

Experimentally one has determined the masses for particles and antiparticles as a test of CPT invariance, and some of the most accurate results are summarized in table 1 below. The most precise test of CPT invariance refers to the measured mass

<table>
<thead>
<tr>
<th>Particle</th>
<th>$\frac{m^+}{m^-} - 1$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^+$</td>
<td>$10^{-4}$</td>
<td>(38)</td>
</tr>
<tr>
<td>$\pi^+$</td>
<td>$0.002 \pm 0.002$</td>
<td>(39)</td>
</tr>
<tr>
<td>$K^+$</td>
<td>$0.00 \pm 0.01$</td>
<td>(39)</td>
</tr>
</tbody>
</table>

Table 1: The mass ratio for some particle-antiparticle systems.

difference between the two neutral $K$-mesons, namely $K^0_L$ and $K^0_S$. If we define $M$ and $\tilde{M}$ to be the diagonal matrix elements of the mass operator in the basis $K^0_L$ and $\tilde{K}^0$, then CPT invariance
of the mass (energy) operator implies

$$M = \langle K^0 | H | K^0 \rangle = \langle \bar{K}^0 | H | \bar{K}^0 \rangle \equiv \bar{M}$$

(180)

From the measured mass difference $$\Delta m \equiv m(K_L^0) - m(K_S^0)$$ one may deduce that the eq. (7.7) holds to the accuracy $$|\Delta m/m(K)| \approx 10^{-14}$$. To the extent that a decomposition

$$H = H_{st} + H_{em} + H_{wk}$$

(according to the type of interaction) is meaningful this result implies that the ratio of the CPT violating amplitude and the CPT conserving amplitude is less than $$10^{-14}$$ for $$H_{st}$$, less than $$10^{-12}$$ for $$H_{em}$$ and less than $$10^{-8}$$ for the $$\Delta S=0$$ part of $$H_{wk}$$.

(ii) Equality for the lifetime of a particle and its antiparticle.

We next consider a state $$|i; \text{in}\rangle$$ corresponding to particle, which can only decay as a result of the weak interactions. With regard to the strong and the electromagnetic interactions it is a steady state. Further, to start with we only consider the case where the final state interactions are negligible, so that the eq. (7.4) for $$\Omega = H_{wk}$$ takes the form

$$\langle \bar{f}; \text{out} | H_{wk} | \bar{i}; \text{in} \rangle = \langle \bar{f}; \text{out} | H_{wk} | \bar{i}; \text{in} \rangle^*$$

(181)

Inserting this result into the expression (10) for the partial decay rate one immediately obtains $$\lambda = \bar{\lambda}$$, where $$\bar{\lambda}$$ refers to the antiparticle channel and $$\lambda$$ to the particle channel with all spins inverted. If we sum over possible spin orientations and hence deal with partial decay rates corresponding to no spin measurements in the initial or the final state then one obtains

$$\lambda(i \to f) = \lambda(\bar{i} \to \bar{f})$$

(182)

Among processes for which this result holds we just mention the
and the \( K_{12} \) and \( K_{13} \) decays. Due to the eq. (182) we conclude that for these processes the energy spectra and partial decay rates (after spin summation) are equal for the \( K^+ \) decay and the \( K^- \) decay.

\[
\lambda (K^+ \rightarrow \mu^+ + \nu_\mu) = \lambda (K^- \rightarrow \mu^- + \bar{\nu}_\mu) \\
\lambda (K^+ \rightarrow \pi^0 + \mu^+ + \nu_\mu) = \lambda (K^- \rightarrow \pi^0 + \mu^- + \bar{\nu}_\mu)
\]

(183)

These predictions of CPT invariance have so far not been tested to any degree of accuracy. For a more complete discussion of these decays in relation to CPT invariance we refer to the article by Lee and Wu 25).

In the case of non-negligible final state interactions the eq. (7.8) is replaced by

\[
\langle \tilde{f}; \text{out} | H_{wk} | \tilde{i}; \text{in} \rangle = \exp(2i\delta_f) \langle \tilde{f}; \text{out} | H_{wk} | \tilde{i}; \text{in} \rangle
\]

(184)

provided \( |\tilde{f}; \text{out} \rangle \) is an eigenstate of the strong (and electromagnetic) interactions. In the more general case when this is not the case one may always expand the final state in terms of such eigenstates and one obtains a sum of terms with different phase factors on the right hand side. The implications of these last remarks are most easily explained in terms of an example. Consider the decays

\[
\Lambda \rightarrow p + \pi^- \\
\rightarrow n + \pi^0
\]

which both occur due to the weak interactions. Since both particles in the final states are hadrons we may not neglect the final state interactions. As a matter of fact, an event which is identified as a decay \( \Lambda \rightarrow p + \pi^- \) may intrinsically be a \( \Lambda \rightarrow n + \pi^0 \) decay in which there is a charge exchange
scattering in the final state. Somehow one must first disentangle the final states. This is done by introducing eigenstates of the strong interactions. Since the strong interactions conserve isospin and parity we must deal with states of definite I and L.

For the $\pi N$ system there are two isospin channels with $I = \frac{1}{2}$ and $I = 3/2$. The weak decay process is not parity conserving and hence the final state can have $L = 0$ and $L = 1$. Neglecting the electromagnetic interactions there will be four eigenstates of the strong interactions involved in this process and there will be four phase shifts $\delta_{2J}(L, I)$ to consider, namely $\delta_{1}(0, \frac{1}{2})$, $\delta_{1}(0, 3/2)$, $\delta_{1}(1, \frac{1}{2})$ and $\delta_{1}(1, 3/2)$. We shall not carry the analysis any further but note that the presence of several terms with different phase factors in the eq. (7.11) does not permit us to deduce equalities between the partial decay rates via particle conjugate channels. For example, it does not follow from CPT invariance that $B = \overline{B}$, where $B$ and $\overline{B}$ are defined by

$$B = \frac{\lambda(\Lambda \rightarrow n + \pi^0)}{\lambda(\Lambda \rightarrow p + \pi^-)}$$

$$\overline{B} = \frac{\lambda(\Lambda \rightarrow \overline{n} + \pi^0)}{\lambda(\overline{\Lambda} \rightarrow \overline{p} + \pi^+)}$$

However, with the additional assumption of C or T invariance this equality holds (41).

In some cases it so happens that the final state always is an eigenstate of the strong interaction $S$ operator due to various independent selection rules. In that case the phase factor in the eq. (7.11) does not lead to any complications. As an example of this we consider the two particle conjugate processes

$$K^\pm \rightarrow \pi^\pm + \pi^0$$

It follows from the generalized Pauli principle (Bose statistics) that the final $2 \pi$ state must be an eigenstate of isospin with $I = 2$. Furthermore, angular momentum conservation
requires that \( L = 0 \) (S-state). Since isospin and parity is
conserved by the strong interactions there is no other possible
final state which can be reached by a strong final state inter-
action. The \( 2\pi \) final state is in this case an eigenstate of
the strong interaction S operator. Neglecting the influence
of electromagnetic interactions one then obtains from CPT in-
variance the following equality

\[
\lambda(K^+ \to \pi^+ + \pi^0) = \lambda(K^- \to \pi^- + \pi^0) \tag{185}
\]

The corrections due to electromagnetic interactions are ex-
pected to be of the order \( 10^{-4} \). In the same fashion one can
deduce. The following relation from CPT invariance (neglec-
ting electromagnetic corrections)

\[
\lambda(K^+ \to \pi^+ + \pi^-) + \lambda(K^+ \to \pi^+ + \pi^0 + \pi^0) = \lambda(K^- \to \pi^- + \pi^- + \pi^+) + \lambda(K^- \to \pi^- + \pi^0 + \pi^0) \tag{186}
\]

with the corrections expected to be of the same order as in
the previous case. If one further assumes that there is no
admixture of \( I = 3 \) in the final state (due to weak interaction
selection rules), then one obtains more restrictive results,
namely

\[
\lambda(K^+ \to \pi^+ + \pi^- + \pi^-) = \lambda(K^- \to \pi^- + \pi^- + \pi^+) \tag{187}
\]

and

\[
\lambda(K^+ \to \pi^+ + \pi^0 + \pi^0) = \lambda(K^- \to \pi^- + \pi^0 + \pi^0) \tag{188}
\]

A more extensive study of the consequences of CPT invariance
for the K decays is found in the article by Lee and Wu to which
we have previously referred \( 25 \).

So far we have been considering partial decay rates
corresponding to specific decay channels. If we sum over all possible channels to obtain the total decay rate (the life-time) then we may always analyze the final state in terms of eigenstates of the non-weak S operator and since in this case the partial rates are equal it follows that the same holds true for the total decay rate. Thus, we conclude that the life-time of a particle equals the life-time of its antiparticle if the theory is CPT invariant. This prediction has been examined experimentally and we list the most accurate results below in table 2, which is taken from reference 29.

<table>
<thead>
<tr>
<th>Particles</th>
<th>$\frac{\tau^+}{\tau^-} - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^+, \mu^-$</td>
<td>$0.000 \pm 0.001$</td>
</tr>
<tr>
<td>$\pi^+, \pi^-$</td>
<td>$0.0018 \pm 0.0040$</td>
</tr>
<tr>
<td>$K^+, K^-$</td>
<td>$-0.0009 \pm 0.0008$</td>
</tr>
</tbody>
</table>

Table 2: The ratio of the life-time for some particles and their antiparticles.

(iii) Equality for the anomalous magnetic moment of a particle and its antiparticle.

We have previously discussed the weak process $\nu_e + n \rightarrow \mu^- + p$ and introduced the concept of weak form factors in order to obtain explicit expressions for the matrix elements of the weak hadronic current. Similarly, for the electromagnetic interactions of hadrons one needs explicit expressions for matrix elements of the electromagnetic current $J_{\mu}^{EM}(x)$. Consider for example the scattering of a proton in an external
field. To lowest order in the electromagnetic interactions this process is described by the diagram of figure 12, and the relevant part of the matrix element for this process is

\[ \langle p | j_{\mu}^{\text{EM}}(0) | p \rangle. \]

From Lorentz invariance and current conservation it is easily shown that the most general form for this matrix element is given by (some irrelevant normalization factors have been omitted)

\[ \langle f; \text{out} | J_{\mu}^{\text{EM}}(0) | i; \text{in} \rangle \equiv \langle \bar{p}', s' | J_{\mu}^{\text{EM}}(0) | \bar{p}, s \rangle = \]

\[ = e^{-i\frac{\bar{p}' \cdot s'}{h} \left[ F_1(q^2)\gamma_\mu + i F_2(q^2)\gamma_\mu \gamma_5 \right]} u(p, s) \]  (189)

with \( q = p' - p \). Considering the static limit, that is \( q \to 0 \), one easily identifies \( eF_1(o) \) as the proton charge and \( eF_2(o) \) as the anomalous magnetic moment of the proton.

Since the charge of the proton equals the charge of the electron it follows that \( F_1(o) = 1 \). From the fact that \( J_{\mu}^{\text{EM}}(o) \) is a hermitian operator it follows that \( F_1(q^2) \) and \( F_2(q^2) \) are real functions.

To investigate the consequences of CPT invariance of the
electromagnetic interaction we first note the transformation property for the current operator in order that this condition be satisfied

\[ \Theta_{\mu}^{EM}(0) e^{-1} = - J_{\mu}^{EM}(0) \]  

(190)

From this we obtain the analogue of the eq. (177)

\[ < \overline{F}; \text{out} \mid J_{\mu}^{EM}(0) \mid F; \text{in} > = - < \overline{F}; \text{out} \mid J_{\mu}^{EM}(0) \mid F; \text{in} >^* \]  

(191)

The right hand side may be evaluated from (189) with the result

\[ < \overline{F}; \text{out} \mid J_{\mu}^{EM}(0) \mid F; \text{in} > = - e \{ \overline{u}(\overline{p}; -\overline{s})'[F_{1}(q^2)\gamma_{\mu} + iF_{2}(q^2)\sigma_{\nu} q^\nu]u(\overline{p}; -\overline{s}) \} \]  

(192)

If we make use of the following spinor relations (cf. appendix 1)

\[ u(\overline{p}; -\overline{s}) = \gamma_{0} \Gamma u(\overline{p}; -\overline{s}) \]  

(193)

\[ \overline{u}(\overline{p}; -\overline{s}) = u(\overline{p}; -\overline{s}) \Gamma^{-1} \gamma_{0} \]

we obtain

\[ < \overline{F}; \text{out} \mid J_{\mu}^{EM}(0) \mid F; \text{in} > = - e \overline{u}(\overline{p}; -\overline{s})'[F_{1}(q^2)\gamma_{\mu} + iF_{2}(q^2)\sigma_{\nu} q^\nu]u(\overline{p}; -\overline{s}) \]  

(194)

The change in sign as compared to the eq. (189) just reflects the opposite charge for a particle and its antiparticle. From (189) and (194) we conclude that CPT invariance implies equality for the particle and the antiparticle form factors. In particular, their anomalous magnetic moments should be the same. This last prediction has been tested experimentally 29)

<table>
<thead>
<tr>
<th>Particles</th>
<th>$\hat{g}(g_+ - g_-)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^+, e^-$</td>
<td>$(1, 5/2)\alpha^2/\pi^2$</td>
</tr>
<tr>
<td>$\mu^+, \mu^-$</td>
<td>$(0, 1.5)\alpha^2/\pi^2$</td>
</tr>
</tbody>
</table>

Table 3: The gyromagnetic ratios for electrons and muons.
In summary we conclude that there is very strong evidence for CPT invariance for the strong, the electromagnetic and the strangeness-conserving non-leptonic weak interactions. Also the purely weak processes lend rather strong support for CPT invariance while the semi-leptonic and the strangeness-changing weak interactions remain to be investigated with higher precision. It should be noted also that in general only some aspects of CPT invariance have been examined. In no case do we have precision measurements directly checking the relation (177) without summation over spin directions. It would certainly be worthwhile to test the equality of the matrix elements for particle conjugate processes with the spins reversed as given by the eq. (177).

8. Concluding remarks.

We summarize the present experimental status of the discrete symmetries C, P, T or CP and CPT with regard to the strong, electromagnetic and weak interactions in table 4. The figures which are quoted refer

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>strong interactions</th>
<th>electromagnetic interactions</th>
<th>weak interactions</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPT</td>
<td>$10^{-14}$</td>
<td>$10^{-12}$</td>
<td>$\Delta S=0$, non-lept. $10^{-8}$</td>
</tr>
<tr>
<td>$T \sim CP$</td>
<td>$10^{-3}$</td>
<td>non-leptonic?</td>
<td>other $\sim 10^{-3}$</td>
</tr>
<tr>
<td>P</td>
<td>$10^{-6}$</td>
<td>$10^{-4}$</td>
<td>violated $\eta_+ \sim 2 \times 10^{-3}$</td>
</tr>
<tr>
<td>C</td>
<td>$10^{-2}$</td>
<td>non-leptonic $10^{-4}$</td>
<td>violated</td>
</tr>
</tbody>
</table>

Table 4: Upper limits for symmetry-violating amplitudes in the different types of interactions.
to the upper limit for the symmetry-violating amplitudes. The electromagnetic interactions of leptons are to a high degree of accuracy C and T invariant and these symmetries are incorporated in quantum electrodynamics.

We end this presentation of the discrete symmetries with some brief remarks concerning two important aspects of symmetry arguments, namely (i) a specific prediction may follow from more than one invariance principle and its verification cannot be uniquely attributed to any one them, and (ii) the whole concept of symmetry when symmetry-violating interactions are present.

8.1 Predictions which follow from more than one invariance principle.

At various occasions we have encountered predictions which can be derived from more than one principle. This is, for example, the case with the vanishing electric dipole moment for the neutron which is a consequence of C or T invariance. A violation of this prediction implies a violation of both C and T invariance. A verification of the prediction lends support to both invariance principles but cannot be taken as evidence for either one. For this reason it is extremely important to clarify under what assumptions a certain prediction may be derived, and if there are more than one alternative. This inherent weakness of symmetry arguments was once again pointed out very clearly by Bernstein et al. 11 who noted that very many predictions of C and T invariance in electromagnetic processes can also be derived from other equally fundamental principles. As an example we consider the electromagnetic form-factors of a nucleon. In section 7.2 we introduced the concept invoking Lorentz invariance and current conservation. If we only assume Lorentz invariance, then the eq. (189) is replaced by

\[
\langle \bar{p}; \bar{s} | J_{\mu}^{EM}(0) | \bar{p}, \bar{s} \rangle =
\]

\[
= e \bar{u}(\bar{p}; \bar{s}')(F_{1}(q^2)\gamma_{\mu}+i F_{2}(q^2) \sigma_{\mu\nu}q^{\nu}+F_{3}(q^2)q_{\mu})u(\bar{p}, \bar{s})
\]

(195)

Current conservation implies
\[ \langle \bar{p}; \bar{s}' | q^\mu J_{EM}^\mu(0) | \bar{p}, \bar{s} \rangle = 0 \]  \hspace{1cm} (196)

and it follows from gauge invariance. Noting that the first term in (195) may be written

\[ \bar{u}(\bar{p}; \bar{s}') \gamma_\mu q^\mu u(\bar{p}, \bar{s}) = \bar{u}(\bar{p}; \bar{s}') \left[ \gamma_\mu p^\mu - \gamma_\mu p^\mu \right] u(\bar{p}, \bar{s}) \]  \hspace{1cm} (197)

it follows from the Dirac equation for the two spinors that this term vanishes. In the second term we obtain \( \sigma_{\mu \nu} q^\mu q^\nu \). Since \( \sigma_{\mu \nu} \) is antisymmetric in \( \mu \) and \( \nu \) while \( q^\mu q^\nu \) is symmetric also this term vanishes. Thus, from the condition (196) we conclude

\[ q^2 F_2(q^2) = 0 \]  \hspace{1cm} (198)

or

\[ F_2(q^2) = 0 \]  \hspace{1cm} (199)

except possibly for \( q^2 = 0 \). This result was anticipated in the eq. (189). However, one reaches the same conclusion by invoking (i) that \( J_{EM}^\mu(0) \) is hermitian and (ii) that under time reversal \( J_{EM}^\mu(0) \rightarrow \epsilon(\mu) J_{EM}^\mu(0) \). From (i) we obtain

\[ \langle \bar{p}; \bar{s}' | J_{EM}^\mu(0) | \bar{p}, \bar{s} \rangle = \langle \bar{p}, \bar{s} | J_{EM}^\mu(0) | \bar{p}; \bar{s}' \rangle^* \]  \hspace{1cm} (200)

or

\[ \bar{u}(\bar{p}; \bar{s}') \left[ F_1 \gamma_\mu + i F_2 \sigma_{\mu \nu} q^\nu + F_3 q_\mu \right] u(\bar{p}, \bar{s}) = \]

\[ = \{ \bar{u}(\bar{p}, \bar{s}) \left[ F_1 \gamma_\mu - i F_2 \sigma_{\mu \nu} q^\nu - F_3 q_\mu \right] u(\bar{p}; \bar{s}') \}^* = \]

\[ = u^+(\bar{p}; \bar{s}') \left[ F_1^* \gamma_\mu^+ + i F_2^* \sigma_{\mu \nu}^+ q^\nu - F_3^* q_\mu^+ \right] \gamma_0 u(\bar{p}, \bar{s}) = \]

\[ = \bar{u}(\bar{p}; \bar{s}') \left[ F_1^* \gamma_\mu + i F_2^* \sigma_{\mu \nu}^* q^\nu - F_3^* q_\mu^* \right] u(\bar{p}, \bar{s}) \]  \hspace{1cm} (201)
We conclude that hermiticity requires that $F_1$ and $F_2$ are real functions and $F_3$ a purely imaginary function. With regard to the condition (ii) we have previously considered the case of a weak current. The computations are in this case very similar and we shall not repeat them.

The result is that the condition

$$\langle \overline{F}_1, \overline{s} | J_\mu \Delta M(0) | F, s \rangle = \varepsilon(\mu) \langle -\overline{p}_1, -\overline{s} | J_\mu \Delta M(0) | -\overline{p}, -\overline{s} \rangle^*$$ (202)

implies that $F_1$, $F_2$ and $F_3$ must all be real functions. In order to satisfy both conditions we must then choose $F_3 = 0$, which is the same result as the eq. (8.5).

We shall not pursue the subject further, but we conclude that a verification of a prediction based on a symmetry argument never constitutes an absolute proof of the invariance principle since (i) the same result may follow from an alternative set of principles as discussed above, or (ii) there may be an accidental cancellation, for example of dynamical origin. An example of this latter possibility is known from the parity-violating asymmetry parameters $\alpha$ in $\Sigma$ decay. For the $\Sigma^+ (\Sigma^+ \rightarrow n + \pi^0)$ and $\Sigma^- (\Sigma^- \rightarrow n + \pi^-)$ decays the asymmetry parameters vanish while for $\Sigma^0 (\Sigma^0 \rightarrow p + \pi^0)$ it is large. On-off hand one would say that the two first decays are parity conserving while the third one is parity violating. For the time being these observations can only be understood as accidental cancellations although they neatly conform with the $\Delta I = \frac{1}{2}$ rule.

Finally, it should also be stated that although the fundamental laws of physics may satisfy certain symmetry properties the same symmetry properties are reflected in the experimental results only for isolated systems with no "external fields" present. For example, if we neglect the very small CP violation it was found that the two neutral $K$ mesons $K_1^0$ and $K_2^0$ retain their identity until the moment of decay since CP conservation forbids the transitions $K_1^0 \leftrightarrow K_2^0$. How then can one explain that a beam of $K_2^0$ regenerates $K_1^0$ mesons copiously when it passes through matter even though all interactions are CP invariant to a high degree of accuracy? The reason is simply that the piece of matter through which the beam passes is not CP invariant and it constitutes an "external field". Quite naturally then the regeneration effect depends on the difference between the scattering amplitudes for KN
and $\bar{K}N$ scattering. In the same spirit it was first attempted to save the CP invariance in the $K_L^0 \to 2\pi$ decay by invoking an external field to account for the apparent CP violation. This possibility has later been ruled out on experimental grounds.

8.2 The concept of symmetry in the presence of symmetry-violating interactions.

We have consistently defined the various symmetry operations by giving their action on certain state vectors. In those cases where there is a classical analogue we have chosen the definitions in such a way that the quantum-mechanical operators transform in the same way as their classical counterparts. This procedure may seem very straight-forward, but in realistic applications there are some inherent difficulties. For one thing, the problem of how to treat unstable states has not been tackled more than superficially, and a completely satisfactory way to do that is not known yet. The other complication arises in the context of approximate symmetries.

We have seen that our definition of a space reflection operator requires a Hilbert space which is larger than the physical Hilbert space (i.e., the space of physical state vectors) since occasionally non-physical vectors appear (e.g., the neutrino states with positive helicity). In a formal approach strictly based on group theory it is difficult to give an exact meaning to the concept of an approximate symmetry, and yet the concept has turned out to be most useful in elementary particle physics. In a hamiltonian approach approximate symmetries emerge in a rather natural way.

Suppose that we can write the total hamiltonian of a physical system in the following way

$$H = H_{\text{free}} + H_{\text{st}} + H_{\text{em}} + H_{\text{wk}}$$

(203)

where the strong, the electromagnetic and the weak parts depend on the corresponding coupling constants. Suppose further that one may define some limiting procedure so that, say, the weak and the electromagnetic part of the hamiltonian vanish. In this limit, if it exists,
it is conceivable that we obtain a model which exhibits full C and P symmetry. This means that the corresponding quantum mechanical operators would not contain any parts, which would lead out of the physical Hilbert space when they act on a physical state vector. On the other hand, in this limit the Hamiltonian will in general be invariant under entirely new transformations which render the C and P transformations ambiguous and some of the conventional state labels are no longer relevant. In the example considered above this happens for the electric charge, since it requires the presence of electromagnetism for an operational definition. Since the strong forces are charge independent one arrives at a model which is invariant under the isospin group. Under these circumstances a space reflection or a charge conjugation may just as well contain isospin transformations. Without the charge label we would retain transformation properties for observables analogous to the classical ones for P, and the C transformation would, for example, transform a p state into a \( \bar{n} \) state etc. In this way one faces the possibility of different C, P and T transformations corresponding to the limits which we conventionally refer to a strong, electromagnetic and weak interactions \(^{43}\). We shall not develop this subject further, but it is clear that once a symmetry is found not to hold rigorously we must approach the subject with an open mind for entirely new possibilities with drastic changes in old concepts. This is, of course, an observation which is particularly pertinent in these times when old symmetries repeatedly fail under closer scrutiny.
REFERENCES


2. For a derivation see for example: G. Källén, Elementary Particle Physics, Addison-Wesley Publishing Company (1964).


6. We use the same metric and the same set of \( \gamma \)-matrices as


   Lobashor et al., JETP \( \overline{3} \), 76 and \( \overline{3} \), 268 (1966).
   H. Paul et al., Report to the 1966 Berkeley Conference.


12. Note that \( \langle p | J^+_\mu(0) | n \rangle = \langle p | c J^+_\mu(0)c^{-1} | n \rangle \) and
    \( c J^+_\mu(0)c^{-1} = J^+_\mu(0) \) if we leave out overall phase factors.
    A similar relation holds for the lepton part.

13. See for example N. Cabibbo and M. Veltman, Weak Interactions, Cern 65-30 (Yellow report).


20. With our phase convention for the time reversal transformation $T$ (cf. section 6) one finds $CPT |K^0> = |\bar{K}^0>$ which is the conventional definition of the relation between the two neutral $K$ mesons. With this latter definition no restriction to the rest system is needed.

21. The same formfactors appear in the matrix elements for other processes such as the beta-decay of neutrons, and the experimental tests refer to other processes than the one considered here.


27. To avoid confusion in the notations we shall denote the transition operator by $t$ rather than by $T$ as one conventionally does (cf. sections 2.2 and 2.5).

29. V.L. Fitch, Proc. XIIith Int. Conf. on High Energy Physics,  

30. See the discussion after the talk by V.L. Fitch (reference 29).


    See also R.G. Glasser et al., Proc. Int. Conf. on Weak Inter-
    actions, Argonne National Laboratory report ANL - 7130 (1965).


34. M.J. Longo et al., See V.L. Fitch, Rapporteur's talk at the  
    Berkeley Conference (reference 29).

35. See for example G. Källén, Elementary Particle Physics, Addison-  

36. O. Overseth and R. Roth, See Rapporteur's talk by V.L. Fitch  
    at the Berkeley Conference (reference 29).

37. G. Lüders, Kongl. Dansk Medd. Fys. 28, No 5 (1954) and  
    W. Pauli, "Niels Bohr and the Development of Physics",  
    J. Schwinger, Phys. Rev. 82, 914 (1951); 91, 714 (1953)  


    and report to the Berkeley Conference (1966).


Appendix 1.

Brief review of the Dirac equation.

Relativistic particles of spin $\frac{1}{2}$ are described by a four-component spinor wave function $\psi(x)$. For free particles this wave function satisfies the following equation of motion – the Dirac equation

$$(i \gamma_{\mu} \frac{\delta}{\delta x_{\mu}} - m) \psi(x) = 0$$

(1)

The $4 \times 4$ matrices $\gamma_{\mu}$ ($\mu = 0, 1, 2, 3$) obey the anticommutation rules

$$\gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = 2g_{\mu\nu}$$

(2)

where $g_{\mu\nu}$ is the metric tensor, which we take to be

$$
(g_{\mu\nu}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(3)

We shall use the following explicit representation of the $\gamma$-matrices

$$
\gamma_0 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix};
\quad
\gamma_k = \begin{pmatrix}
0 & \sigma_k \\
-\sigma_k & 0
\end{pmatrix}
$$

(4)

where $1$ denotes the $2 \times 2$ unit matrix and $\sigma_k$ ($k = 1, 2, 3$) is a Pauli matrix

$$
\sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix};
\quad
\sigma_2 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix};
\quad
\sigma_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
$$

(5)

With these conventions it is easily shown that $\gamma_0$ is hermitian and $\gamma_k$ antihermitian. These properties are summarized by the relation
\( \gamma^\dagger_\mu = \gamma_0 \gamma_\mu \gamma_0 \)  \( \text{(6)} \)

If the adjoint wave function \( \overline{\psi}(x) \) is defined by

\[ \overline{\psi}(x) = \psi^\dagger(x) \gamma_0 \]  \( \text{(7)} \)

then it follows from the eq. (1) that in the case of free particles it satisfies the equation

\[ i \frac{\partial}{\partial x_\mu} \overline{\psi}(x) \gamma_\mu + m \overline{\psi}(x) = 0 \]  \( \text{(8)} \)

The following two matrices occur frequently in the Dirac theory

\[ \sigma_{\mu\nu} = \frac{1}{2i} ( \gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu ) \]  \( \text{(9)} \)

\[ \gamma_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \]

The matrix \( \sigma_{\mu\nu} \) satisfies the relation

\[ \sigma^\dagger_{\mu\nu} = \gamma_0 \sigma_{\mu\nu} \gamma_0 \]  \( \text{(10)} \)

while \( \gamma_5 \) is hermitian. Furthermore, the matrix \( \gamma_5 \) anticommutes with the four \( \gamma \)-matrices \( \gamma_\mu \)

\[ \gamma_\mu \gamma_5 + \gamma_5 \gamma_\mu = 0 \]  \( \text{(11)} \)

In the standard representation (4) \( \gamma_5 \) is given by

\[ \gamma_5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \]  \( \text{(12)} \)

The Dirac equation (1) is satisfied by plane wave solutions
of the form

\[ \psi(x) = u(p) \exp[-ip_\mu x^\mu] \]

\[ \psi(x) = v(p) \exp[+ip_\mu x^\mu] \]

provided the coordinate-independent spinors \( u(p) \) and \( v(p) \) satisfy the following set of equations

\[ (\gamma_\mu p^\mu - m) u(p) = 0 \]

\[ (\gamma_\mu p^\mu + m) v(p) = 0 \]

The solutions with \( u(p) \) are identified with the particle solutions while those with \( v(p) \) describe antiparticles. It is easily seen that the eqs. (14) have two independent solutions each corresponding to two possible orientations of the spin. With the standard representation (4) the solutions are given by

\[ u_i(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} x_i \\ \frac{\bar{\sigma} \cdot p}{E+m} x_i \\ \frac{E+m}{x_i} \end{pmatrix} \]

\[ v_i(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} -\frac{\bar{\sigma} \cdot p}{E+m} \xi_i \\ \xi_i \\ 1 \end{pmatrix} \]

with \( i = 1,2 \) and

\[ x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\xi_2; \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \xi_1 \]

The normalization has been chosen in such a way that the following orthonormality conditions hold
\[ \bar{u}_1(\vec{p}) \ u_j(\vec{p}) = \delta_{ij} \]
\[ \bar{v}_1(\vec{p}) \ v_j(\vec{p}) = -\delta_{ij} \]  
\[ \bar{u}_1(\vec{p}) \ v_j(\vec{p}) = \bar{v}_1(\vec{p}) u_j(\vec{p}) = 0 \]  

The adjoint spinors \( \bar{u}(\vec{p}) \) and \( \bar{v}(\vec{p}) \) are defined by

\[ \bar{u}(\vec{p}) = u^\dagger(\vec{p}) \gamma_0 \]  
\[ \bar{v}(\vec{p}) = v^\dagger(\vec{p}) \gamma_0 \]  

and they satisfy equations analogous to (14)

\[ \bar{u}(\vec{p}) (\gamma_\mu p^\mu - m) = 0 \]  
\[ \bar{v}(\vec{p}) (\gamma_\mu p^\mu + m) = 0 \]  

In the discussions of charge conjugation and time reversal two more matrices \( C \) and \( T \) are introduced. They have the following properties

\[ C^{-1} \gamma_\mu C = -\gamma^T_\mu \]  
\[ C^{-1} \gamma_5 C = \gamma^T_5 \]  
\[ C^\dagger = C^{-1} \]  
\[ C^T = -C \]  

and

\[ T^{-1} \gamma_\mu T = \epsilon(\mu) \gamma^T_\mu \]  
\[ T^{-1} \gamma_5 T = -\gamma^T_5 \]  
\[ T^\dagger = T^{-1} \]  
\[ T^T = -T \]
By direct inspection it is seen that

$$F = -i \gamma_0 \gamma_2$$  \hspace{1cm} (23)

and

$$T = i \gamma_2 \gamma_5$$  \hspace{1cm} (24)

satisfy these conditions. In the standard representation (4) they are given by

$$C = -i \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$  \hspace{1cm} (25)

$$T = i \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

For applications regarding $P$, $C$ and $T$ we note the following relations, which can be obtained, for example, by direct inspection from the eqs. (15) and (16) with the explicit representation (4) for the $\gamma$-matrices

$$u(-\bar{p}, \mp \bar{s}) = \gamma_0 u(\bar{p}, \pm \bar{s})$$  \hspace{1cm} (26)

$$v(-\bar{p}, \mp \bar{s}) = -\gamma_0 v(\bar{p}, \pm \bar{s})$$

with

$$u(\bar{p}, \bar{s}) \equiv u_1 (\bar{p})$$

$$u(\bar{p}, -\bar{s}) \equiv u_2 (\bar{p})$$  \hspace{1cm} (27)

etc. Similarly

$$v(\bar{p}, \bar{s}) = g \bar{u}^T (\bar{p}, \bar{s})$$  \hspace{1cm} (28)

$$\bar{v}(\bar{p}, \bar{s}) = -u^T (\bar{p}, \bar{s}) g^{-1}$$

and
\begin{align*}
    u(-\bar{p}, -\bar{s}) &= t^T \bar{u}^T (\bar{p}, \bar{s}) \\
    \bar{u}(-\bar{p}, -\bar{s}) &= u^T (\bar{p}, \bar{s}) t^{-1}
\end{align*}
\quad (29)
Appendix 2.

Free particle states.

The basis vectors of the physical Hilbert space are labeled by the eigenvalues of a complete set of commuting observables, and any state vector representing a physical system can always be expanded in such a basis. Corresponding to different sets of commuting observables one obtains different sets of basis vectors, which are related by unitary transformations (Clebsch-Gordan type expansions). It is often important to know the explicit form of these transformations for the sets which are most frequently used in physics. It is also very important that a consistent set of phase conventions is used. Since there are many different conventions used in the literature special care must be exercised in comparing results from different authors.

There are essentially two types of basis vectors which are used in particle physics; (i) plane wave states of definite momentum and (ii) angular momentum states of various kinds. In this appendix we shall define the sets which have been used in this series of lectures and give the relations between them. For a more complete discussion of these questions we refer to (1). We shall treat single-particle states, two-particle states etc. separately since the complexity of labeling the state vectors increases with increasing number of particles. We shall further restrict the discussion to free particle states for which single-particle labels can be used.

The classification of relativistic particle states is based on the Poincaré group, that is, the group of proper orthochronous Lorentz transformations. Within this scheme the kinematics of a free particle of mass m and spin s is related to a unitary irreducible representation (UIR) of the Poincaré group. These UIR's are characterized by the value of the two invariants (Casimir operators) \( p^\mu p_\mu \) and \( \mathcal{W}^\mu \mathcal{W}_\mu \) which in turn are related to the rest mass and the spin of the particle

\[
\begin{align*}
    p^\mu p_\mu &= m^2 \\
    \mathcal{W}^\mu \mathcal{W}_\mu &= -m^2 s(s+1)
\end{align*}
\]
with

\[ W^\mu = \frac{i}{2} \epsilon^{\mu \nu \rho \sigma} J_{\nu \rho} P_{\sigma} \]

and \( \epsilon^{0123} = - \epsilon_{0123} = -1 \). More precisely, the manifold of state vectors representing the possible states which the particle can occupy form a representation space of the UIR \((m,s)\) of the Poincaré group. Alternatively, given the state vector representing any possible state of the particle, all other physical state vectors representing the same particle (in different states of motion) are obtained by means of a Lorentz transformation acting on the original state vector. This property provides us with the means to construct the state vectors representing a particle from some standard state vector. We shall outline the procedure below but first we summarize the basic properties of the Poincaré group and its Lie algebra spanned by the infinitesimal generators \( I^\mu \) and \( J_{\mu \nu} \).

(i) \( (a, \Lambda) : x^\mu = \Lambda^\mu_\nu \, x^\nu + a^\mu \)

(ii) \( (a_1, \Lambda_1) \, (a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2) \)

(iii) \([p^\mu, p^\nu] = 0\)

\[ [J^{\mu \nu}, p^\rho] = -i (\epsilon^{\mu \rho \nu} p^\sigma - g^{\nu \rho} p^\mu) \]

\[ [J^{\mu \nu}, J^{\rho \sigma}] = -i (\epsilon^{\mu \rho \nu} J^{\sigma \rho} + g^{\nu \sigma} J^{\mu \rho} - g^{\mu \sigma} J^{\nu \rho} - g^{\nu \rho} J^{\mu \sigma}) \]

(iv) If we define a new set of generators by

\[ J^{kl} = \sum_m \epsilon^{klm} J^m ; \quad k, l, m = 1, 2, 3 \]

\[ J^{0k} = -J^{k0} = K^k \]

then the commutation rules read
\[ [ p^\mu, p^\nu ] = 0 \]
\[ [ j^k, p^o ] = 0 \]
\[ [ j^k, p^l ] = i \varepsilon^{klm} p^m \]
\[ [ j^k, j^l ] = i \varepsilon^{klm} j^m \]
\[ [ k^k, p^l ] = -i \delta^{kl}_{kl} p^o \]
\[ [ j^k, k^l ] = i \varepsilon^{klm} k^m \]
\[ [ k^k, k^l ] = -i \varepsilon^{klm} j^m \]
\[ [ k^k, p^o ] = -i r^k \]


Consider a particle of mass \( m > 0 \) and spin \( s \) at rest. The state of this particle is completely specified by the spin component \( m_s \) along the z-axis, since \( p^\mu, p^\mu, W^\mu W^\mu, \bar{P} \) and \( J^2 \) form a complete set of commuting observables. In the rest system the angular momentum operator \( \bar{J} \) is identical with the spin operator, for which we shall use the notation \( \bar{S} \). Altogether there are \((2s + 1)\) linearly independent states corresponding to \( m_s = s, s-1, \ldots, -s \). We denote the state vectors by \( | p = 0, m_s \rangle \) and we suppress the labels \( m \) and \( s \). The transformations between the different states are accomplished with the raising and lowering operators \( J^\pm = (J^1 \pm i J^2) \) which satisfy the following relations

\[ J^\pm | p = 0, m_s \rangle = \sqrt{s - m_s} (s + m_s + 1) | p = 0, m_s \pm 1 \rangle \quad (1) \]

while \( J^3 \) is diagonal

\[ J^3 | p = 0, m_s \rangle = m_s | p = 0, m_s \rangle \quad (2) \]

A finite rotation \( R(\bar{Q}) \) is given by
\[ R(\vec{q}) = \exp \left[ -i \vec{J} \cdot \vec{q} \right] \]  

(3)

and acting on the state \( |\vec{p} = 0, m_s> \) one obtains

\[ R(\vec{q}) |\vec{p} = 0, m_s> = \sum_{m_s'} D^S_{m_s'm_s}(\vec{q}) |\vec{p} = 0, m_s'> \]

(4)

If we parametrize the rotation by means of the Euler angles \((\alpha, \beta, \gamma)\) then we obtain \((j = s)\)

\[ D^j(\alpha, \beta, \gamma) = \exp \left[ -i J^3 \alpha \right] \exp \left[ -i J^2 \beta \right] \exp \left[ -i J^3 \gamma \right] \]

(5)

and hence

\[ D^j_{m',m}(\alpha, \beta, \gamma) = <j, m'|\exp\left[-i J^3 \alpha\right] \exp\left[-i J^2 \beta\right] \exp\left[-i J^3 \gamma\right]|j, m> \]

(6)

with

\[ d^j_{m'm}(\beta) = <j, m'|\exp\left[-i J^2 \beta\right]|j, m> \]

(7)

These \(d\)-functions can be expressed in elementary funtions (1). The results so far are based on the fact that the state vectors representing the particle at rest form a \((2s + 1)\)-dimensional UIR of the rotation group \(SO(3)\).
Starting from a state at rest we may now proceed to define various states in motion. We introduce

\[ |\vec{p}, m_s \rangle \equiv R(\varphi, \theta, 0) L_z(v) R^{-1}(\varphi, \theta, 0) |\vec{p} = 0, m_s \rangle \]  \hspace{1cm} (8)

and

\[ |\vec{p}, \lambda \rangle \equiv R(\varphi, \theta, 0) L_z(v) |\vec{p} = 0, \lambda \rangle \]  \hspace{1cm} (9)

where \( \varphi \) and \( \theta \) are the polar angles of \( \vec{p} \) and \( \vec{v} = \vec{p}/p_0 \). We shall refer to the vectors \( |\vec{p}, \lambda \rangle \) as helicity states since they are eigenvectors of the helicity operator \( \vec{J} \cdot \vec{p}/|\vec{p}| \). The Lorentz transformation \( L_z(v) \) along the \( z \)-axis can be expressed in terms of the infinitesimal generator \( K^3 \).

\[ L_z(v) = \exp[-i K^3 u] \]  \hspace{1cm} (10)

and \( \tanh u = v \). The normalizations of the vectors (8) and (9) are given by

\[ <\vec{p}', m_s' |\vec{p}, m_s \rangle = p_0 \delta^3(\vec{p}' - \vec{p}) \delta_{m_s' m_s} \]  \hspace{1cm} (11)

\[ <\vec{p}', \lambda' |\vec{p}, \lambda \rangle = p_0 \delta^3(\vec{p}' - \vec{p}) \delta_{\lambda' \lambda} \]  \hspace{1cm} (12)

To demonstrate the consistency of our procedure we shall compute the \( x \)-component of the momentum for a particle represented by \( |\vec{p}, \lambda \rangle \equiv |\varphi, \theta, p, \lambda \rangle \). Thus

\[ p^1 |\varphi, \theta, p, \lambda \rangle = p^1 R(\varphi, \theta, 0) L_z(v) |\vec{p} = 0, \lambda \rangle = \]

\[ = p^1 \exp[-i J^3 \varphi] \exp[-i J^2 \theta] \exp[ -i K^3 u ] |\vec{p} = 0, \lambda \rangle = \]
= \exp[-i J^3 \varphi] \exp[-i J^2 \theta] \exp[-i K^2 u] \times \{ \exp[i K^3 u] \} \exp[i J^3 \varphi] \exp[-i J^2 \theta] \exp[-i J^3 \varphi] \exp[-i J^2 \theta].

\exp[-i K^3 u] \} | p = 0, 3>

We first compute the expression within the parenthesis and make use of the following relations, which easily can be derived from the commutation rules:

\exp[i J^3 \varphi] P^1 \exp[-i J^3 \varphi] = P^1 \cos \varphi - P^2 \sin \varphi

\exp[i J^2 \theta] P^1 \exp[-i J^2 \theta] = P^1 \cos \theta + P^3 \sin \theta

\exp[i J^2 \theta] P^2 \exp[-i J^2 \theta] = P^2

\exp[i K^3 u] P^1 \exp[-i K^3 u] = P^1

\exp[i K^3 u] P^2 \exp[-i K^3 u] = P^2

\exp[i K^3 u] P^3 \exp[-i K^3 u] = P^3 \cosh u + P^0 \sinh u

Hence, we arrive at the following result:

\{ \exp[i K^3 u] \exp[i J^2 \theta] \exp[i J^3 \varphi] P^1 \exp[-i J^3 \varphi] \exp[-i J^2 \theta] \exp[-i K^3 u] \} = P^0 \cos \varphi \sin \theta \sinh u + P^1 \cos \varphi \cos \theta - P^2 \sin \varphi + P^3 \cos \varphi \sin \theta \cosh u
Acting on $| \overrightarrow{p} = 0, \lambda \rangle$ only the first term contributes

$$p^0 | \overrightarrow{p} = 0, \lambda \rangle = m | \overrightarrow{p} = 0, \lambda \rangle$$

and we obtain

$$p^1 | \phi, \theta, p, \lambda \rangle = m \cos \phi \sin \theta \sinh u | \phi, \theta, p, \lambda \rangle$$

We finally take notice of the following identity

$$m \sinh u = m \frac{\tanh u}{\sqrt{1 - \tanh^2 u}} = m \sqrt{\frac{p/E}{E^2 - p^2}} = p$$

so that

$$p^1 | \phi, \theta, p, \lambda \rangle = p \cos \phi \sin \theta | \phi, \theta, p, \lambda \rangle$$

which is the correct result.

The two sets of basis vectors given by (8) and (9) are clearly closely related. The transformation from one set to the other is immediately seen to be

$$| \overrightarrow{p}, \lambda \rangle = \sum_{m_s=-s}^{+s} D^s_{m_s \lambda} (\phi, \theta, 0) | \overrightarrow{p}, m_s \rangle \quad \text{(13)}$$

It should be noted that the angles $\theta$ and $\phi$ in the definitions (8) and (9) range over the intervals

$$0 < \theta < \pi$$

$$-\pi < \phi < \pi$$
for which the definitions are unique. For \( \varphi = 0 \) and \( \varphi = \pi \) this is no longer the case and similarly with half-integer spin representations for \( \varphi = -\pi \). We define the state vectors corresponding to the momentum parallel or antiparallel with the z-axes in the following way

\[
|0, 0, \mu, \lambda\rangle \equiv \mathcal{L}_z(\varphi) |0, 0, 0, \lambda\rangle
\]

(14)

and

\[
|0, \pi, \mu, \lambda\rangle \equiv \mathcal{L}_{-z}(\varphi) |0, 0, 0, -\lambda\rangle = \exp[-i\pi s] \mathcal{R}(\pi, \pi, 0) \mathcal{L}_z(\varphi) |0, 0, 0, \lambda\rangle
\]

(15)

To derive the transformation properties of the vectors \(|\bar{p}, \lambda\rangle\) under a Lorentz transformation we introduce the abbreviated notation (cf. the eq. (9))

\[
|\bar{p}, \lambda\rangle = \mathcal{L}(\bar{p}) |0, \lambda\rangle
\]

(16)

and thus the operator \(\mathcal{L}(\bar{p})\) takes a state at rest into a state of momentum \(\bar{p}\). To deduce the action of an arbitrary Lorentz transformation \(\mathcal{L}(A)\) on \(|\bar{p}, \lambda\rangle\) we note the following decomposition of \(\mathcal{L}(A)\)

\[
\mathcal{L}(A) = \mathcal{L}(A\bar{p}) \mathcal{R}(p, A) \mathcal{L}^{-1}(\bar{p})
\]

(17)

where \(\mathcal{R}(p, A)\) is a pure rotation, the Wigner rotation, and \(A\bar{p}\) is defined by

\[
p_{\mu} \rightarrow (A\bar{p})_{\mu} = A_{\mu}^{\nu} p_{\nu}
\]
and $\Lambda^\nu_\mu$ is the $4 \times 4$ matrix defining the Lorentz transformation. This decomposition yields

$$L(\Lambda) | \bar{p}, \lambda \rangle = L(\Lambda \bar{p}) R(p, \Lambda) | \bar{p} = 0, \lambda \rangle =$$

$$= \sum_{\lambda'} D^S_{\lambda' \lambda} [R(p, \Lambda)] | \Lambda \bar{p}, \lambda' \rangle \tag{16}$$

and corresponds to first transforming the state to rest and then from there to the state of final momentum $\Lambda \bar{p}$. However, in general there is an extra rotation for the "intermediate" state at rest. The explicit form of the rotation $R(p, \Lambda)$ is given by the eq. (17). We shall not derive it but note that the transformation rule (18) for $s \neq 0$ is rather complicated.

We shall finally derive the transformation rule for the vector $| \bar{p}, \lambda \rangle$ under a space reflection (Cf. the eq. (48) of section 3.1). To this end we start from the state $| 0, 0, 0, \lambda \rangle$. Since the space reflection operator $P$ commutes with $\bar{J}$ we have

$$P | 0, 0, 0, \lambda \rangle = \eta_p | 0, 0, 0, \lambda \rangle \tag{19}$$

and further $P$ clearly also commutes with the raising and lowering operators $J^\pm$ so that $\eta_p$ is independent of $\lambda$. The operator $P$ is unitary and hence the intrinsic parity must be a phase factor. Next consider the operator $R_{xz}$ of reflection in the xz-plane

$$R_{xz} = \exp[-i \pi J^2] P = R(0, \pi, 0)P \tag{20}$$

It commutes with $L_z(\nu)$ and further

$$R_{xz} | 0, 0, 0, \lambda \rangle = \eta_p R(0, \pi, 0) | 0, 0, 0, \lambda \rangle =$$
\[
\eta_p \sum_{\lambda', \lambda} D_{\lambda'}^s (0, \pi, 0) |0, 0, 0, \lambda'\rangle
\]

It can be shown that

\[
D_{\lambda'}^s (0, \pi, 0) = (-1)^{s-\lambda} \delta_{\lambda', \lambda}
\]

so that

\[
R_{xz} |0, 0, 0, \lambda\rangle = \eta_p (-1)^{s-\lambda} |0, 0, 0, -\lambda\rangle
\]

and thus

\[
P |0, 0, p, \lambda\rangle = R^{-1}(0, \pi, 0) R_{xz} L_z(v) |0, 0, 0, \lambda\rangle =
\]

\[
= \eta_p (-1)^{s-\lambda} R^{-1}(0, \pi, 0) |0, 0, p, -\lambda\rangle
\]

Recalling that P commutes with rotations and noting the following relation

\[
R(\varphi, \theta, 0) R^{-1}(0, \pi, 0) = R(\varphi + \pi, \pi - \theta, -\pi)
\]

we finally arrive at

\[
P |\varphi, \theta, p, \lambda\rangle = \eta_p \exp[-i\pi s] |\varphi + \pi, \pi - \theta, p, -\lambda\rangle
\]
which is the result previously quoted.

Beyond the vectors \( | \bar{p}, \lambda > \) and \( | p, m_s > \) we shall also encounter the angular momentum states \( | j, m, p, \lambda > \) defined by the partial wave expansion

\[
| \bar{p}, \lambda > = \sum_{j=0}^\infty \sum_{m=-j}^{+j} \frac{2j+1}{4\pi} d_{\lambda}^{j} (\varphi, \theta, 0) | j, m, p, \lambda >
\]

(26)

To invert this relation we recall the following orthogonality relation

\[
\frac{2j+1}{4\pi} \int d\Omega D_{mn}^{j*} (\varphi, \theta, 0) D_{m' n'}^{j'} (\varphi, \theta, 0) = \delta_{jj'} \delta_{mm'} \delta_{nn'}
\]

(27)

where \( d\Omega = \sin\theta \ d\theta \ d\varphi \) is the volume element. We then obtain

\[
| j, m, p, \lambda > = \sqrt{\frac{2j+1}{4\pi}} \int d\Omega D_{m\lambda}^{j*} (\varphi, \theta, 0) | \varphi, \theta, p, \lambda >
\]

(28)

An arbitrary single-particle state vector \( | \psi > \) can now always be expressed as an expansion in whichever base is more convenient. For example, we may express it as a plane wave expansion

\[
| \psi > = \sum_{\lambda} \int \frac{d^3 p}{E} \psi (\bar{p}, \lambda) | \bar{p}, \lambda >
\]

(29)

where the weight function \( \psi \) must be square integrable in order that the norm of \( | \psi > \) be finite (cf. the eq. 12)

\[
< \psi | \psi > = \sum_{\lambda} \int \frac{d^3 p}{E} \psi^* (\bar{p}, \lambda) \psi (\bar{p}, \lambda) < \infty
\]

(30)
We shall not pursue the subject further and we consider next the two-particle states.

We finally note that the previous analysis is restricted to the case $m > 0$. Of practical importance is also the case $m = 0$ because there exist massless particles. For $m = 0$ the eq. (8) does not make sense since we cannot define a rest system in this case. However, we may start from some other standard state to generate the whole space. As a reference state one usually chooses one in which one has $p^\mu = (p, 0, 0, p)$, that is, the particle travels along the positive $z$-axis. An irreducible representation is characterized by $m = 0$ but not any more by the spin invariant $W^\mu W^\nu$ for the physically important case where $W^\mu W^\nu = 0$. Instead then one finds $W^\mu = \lambda p^\mu$, where $\lambda$ is the helicity eigenvalue. The helicity is there the second label which characterizes the UIR's, and we write the reference state vector $| (0, \lambda); 0, 0, p >$. An arbitrary state is then defined by the relation

$$| (0, \lambda); \varphi, \theta, p > = R(\varphi, \theta, 0) L_z(v) | (0, \lambda); 0, 0, p >$$

with $p^\mu = \{ R(\varphi, \theta, 0) L_z(v) \}^\mu \nu p^\nu$. Since we shall not make explicit use of massless state vectors we shall not pursue the subject any further.

2. Two-particle states.

In most experiments involving two-particle or many-particle states the experimental arrangements are designed to permit a rather accurate determination of the individual momenta. Therefore, it is from this point of view natural to work with states represented by state vectors of the following type

$$| \vec{p}_1, \lambda_1; \vec{p}_2, \lambda_2 > \equiv | \vec{p}_1, \lambda_1 > | \vec{p}_2, \lambda_2 >$$

These state vectors span a space for which the value of $P^\mu = (P^\mu_1 + P^\mu_2) (P^{1\mu}_1 + P^{2\mu}_2)$ is no longer constant and hence, we are no longer dealing with a UIR of the Poincaré group. If we denote the single-particle
Hilbert space $H(m, s)$, then we have

$$H(m_1, s_1) \otimes H(m_2, s_2) = \int_{m_1 + m_2} \ dM \sum_{S \ a} H(M, S, a)$$

where $a$ is a degeneration index accounting for the possibility that the same irreducible space $H(M, S)$ appears more than once in the decomposition.

For a theoretical analysis it is often advantageous to work with basis vectors different from the ones given by (31), and we shall find it useful to introduce several alternative sets, some of which in fact correspond to irreducible subspaces $H(M, S)$. We shall exclusively consider states in the center of mass system and thus we shall introduce the total momentum $\vec{P} = \vec{P}_1 + \vec{P}_2$ and the relative momentum $\vec{p}$. The restriction to the center of mass system then corresponds to choosing $P = 0$. We now define the state $| \vec{P} = 0; \varphi, \theta, p; \lambda_1, \lambda_2 \rangle$ in the following way

$$| \vec{P} = 0; 0, 0, p; \lambda_1, \lambda_2 \rangle = \sqrt{\frac{E}{E_0}} | 0, 0, p, \lambda, \lambda_0 \rangle | 0, 0, -p, \lambda_2 \rangle$$

(33)

$$| \vec{P} = 0; \varphi, \theta, p; \lambda_1, \lambda_2 \rangle = R(\varphi, \theta, 0) | \vec{P} = 0; 0, 0, p; \lambda_1, \lambda_2 \rangle$$

From the normalization of the single-particle state vectors involved in the definition it follows that

$$< \vec{P} = 0; \varphi, \theta, p; \lambda_1', \lambda_2' | \vec{P} = 0; \varphi, \theta, p; \lambda_1, \lambda_2 > =$$

$$= \delta^4 (\vec{p}_\mu - \vec{p}_\mu) \delta^2(\omega' - \omega) \delta \lambda_1' \lambda_1 \delta \lambda_2' \lambda_2$$

(34)
where \( \omega = (\theta, \varphi) \) is the solid angle so that

\[
\int d^2 \omega \delta^2(\omega - \omega') = 1; \quad d^2 \omega = d(\cos \theta) d\varphi
\]

if integrated over the whole solid angle.

By the definition (33) we have established the phase relations between single-particle and two-particle state vectors and we may now proceed from there and on to define new and suitable sets of two-particle state vectors. We shall leave out the detailed computations and just quote the definitions essentially.

In some cases we shall use angular momentum vectors of the following type

\[
| \vec{P} = 0; J, M, p; \lambda_1, \lambda_2 > = \frac{1}{2\pi} \sqrt{\frac{2J + 1}{4\pi}} \int dR D^*_{MLJ}(R) R | \vec{P} = 0; 0, 0, p; \lambda_1, \lambda_2 >
\]

(35)

where \( \lambda = \lambda_1 - \lambda_2 \), \( R \) stands for an arbitrary rotation and \( dR \) is the corresponding volume element in the parameter space. For example, if \( R \) is defined by the Euler angles then \( R = R(\alpha , \beta , \gamma) \) and \( dR = d\alpha d(\cos \beta) d\gamma \).

It is easily shown that the vectors \( | \vec{P} = 0; J, M, p; \lambda_1, \lambda_2 > \) for fixed \( J \) span a UIR of the rotation group \( SO(3) \) corresponding to the angular momentum \( J \). Integrating (35) over \( \gamma \) one finds

\[
| \vec{P} = 0; J, M, p; \lambda_1, \lambda_2 > = \sqrt{\frac{2J + 1}{4\pi}} \int d^2 \omega D^*_{M\lambda J}(\varphi, \theta, 0)
\]

\[
| \vec{P} = 0; \varphi, \theta, p; \lambda_1, \lambda_2 >
\]

(36)

If we make use of the eq. (27) we may invert (36) to obtain the following partial wave expansion
\[
\left| \vec{F} = 0 ; \varphi, \theta, p ; \lambda_1, \lambda_2 \right> = \sum_{J} \sum_{M=-J}^{+J} \frac{2J+1}{4\pi} D^J_M (\varphi, \theta, 0) \left| \vec{F} = 0 ; J, M, p ; \lambda_1, \lambda_2 \right>
\]

(37)

and as before \( \lambda = \lambda_1 - \lambda_2 \).

For discussions of selection rules such as space parity it is often convenient to use a somewhat different angular momentum basis in which one first couples the two individual spins \( s_1 \) and \( s_2 \) to a resultant spin \( \vec{s} \). Then \( \vec{s} \) is coupled to the relative orbital angular momentum \( \vec{l} \) so that \( \vec{J} = \vec{l} + \vec{s} \) in the center of mass. With this in mind we define the vectors \( \left| \vec{F} = 0 ; J, M, p ; l, \sigma \right> \) by the following relation

\[
= \sum \frac{(2l+1)}{\lambda_1, \lambda_2, 2J+1} \left| s_1 s_2, \lambda_1, -\lambda_2 \right| \sigma, \lambda > x
\]

\[
\left| \vec{p} = 0 ; J, M, p ; \lambda_1, \lambda_2 \right>
\]

(38)

where \( \left< j_1, j_2, m_1, m_2 \right| j, m \) are Clebsch-Gordan coefficients with phase conventions as given for example by Werle (1).

3. Three- and many-particle states.

Just as in the case of two-particle states we may form multiparticle states as outer products of single-particle states, that is

\[
\left| \vec{p}_1, \lambda_1 ; \vec{p}_2, \lambda_2 ; \ldots ; \vec{p}_n, \lambda_n \right> \equiv \left| \vec{p}_1, \lambda_1 \right> \left| \vec{p}_2, \lambda_2 \right> \ldots \left| \vec{p}_n, \lambda_n \right>
\]

(39)
and an arbitrary state $| \psi_n \rangle$ can then always be written as

$$
| \psi_n \rangle = \sum_{\lambda_1, \ldots, \lambda_n} \int \frac{d^3p_1}{E_1} \cdots \frac{d^3p_n}{E_n} \psi(\vec{p}_1, \lambda_1; \ldots; \vec{p}_n, \lambda_n)
$$

$$
| \vec{p}_1, \lambda_1; \ldots; \vec{p}_n, \lambda_n \rangle
$$

(40)

where $\psi(\vec{p}_1, \lambda_1; \ldots; \vec{p}_n, \lambda_n)$ must be square integrable in order that $| \psi_n \rangle$ be normalizable.

For theoretical discussions the states (39) are often complicated since they do not correspond directly to UIR for the Poincaré group. Of course, we have just as in the case of two-particle states the following reduction of the Hilbert space with regard to the Poincaré group

$$
H(m_1, s_1) \otimes H(m_2, s_2) \otimes \cdots \otimes H(m_n, s_n) = \int d\mu \sum_{\Sigma} \sum_{a} H(\mu, S; a)
$$

(41)

For a n-particle state there are in general $(4n - 6)$ degeneracy labels a, some of which range over a continuous spectrum so that the summation is replaced by an appropriate integration. The $(4n - 6)$ additional quantum numbers which are necessary to remove the degeneracy can be chosen in many different ways, of course. We shall not discuss this rather complicated problem which is treated by several authors (1). We just note that the technique used for the two-particle states can be extended in the sense that we first consider for example the particles 1 and 2 and reduce the corresponding product space. We then proceed and consider each of the components in this reduction coupled to particle 3 and reduce these products and so on. Physically we may view this as a successive coupling of n particles into states of definite relative angular momenta and invariant masses. Thus, first we consider the particles 1 and 2 in their center of mass and couple them to states of definite relative
angular momentum $j_{12}$ and invariant mass $m_{12}$. Then we perform a suitable Lorentz transformation to the center of mass of the systems (12) and 3 and couple these subsystems to a state of definite relative angular momentum $j_{123}$ and invariant mass $m_{123}$ and so on. Of course, this prescription will quickly become very complicated, but in most cases of practical importance $n = 3$ and it is not too elaborate a task to carry out the analysis. We shall not do it but refer the interested reader to the standard literature, for example the excellent book by Werle (1).
WEAK INTERACTIONS AND HIGHER SYMMETRIES

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The purpose of these two lectures is to give an introduction to the presently important domain of particle physics which deals with the consequences of higher symmetries in weak interaction processes. It might sound surprising at first that these symmetries, which most naturally refer to strong interactions, are of so great an interest in understanding weak interactions. Nevertheless weak interactions proper are now assumed to be well described at least for low momentum transfer reactions, in the framework of the Fermi "V−A" theory. It is because strong interactions deeply modify many weak processes in which hadrons are involved, that the corresponding observed reactions show so much variety and, a priori, so many complications. This is opposed to the basic simplicity of purely leptonic interactions. Even though strong interactions are yet poorly amenable to practical calculation, the existence of large symmetries may however be taken care of in a very satisfactory way in many particular cases. In particular, when applied to weak interactions processes, this leads to very interesting relations among reaction amplitudes as well as to some precise predictions. They can be obtained irrespectively of the detailed and as yet poorly known structure of strong interactions.

If the consequences of an exact symmetry are relatively easy to reach, the introduction of a manifest but imperfect symmetry, such as unitary symmetry, immediately brings up delicate questions. The great interest of the "Current Algebra" approach to higher symmetries, advocated by Gell-Mann, is that it provides a non ambiguous procedure to deal with a broken symmetry. The key idea is to use as a basic assumption the relations among observables, which hold for an exact symmetry, and may remain exact when the symmetry is broken. With such a powerful and precise tool at hands it is tempting to try higher and higher symmetry groups. Nevertheless, in this introduction, we will limit ourselves to the SU(3) × SU(3) symmetry group, one subgroup
of which is the SU(3) group of unitary symmetry. Even though the pertinent symmetry is so badly broken in the actual world, that it is no longer strikingly apparent, as SU(3) now is, its consequences are much richer than those of unitary symmetry alone, and they provide extremely valuable relations.

This particular higher symmetry brings together the unitary symmetry of strong interactions and the Vector–Axial symmetry of weak interactions. As a matter of fact it introduces itself rather naturally as a further general property of the weak hadronic currents. The purpose of this first lecture is therefore to introduce the basic relations known under the name of current algebra through a general and rapid survey of the present understanding of semi-leptonic interactions, that is the known properties of the weak hadronic currents. We shall see how a few general properties may in effect summarize our present knowledge of weak interactions, or at least of the leptonic and semi-leptonic ones. In so doing we shall aim at simplicity and present the most simple picture possible which is compatible with present experimental knowledge. In the second lecture we shall start from current algebra and derive the Adler–Weisberger relation, which gives the renormalization constant for the β-decay axial coupling in terms of a sum rule over π-nucleon cross sections. This is the prototype of several similar relations which hold for the various non-leptonic decays. I would like to show here in some details how such a relation appears most naturally as a low energy theorem when current algebra is combined with the PCAC (partially conserved axial current) hypothesis, which we will also come across in the first lecture. Such a low energy theorem gives a precise piece of information on an hadronic scattering amplitude (in this particular case, the first derivative of the π-nucleon forward elastic scattering amplitude) but limited to zero incident energy. This is obtained irrespectively of the detailed structure of strong interactions. Some general properties of the π-nucleon amplitudes will then allow us to translate this relation in terms of a sum rule by means of a dispersion integral, thus reaching the Adler–Weisberger sum rule in its usual form.
Even though such a calculation can be almost duplicated with only slight modifications for any semi-leptonic decay, this does not exhaust at all the applications of higher symmetries to weak interactions or, at least, a review of their present successes. Many interesting results can also be obtained for non-leptonic interactions. They relate among themselves many decay amplitudes which could hitherto be studied only through plain and specialized phenomenology. They also provide some understanding of the $\Delta I = 1/2$ rule for non-leptonic processes. This would however be outside of the scope of these two introductory lectures. We list in reference\textsuperscript{1} some review articles in which these questions could be studied in details. We also refer to some articles\textsuperscript{2} in which the topics covered hereby could be further studied.

B - GENERAL PROPERTIES OF WEAK CURRENTS

1) Weak Currents

Leptonic and semi-leptonic decays are very well described in terms of a phenomenological weak hamiltonian with a local current-current structure. The decay amplitudes are always considered to lowest order in the weak coupling. The weak hamiltonian reads:

$$H_W = \frac{G_V}{\sqrt{2}} j^+ \gamma^\mu j^\mu,$$  \hspace{1cm} (1)

where $G$ is the weak coupling constant:

$$G_V = (1.023 \pm 0.002) \frac{10^{-5}}{m^2},$$  \hspace{1cm} (2)

with $m$ standing for the proton mass.
The weak current $J_\mu$ is decomposed into two parts: a leptonic part $J_{\mu}^{\text{lept}}$ and an hadronic part $J_{\mu}^{\text{had}}$:

$$J_\mu = J_{\mu}^{\text{lept}} + J_{\mu}^{\text{had}}.$$  \hfill (3)

The leptonic current is well known. It can be written in terms of local lepton fields. This reads:

$$J_{\mu}^{\text{lept}} = \overline{\psi}_e \gamma_\mu (1 + \gamma_5) \psi_e + \overline{\psi}_\mu \gamma_\mu (1 + \gamma_5) \psi_\mu.$$  \hfill (4)

In so doing one actually summarizes several properties of $J_{\mu}^{\text{lept}}$. They are the following:

a) Electron-muon symmetry

b) Existence of two types of neutrinos. The $\nu_e$ neutrino, associated with the electron and the $\nu_\mu$ neutrino associated to the muon. Relation (4) includes the conservation of the muonic quantum number as well as the conservation of the total number of leptons.

c) The Vector-Axial coupling. Parity is violated in a "maximal" way.
The leptons are coupled with negative helicity and the anti-leptons with positive helicity. This implies the two component neutrino theory.

Since the leptonic part of the weak coupling is well known, the calculation of any semi-leptonic amplitude, as obtained from (1) and (3), may now proceed along standard lines. One needs to know the matrix element of the hadronic current between the initial and final hadronic states considered. To be more precise one needs actually the Fourier transform of the current density:

\[
\int e^{-i q x} d^4 x \langle B(p') | j^{\text{had}}_{\mu}(x) | A(p) \rangle = (2\pi)^4 \delta(p'-p-q) \langle B(p') | j^{\text{had}}_{\mu}(0) | A(p) \rangle ,
\]

where \(-q\) is the momentum transferred to the lepton pair.

The great simplicity of the weak leptonic current is not obviously shared by the hadronic part of the current. Nevertheless one is now lead to write it in the following way.

\[
j^{\text{had}}_{\mu} = \cos \theta \left\{ (V^1_{\mu} + i V^2_{\mu}) + (A^1_{\mu} + i A^2_{\mu}) \right\} \\
+ \sin \theta \left\{ (V^4_{\mu} + i V^5_{\mu}) + (A^4_{\mu} + i A^5_{\mu}) \right\} ,
\]

(5)
where $\theta$ is the Cabibbo angle. In order to simplify things as much as they can be, in view of the present experimental situation, we have taken the same angle for the Vector (V) and Axial (A) part of the current, when two slightly different angles could be in order. The value of $\theta$ is of the order of 0.24 radians. The indices 1, 2, 4 and 5 are unitary spin indices.

According to (5) the hadronic current is separated into a strangeness conserving part (proportional to $\cos \theta$) and a part which changes strangeness by one unit (proportional to $\sin \theta$). Each part is further separated into a Vector and an Axial part.

In order to illustrate what the indices associated to the different parts of the current actually mean, it is useful to have detailed expressions for the Vector and Axial Currents in terms of fields, as it is the case for the leptonic current. We have however to appeal to a particular model and we shall consider here the presently simplest one, that is the quark model.

We write the hadronic current in terms of three Dirac quark fields which refer respectively to three fictitious quarks, the properties of which are summarized on table 1. They are denoted by $p$ and $n$ (an isotopic spin doublet) and $\lambda$ (an isotopic spin singlet). We may also write the quark field in terms of a twelve component spinor (4 component Dirac field with 3 unitary components):

$$q(x) = \begin{pmatrix}
\psi_p(x) \\
\psi_n(x) \\
\psi_\lambda(x)
\end{pmatrix}.$$ 

As Dirac fields they satisfy the canonical commutation relations:
\[ \delta(x_o - y_o) \{ q^a(x), q^+\beta(y) \} = \delta_{a\beta} \delta(x-y). \] (7)

Following Gell-Mann, one introduces the eight 3x3 matrices \( \lambda_i \) of the Algebra of SU(3). They are listed on table 2. The first three ones correspond to isospin. They reduce to the Pauli matrices in the subspace of the isotopic spin doublet \((p,n)\). The \( \lambda_4 \) and \( \lambda_5 \) matrices play the same role in the subspaces \((p,\lambda)\) and \((n)\) as \( \lambda_1 \) and \( \lambda_2 \) do in the subspaces \((p,n)\) and \((\lambda)\). So do the \( \lambda_6 \) and \( \lambda_7 \) matrices in the subspaces \((n,\lambda)\) and \((p)\). The \( \lambda_8 \) matrix corresponds to the hypercharge \( Y = \frac{\lambda_8}{\sqrt{3}} \).

We may now construct explicitly the Vector and Axial currents in the quark model:

\[ V^i_\mu(x) = \overline{q}(x) \gamma^i_\mu \frac{\lambda_1}{2} q(x) \]  
\[ A^i_\mu(x) = \overline{q}(x) \gamma^i_\mu \gamma_5 \frac{\lambda_1}{2} q(x), \] (8)

the electromagnetic current would read:

\[ V^{E,M}_\mu(x) = \overline{q}(x) \gamma^\mu \left( \frac{\lambda_3}{2} + \frac{1}{2\sqrt{3}} \lambda_8 \right) q(x), \]

that is the sum of an isovector and an isoscalar component. This is a general property of the electromagnetic current which follows from the principle of minimal electromagnetic coupling. \( \mu \) is a Lorentz index \( \mu = 0, 1, 2, 3 \) and \( i \) a unitary spin index \( i = 1, 2, \ldots, 8 \), we next write the hadronic current in the quark model:

\[ \gamma^{\text{had}}_\mu = \cos\theta \left( \overline{\psi} \gamma^\mu (1 + \gamma_5) \psi_n \right) + \sin\theta \left( \overline{\psi} \gamma^\mu (1 - \gamma_5) \psi_n \right). \] (9)

From (8) and (9), one reads off directly some general properties of the hadronic current, namely: the strangeness preserving part transforms like an isovector; the strangeness changing part obeys the rules \( \Delta S = 1 \) and \( \Delta I = 1/2 \); furthermore the change in charge and in strangeness are related through \( \Delta S = \Delta Q \). These
properties are obvious in the quark model which we have used. They are general properties of the weak current.

We neglect here the PC violating effects which could appear in semi-leptonic interactions. The interaction which we have written is PC conserving.

2) Conserved Vector Current and partially conserved Axial Current

We shall now consider two general properties which are not explicitly contained in (1), (5) and (8). The first one is the Conserved Vector Current (CVC) theory \[4\].

The vector current which we wrote in (8) for \(i = 1, 2, 3\) is the isotopic spin current in the quark model. It has been in fact recognized that the hadronic vector current which enters in weak interaction is the isotopic spin current. It is a conserved current, up to electromagnetic corrections, which satisfies the relation:

\[ \partial^\mu j^i_\mu = 0 \quad i = (1, 2, 3) . \] (10)

The current component with isotopic spin index \(i = 3\) is the isovector part of the electromagnetic current. The matrix elements of the weak current are then simply related through a rotation in isotopic spin space to the matrix elements of the isovector electromagnetic current. This predicts relations between semi-leptonic decay amplitudes such as \(n \rightarrow p + e + \bar{\nu}\) and \(\pi^- \rightarrow \pi^0 + e + \bar{\nu}\) which merely translate the fact that the proton (a member of an isodoublet) and the \(\pi^+\) (a member of an isotriplet) have the same charge. This also predicts the existence of weak coupling associated with the anomalous
magnetic moment of the nucleon. Both results have been beautifully verified by experiment.

A further implication of the CVC theory is that the weak coupling constant is not "renormalized" by strong interactions in much the same way as the charge of the proton is +1, as opposed to −1 for the electron, irrespectively of the fact that the proton has strong interactions which give it a complicated electromagnetic structure. This further explains the fact that the weak vector coupling constant for β decay, defined with a phenomenological expression for the current in terms of the nucleon field;

$$J^1_\mu = \overline{\psi}_N \gamma_\mu \frac{\tau^1_i}{2} \psi_N$$  \hspace{1cm} (11)

was found to have the same value (not withstanding the presence of \(\cos \delta \neq 1\)) as the coupling constant observed in the pure leptonic interaction of muon decay. There are no renormalization corrections.

Except for this last remark, the same conclusions would have been reached with a weak current proportional, instead of identical, to the isotopic spin current. This is an important question in connection with the universality of the weak coupling, that is the same value of \(G_V\) in (1) whether we take the leptonic or the hadronic part of \(J\). It is in fact possible to introduce this universality irrespectively of any field model for the hadron, as the one taken in (11). This is done as follows.

We can associate to the components of the isovector current charges which are defined as:

$$I_i = \int d^3x \, V_i^V(x) \quad (i = 1, 2, 3) \,.$$

(12)
Relation (10) implies that the charges, defined in (12), are time independent operators, they are the isotopic spin operators (the generators of the SU(2) group of charge independence) and they obey the familiar commutation rules:

\[ [I_i, I_j] = i \varepsilon_{ijk} I_k \]

(13)

where \( \varepsilon_{ijk} \) is the totally antisymmetrical tensor.

The hermitian operator \( I_i \) commute with the hamiltonian. They are the generators of the isospin symmetry group.

The non linear relations (13) which are readily transformed into non linear relations among amplitudes, provide a definition of the isotopic spin current, as opposed to the isotopic spin current times a constant. One can attach a precise meaning to the universality (not withstanding so far strangeness changing term and even the presence of \( \cos \theta \)) of the weak coupling through (1), (3), (4), (5) and (13), and this irrespectively of the notion of fields attached to the hadrons.

In order to be more precise, we can take the matrix element of (13) between two hadronic states and sum over a complete set of intermediate states. This reads:

\[
\sum_{n,n'} \left< B | I_i | n \right> \left< n | I_j | n' \right> \left< n' | I_i | A \right> = \left< B | I_k | A \right> \varepsilon_{ijk}.
\]

We relate this way terms which are quadratic in the charge, on the left, to a term which is linear on the right, when each term can be attached in principle to a weak or electromagnetic transition amplitude. This clearly differentiate the isotopic spin current as opposed to the isotopic spin current times a cons-
tant, as the current entering weak interactions. This gives therefore a precise meaning to the presence of the same constant \( G_N \) for leptonic and semi leptonic decays in (1), as well as for the presence of \( \cos \theta \) in (5).

If the symmetry is exact the sum over the \( n, n' \) states is of course limited to members of the symmetry multiplet to which both \( A \) and \( B \) should belong.

The CVC theory is extended to the strangeness changing part of the current. The four different terms which appear in (5) \( \langle V_i \rangle \), with \( i = 1, 2, 4, 5 \) are in fact components of the unitary spin current. The associated charges are now four generators chosen among the eight generators of the SU(3) group of unitary symmetry. In this limit, the matrix elements of all the vector current unitary components, between any two baryons, can be calculated in terms of one of them: the isovector electromagnetic current of the proton, for instance, which is well known. The general consequences of CVC which we have just listed are thus extended in a straightforward way to strangeness changing amplitudes\(^4\), this is of course true only in the limit where SU(3) is a good symmetry, this is done except for the notion of universality which may seem to be a priori lost, with the two different terms \( \cos \theta \) and \( \sin \theta \), present in (5). We shall however come back to this particular point later on.

We may now define eight unitary charges, as we did in (12) but with \( i \) running from 1 to 8, 4 of these charges being associated to weak currents. Being associated with conserved current (neglecting first medium strong and electromagnetic interactions) they commute with the Hamiltonian. They are the generators of the SU(3) symmetry group. As generators of the SU(3) group, they satisfy the commutation rules:

\[
[I_i, I_j] = f_{ijk} I_k
\] (14)
where the $f_{ijk}$ are the structure constants of SU(3).

These commutation properties are of course true in the quark model where they can be readily worked out from (7), (8) and (12). They of course hold irrespectively of the model.

The axial current does not have these properties. Nevertheless its various components ($1 = 1, 2, 4$ and $5$) are known to transform like members of an octet under unitary (SU(3)) transformations. The vector current has the same property but with further the much important one of being the unitary current. It is also no longer possible to obtain the matrix elements of the current between any two hadrons in exact SU(3) in terms of a simple Clebsch-Gordan coefficient of SU(3). One needs two parameters which characterize the strengths of the two possible octet couplings. There are two ways to couple the axial current to the initial and final baryon octets (symmetrical and antisymmetrical). They are respectively referred to as D and F couplings. For instance the nucleon $\beta$ decay axial coupling is written $\cos^0(D+F)$ while the $\Xi^-$'s one is written as $\cos^0(D-F)$. The vector charges, as generators of the SU(3) group correspond to pure F coupling. It is not possible here to go into the details of the calculation of such matrix elements in the exact SU(3) limit. All the necessary material can be found, beautifully presented, in the article of Gell-Mann CTSL 20, which has been published in "The Eightfold Way" by Gell-Mann and Neeman (Benjamin Frontiers series). Anyway, with three parameters only, $\theta, F$ and $D$, it is then possible to calculate all decay rates and the results obtained are extremely good. This approach is known as the Cabibbo theory. A comparison between the theoretical and experimental values is given on table 3.
With respect to the detailed verification of the Cabibbo theory, there are in effect two schools of thought. One may either fit present data with one single angle. This is the simplest assumption which we have followed here. C. Carlson \([3]\) gets very good results with \( \theta = 0.245 \pm 0.010 \), \( F = -0.415 \pm 0.035 \) and \( D = -0.766 \pm 0.037 \). One may also reach a fit with two slightly different angles (vector and axial).

N. Brene et al. \([3]\) also get very good agreement preferring \( \theta_V = 0.212 \pm 0.004 \) and \( \theta_A = 0.268 \pm 0.001 \).

The octet transformation property of the axial current may be translated into commutation relations between the \( I_1 \) charges, previously defined, and axial charges \( D_i \) which are defined as:

\[
D_i = \int d^3x \ A^i_o(x)
\]

\( (i = 1, 2, \ldots, 8) \) ,

(15)
they read:

$$[I_i^1, D_j^1] = i f^{ijk} D_k^1.$$  \hfill (16)

They can also be worked out readily in the quark model. They hold however irrespectively of the model and stand for the fact that the axial charges transform like the members of an octet while the vector charges are the generator of the \( SU(3) \) symmetry group.

We now turn to the partially conserved character (PCAC) of the axial current\(^4\). The axial current is not conserved and it may therefore seem puzzling that the "renormalization" brought by strong interaction is only a rather small effect. The \( \beta \) decay axial coupling constant is only 1.2 times the vector one\(^*\). This leads to the idea that in a particular model, not too different from the actual world, the axial current also would be conserved\(^4\).

In order to illustrate this, we shall consider the matrix element of the axial current \((i = 1 \text{ or } 2)\) between two one nucleon states. Lorentz invariance imposes:

$$\langle N(p')|A_{\mu}^1(0)|N(p)\rangle = \frac{i}{2} \bar{u}(p') \left[ \gamma_{\mu} \gamma_5 s_A(t) + k_{\mu} \gamma_5 h_A(t) \right] u(p), \tag{17}$$

where \( k = p' - p \), \( t = k^2 \). \( \tau^4 \) is the Pauli matrix. This defines a vertex function which can be represented by the following graph:

\[\text{Diagram} \]

\(^*\)For historical reasons the axial coupling is defined with a minus sign \(((D_{\mu} F_{\mu}) \cos \theta = -1.18)\).
We can calculate from (17) the matrix element of the divergence of the current. It reads:

\[
\langle N(p') | p^\mu A^- \mu | N(p) \rangle = i(p' - p)^\mu \langle N(p') | A^- \mu (0) | N(p) \rangle \\
= (2m g_A(t) + t h_A(t)) \bar{u}(p') i \gamma^\mu \frac{\tau^i}{2} u(p).
\]

(18)

If the axial current were conserved, one would have:

\[
G(t) = 2m g_A(t) + t h_A(t) = 0.
\]

(19)

\(G(t)\) is known to be an analytic function with singularities given by generalized unitarity. It has a pole at \( t = \frac{m^2}{\pi} \) and it is defined with a cut along the real axis from \( 9m^2/\pi \) to \( \infty \).

This is clearly incompatible with (19). Nevertheless the following assumptions can be made:

a) \(G(t)\) satisfies an unsubtracted dispersion relation. We may therefore write:

\[
G(t) = \frac{R}{m_\pi^2 - t} + \frac{1}{\pi} \int_{9m^2/\pi}^{\infty} \frac{\text{Im} G(t')}{t' - t - i\epsilon} \, dt'.
\]

(20)
where $R$ is the residue of the pion pole. This pole is associated with the one pion intermediate state:

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and its residue reads:

\[ R = 2f_\pi m_\pi^2 g . \] (21)

It combines the pion axial coupling appearing in the upper blob with the pion-nucleon coupling constant present at the lower blob. $f_\pi$ is the pion-axial coupling constant defined by:

\[ \langle 0 | A^{(1)}_\mu (0) | \pi^i (q) \rangle = i f_\pi q_\mu , \]

and it is related to the charged pion decay rate ($\mu\nu$ decay) through the following relation:

\[ \Gamma = f_\pi^2 G_V^2 \cos^2 \theta \frac{m_\pi^2 m_\mu^2}{4\pi} \left( 1 - \frac{m_\mu^2}{m_\pi^2} \right)^2 , \]

$g$ is the pion nucleon coupling constant $\left( \frac{g^2}{4\pi} = 14.7 \right)$.

b) It is further assumed that, in the physical region of interest ($t$ negative and close to 0), the pole contribution is dominant:

\[ G(t) \propto \frac{2f_\pi m_\pi g}{m_\pi^2 - t} . \] (22)

This is the pion dominance of the divergence of the axial current (PDDAC).
form of the PCAC hypothesis.

Combining (19) and (22) at $t = 0$, one gets:

$$ f_{\pi} = \frac{m}{g} \frac{G_A}{G_V} $$

(23)

with

$$ g_A(0) = \frac{G_A}{G_V} $$

that is the Goldberger-Treiman relation.

In this picture, the divergence of the axial current would be zero if the $\pi$-meson had a zero mass, a situation which may be not too far away from physics to still have consequences in the actual world. Many tests of the PCAC hypothesis are available. They all give satisfactory results, that is within a 10% accuracy.

Another, and often more practical way, to introduce the PDDAC hypothesis is to identify the divergence of the axial current with the pion field:

$$ \not{\partial} A_\mu^{\dagger}(x) = \frac{m m_\pi^2}{g K(0)} \frac{G_A}{G_V} \not{\bar{\psi}}(x) $$

(24)

where $K(0)$ is the pion nucleon form factor at zero momentum transfer. Relation (23) is meaningful only to the extent that matrix elements of both sides are found to have almost identical off mass shell variations. $K(0)$ is unknown. It should be a priori not too different from 1. This is found to be a very reasonable approximation. We shall not write explicitly $K(0)$ where it should normally appear in the following.

Rather than using PCAC in the neat form provided by (23), we prefer here to follow the pole dominance picture which we used so far and which is more readily visualized. When dealing with several variables, ambiguities may arise this way. We shall not mention them here. They can be argued away in the cases considered as a consequence of the small pion nucleon mass ratio. The PCAC hypothesis can be extended to strangeness changing current. Kaon dominance replaces pion dominance. It is harder to find a rationale for neglecting the cut contribution. Nevertheless it is found to lead to satisfactory results.
As obvious from (23) the axial current would be conserved if the pseudoscalar meson masses were zero. This is perhaps not too far away a limit from physics so that some interesting consequences may appear.

3) Current algebra and universality of weak coupling

Starting from the quark model (relations (7) (8)) it is easy to obtain the equal time commutation relations between vector-vector and vector axial current densities:

\[ \delta(x_0 - y_0) \left[ V^i_\mu(x), V^j_\mu(y) \right] = i f_{ijk} V^k_\mu(x) \delta(x-y) \]  
\[ (24-a) \]

\[ \delta(x_0 - y_0) \left[ V^i_\mu(x), A^j_\mu(y) \right] = i f_{ijk} A^k_\mu(x) \delta(x-y) \]  
\[ (24-b) \]

which, once integrated over \( d^3x \) and \( d^3y \), give, with \( \mu \) taken as 0, the equal time commutation relations among charges (14) and (16). The quark model allows one further relation which is the equal time commutation between two axial densities:

\[ \delta(x_0 - y_0) \left[ A^i_\mu(x), A^j_\mu(y) \right] = i f_{ijk} V^k_\mu(x) \delta(x-y) \]  
\[ (24-c) \]

which integrated over space gives the equal time relation:

\[ [D^i, D^j] = i f_{ijk} I^k \]  
\[ (25) \]

the operators \( I \) and \( D \) build up a closed system under equal time commutation. The algebra thus obtained is the algebra of \( SU(3) \times SU(3) \)\(^5\). Gell-Mann has postulated that (25) holds irrespectively of the quark model. Relations (14) (16) and (25), namely:

\[ [I^i, I^j] = i f_{ijk} I^k \]  
\[ (26-a) \]
\[ [I^i, D^j] = i f_{ijk} D^k \quad (26-b) \]
\[ [D^i, D^j] = i f_{ijk} i^k \quad (26-c) \]

are then exact relations among operators. If the axial and vector currents were conserved, that is would satisfy \((10)\), the axial and vector charges, defined through \((12)\) and \((15)\) would be time independent. They would commute with the hamiltonian and would therefore be the generators of a symmetry group. We would reach SU(3)×SU(3)\(^5\). We would get only SU(3) symmetry keeping only the vector currents as conserved ones and only SU(2) symmetry if we were to keep only the non strange vector currents. The SU(3)×SU(3) group is however not a good symmetry group; the axial current is not conserved. Nevertheless it is readily verified that if the \(I\) and \(D\) operators vary with time (as a result of the non conservation of the pertinent current) the equal time relation \((26)\) still remains true. They can be taken as the only exact remain of a broken symmetry. Their matrix elements, taken between physical states, and this irrespectively of their association with symmetry multiplets, provide exact relations which can be tested against experiment. These multiplets no longer exist in broken symmetry. In the next lecture we shall see how the Adler-Weisberger relation clearly shows how fruitful relation \((26-c)\) actually is. Relations \((26)\) represent a weaker assumption than relations \((24)\). In this introduction we shall in effect stop at \((26)\) even though we might write them as \((24)\) for convenience.

Before we turn to this most interesting application, we may see how \((26)\) provides a precise definition for the universality of weak coupling. This is due to the fact that it provides non linear relations for both the vector and axial currents so that all coupling constants can be related.
In other words we need only one constant $G_V$ and one angle $\theta$ to fix the strength of both the vector and axial couplings.

One may consider the two leptons $e$ and $\nu_\theta$, (or $\mu$ and $\nu_\mu$) as forming an "iso" doublet and write the electronic (or muonic) leptonic current as the (I+II) component of an "iso" triplet:

$$\gamma_{\mu}^{\text{lept}} = 2(f_{\mu}^I + i f_{\mu}^{II}) = 21 \overline{\psi}_e \gamma_\mu \frac{1 + \gamma_5}{2} \psi_\mu ,$$

with a similar muonic term.

We may introduce leptonic charges, as done in (12) and commute them at equal time using the canonical commutation rules of the lepton fields (7), a safe thing to do with leptons. One readily gets this way the algebra of SU(2), that is relation (13).

Now let us rewrite (5) as

$$\gamma_{\mu}^{\text{had}} = \alpha \left( V_{\mu}^I + i V_{\mu}^{II} + A_{\mu}^I + i A_{\mu}^{II} \right) + \beta \left( V_{\mu}^I + i V_{\mu}^{III} + A_{\mu}^I + i A_{\mu}^{III} \right)$$

$$= \left( \alpha V_{\mu}^I + \beta V_{\mu}^{II} \right) + \left( \alpha A_{\mu}^I + \beta A_{\mu}^{II} \right) + i \left[ \left( \alpha V_{\mu}^I + \beta V_{\mu}^{III} \right) + \left( \alpha A_{\mu}^I + \beta A_{\mu}^{III} \right) \right]$$

$$= 2 \left( f_{\mu}^I + i f_{\mu}^{II} \right) .$$

(28)
Defining $Q^i$ as:

$$Q^i = \int d^3x \tilde{F}_o^i(x), \quad i = I, II$$

we then use (26) to calculate the equal time commutator $[Q^I, Q^{II}] = 2\Gamma Q^{III}$, namely:

$$2Q^{III} = (a^2 + \frac{\beta^2}{2})(I^3 + D^3) - a\beta(I^6 + D^6) + \frac{\sqrt{3}}{2} \beta^2 (I^8 + D^8)$$

and then:

$$[Q^{III}, Q^I] = i(a^2 + \beta^2) Q^{II}$$

$$[Q^{II}, Q^{III}] = i(a^2 + \beta^2) Q^I$$, (29)

with $a = \cos \theta$, $\beta = \sin \theta$ we also reach the algebra of SU(2) for charge operators which lump together vector and axial, strangeness preserving and strangeness changing currents. This provides a precise definition of the coupling strength associated to each component and at the same time of the universality of the weak coupling, as written in (1) and (5) according to the Cabibbo theory.

We have for this the same set of non linear relations for the hadronic charges and the leptonic charges. This allows a unique definition for the strength of the weak coupling for the leptonic as well as the hadronic terms. This gives full meaning to the universality property of weak interactions (leptonic and semi-leptonic).

We forced the same angle for the vector and axial currents. Symmetry breaking effects might however result in two slightly different angles.

C - THE ADLER-WEISBERGER RELATION

1) Current algebra and axial coupling

As an illustration of the usefulness of relations (26), and espe-
cially of relation (26-\(c\)), which is the only one which does not result from already known properties of the weak currents, we shall now give a derivation of the Adler-Weisberger relation. We shall calculate \(|G_A/G_V|\) which, as we mentioned earlier, is determined when we write a non linear relation for the axial current. In so doing we shall have to use the PDDAC hypothesis as well as current algebra and we have therefore chosen a method in which some special consequences of PDDAC alone are first clearly exhibited. We shall reach the Adler-Weisberger relation as a low energy theorem for \(\pi\)-nucleon scattering. Using known analytic and asymptotic properties of the \(\pi\)-nucleon scattering amplitude, we shall further transform it into a sum rule which provides a practical test.

The end result is:

\[
1 - \left(\frac{G_V}{G_A}\right)^2 = \left(\frac{2m}{\mu}\right)^2 \frac{1}{2\pi} \int_{(m+\mu)^2}^{\infty} ds \frac{d\sigma^+(s)}{s-m^2} (\sigma^+(s) - \sigma^-(s))
\]  

(30)

where \(\sigma^+(s)\) are the total \(\pi^+p\) cross section at center of mass energy squared \(s\), extrapolated to zero mass pion. Off mass shell correction are difficult to estimate. They can however be rather safely bounded and the result obtained is extremely satisfactory\(^{[6]}\). Relation (30) is known as the Adler-Weisberger sum rule.

In order to introduce a quadratic relation it is most natural to consider double scattering (of a proton say) in an external axial weak field. This is represented by the following graph:

\[
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\]
and we write it in terms of the retarded product of two axial currents \([7]\). We shall limit ourselves here to non-strange ones with isotopic spin indices \((i,j,k = 1,2,3)\):

\[
T^{ij\mu}_{\mu\nu}(p', q'; p; q) = \int d^4 x e^{iq'x} \langle p' | R \left( A^i_{\mu}(x), A^j_{\nu}(O) \right) | p \rangle,
\]

(31)

the retarded product is defined through:

\[
R(A(x), B(y)) = i\delta(x_o - y_o)[A(x), B(y)].
\]

We can integrate by part the right hand side of (31). This gives:

\[
q'^\mu \tilde{T}^{ij}_{\mu\nu} = i \int d^4 x e^{iq'x} \langle p' | R \left( \delta^{\mu}_{\nu} A^i_{\mu}(x), A^j_{\nu}(O) \right) | p \rangle
\]

\[
- \int d^4 x e^{iq'x} \delta(x_o) \langle p' | [A^i_{\mu}(x), A^j_{\nu}(O)] | p \rangle = t^{ij}_{\nu} - F^{ij}_{\nu} \tag{32}
\]

the second term in (32) comes from the derivative of the \(\delta\) function which appears in the definition of the retarded product. It can be readily transformed using current algebra (24). Before we do that however a general remark is in order.

To simplify things as much as possible, we shall make here a thorough use of the familiar polology technique. Relation (32) brings together three amplitudes, with PCAC having a lot to say about \(t\), and locality and then current algebra a lot to say about \(F\). Poles terms, which are associated in a straight-forward way with one particle intermediate states appearing in the different terms, should then cancel against each other in (32), thus providing relations among their residue.

\(T\) and \(t\) are functions of four independent variables which can be chosen as \(s = (p+q)^2 = (p'+q')^2\); \(t = (p'-p)^2\) and \(q^2\) and \(q'^2\). They are usually assumed to have singularities in all these variables which are given by generalized unitarity. In particular the pole terms are readily obtained from
allowed one particle intermediate states. On the other hand $F$ which is proportional
to an equal time commutator, is a polynomial in $q'$. It has only singularities
in $t$. Rewriting (32) as:

$$t^{ij}_\nu - q'_{i\mu} T^{ij}_{\mu\nu} = F^{ij}_\nu,$$

we see that singularities in $s$, $q^2$, and $q'^2$, present in both terms of the
left hand side, have to cancel out identically as a result of the locality
of the currents. This becomes a very stringent constraint when we introduce
the PDDAC hypothesis.

2) Consequences of PDDAC alone

We can first separate out some well known singular terms of $T$ and $t$. They correspond to near-by poles and are simply given by polology.
By polology we mean that a near-by pole, given by generalized unitarity is
not actually cancelled out in the physical amplitude by complicated terms
we are not yet aware of. We may therefore single out the pion and nucleon
pole terms in $T$. Let us assume that $i$ and $j$ respectively stand for $(-)$
and $(+)$ so that the nucleon pole occurs only in the direct channel. We
de decompose $T$ into:

\[\text{(a)} + \text{(b)} + \text{(c)} + \text{(d)}\]

\[\text{(e)} + \text{(f)} + \text{(g)} + \text{(h)}\]

* We define $D^\pm = 1/\sqrt{2} (D^1 \pm i D^2)$. 
this can be written as:

\[
T_{\mu\nu}^{ij} = \frac{q_\mu q_\nu}{(q^2 - m_\pi^2)(q^*^2 - m_\pi^2)} - \frac{q_\mu^* q_\nu}{q^*^2 - m_\pi^2} \Lambda_{\nu}^{ij} - i\frac{q_\nu}{Q^2 - m_\pi^2} \Lambda_{\mu}^{ij} + R_{\mu\nu}^{ij},
\]

in which we have brought together (a) and (b); (c) and (e); (d) and (f); (g) and (h). \(\Lambda\) is the \(\pi^-p\) elastic scattering amplitude, \(\Lambda\) is the pion production amplitude in an external axial field; \(\Lambda\) refers to the crossed amplitude. \(R\) is the double scattering amplitude when all pion pole terms have been subtracted. We decompose \(t\) in a similar way, but now use PDDAC. In so doing we keep only the contribution from the pion pole terms. It is of course difficult to introduce pole dominance in a function of four variables \((s,t,q^2, q^*^2)\), when PDDAC was introduced for a vertex function in the first place. Nevertheless, the pion nucleon mass ratio being rather low, the approximations made here can be justified. This reads:

![Diagram](image)

where the dot stands for the pion axial divergence coupling. Grouping again (a) and (b); (c) and (e):

\[
t_{\nu}^{ij} = \frac{m_\pi^2 q_\nu}{(q^2 - m_\pi^2)(q^*^2 - m_\pi^2)} - i\frac{m_\pi^2}{q^*^2 - m_\pi^2} \Lambda_{\nu}^{ij}.
\]

Combining (32), (33) and (34) we now get:
\[ f(\pi) \frac{q^2}{q^2 - m^2_\pi} \left( f^{i\mu}_\pi A^i_j - q^{i\mu} \frac{A^{i\mu}}{p} \right) - f(\pi) A^i_j + \frac{q^{i\mu}}{p} R^{i\mu}_j + R^{i\mu}_j = 0 \quad (35) \]

the pion pole term, at \( q^2 = m^2_\pi \), has cancelled out between the two terms, as it should. The pion pole term which still remains in (35) should disappear. Therefore:

\[ f^{i\mu}_\pi A^{i\mu} = \frac{q^{i\mu}}{p} A^{i\mu} \text{ (at } q^2 = m^2_\pi) \quad (36) \]

this is a relation which follows from PDDAC.

Since both sides have no near-by singularity in \( q^2 \), we shall assume that (36) hold down to \( q^2 = 0 \). The reason for doing so is that there is no conspicuous low mass hadronic states with the same quantum number as the pion. We do not expect therefore both sides to vary much with \( q^2 \), at least in the range \( q^2 \geq 0 \). This is a sound approximation here, the small pion nucleon mass ratio justifies it.

Both terms contain a one nucleon pole term \((a)\) on the left and \((d)\) on the right. Their contributions can be readily evaluated and are left as an exercise. One sees readily that the equality of the pole residues implies a relation between \( f_\pi g^2 \) in \((a)\) and \( g_{A^*} / g_V \) in \((d)\). This is the Goldberger-Treiman relation. Furthermore the value of the \( \pi \)-nucleon scattering amplitude at \( s = m^2_\pi, t = 0 \), that is \( q = q' = 0 \) may only come from the singular part of \( \tilde{A} \). The nucleon pole term in \((d)\) gives a singular term in \( q' \). \( \tilde{A} \), which is cancelled by the singular term in \( \tilde{M} \), as implied by the Goldberger relation, and also a regular term, which is also readily evaluated. This is the Adler relation*

\[ \tilde{M} \frac{m^2}{m^2} = \frac{2}{m} \]

where \( \tilde{M} (s,t) \) is the pion nucleon amplitude from which the pion pole terms have been subtracted.

At \( q = q' = 0 \) it is easy to see that \( \frac{A^{i\mu}}{p} \), where the nucleon pole term has been removed, can be written as \( \frac{\pi}{p} p^{i\mu} \), so that, at \( s = m^2_\pi \), we have:

* Up to an off-mass shell correction term.
\[ \frac{\partial}{\partial s} \tilde{\mathcal{K}}(s, o) \mid_{s = m^2} = \frac{1}{2\pi^2} \tilde{\lambda}(m^2). \]

We now go back to (35). Both \( A_\mu \) and \( q^\mu R_{\mu\nu} \) have a nucleon pole term. Its cancellation implies a relation between \( \frac{f_\pi}{G_A} g \frac{G_A}{G_V} \) in \( A_\nu \), and \((\frac{G_A}{G_V})^2\) in \( q^\mu R_{\mu\nu} \). Here again one gets the Goldberger-Treiman relation; the cancellation of the two singular terms which are readily written explicitly leaves a regular term which is found to be:

\[ - \frac{f_\pi}{2\pi} \tilde{\lambda}_\nu - \frac{1}{2} \left( \frac{G_A}{G_V} \right)^2 \frac{p_\mu}{m}. \]

We have written \((\frac{G_A}{G_V})^2\), without any form factor, since, in order to get information from (35), we need to eliminate all unknown terms, that is \( q^\mu R_{\mu\nu} \) when the one nucleon contribution is removed, and the finite contribution from the first term. These terms are unknown but regular by construction (since they have no near-by singularities they are further expected to vary very smoothly) and one can safely take the limit \( q = q' = 0 \) to eliminate them all. If we write at \( t = 0 \) \( F_\nu = (f/2m + \text{polynomial in } q') p_\nu \), as it follows from locality, we get, at \( q = q' = 0 \) (\( t = 0 \)):

\[ \frac{f_\pi}{2\pi} \mathcal{L} + \frac{1}{2m} \left( \frac{G_A}{G_V} \right)^2 = \frac{f}{2m} \tag{37} \]

where, at this particular point we have written \( A_\nu \equiv \lambda p_\nu \), \( p \) being the only non zero vector left, \( \lambda \) and \( \mathcal{L} \) are eliminated from (38) and (37) from crossing symmetry. At \( q = q' = 0 \), they are equal. We further use the Goldberger-Treiman relation and get:

\[ 1 - f \left( \frac{G_V}{G_A} \right)^2 = - \left( \frac{2m}{f} \right)^2 \frac{\partial}{\partial s} \tilde{\mathcal{K}}_{\pi-p}(s, o) \mid_{s = m^2} \tag{38} \]
this is a second low energy theorem which, together with (37) gives the value and the first derivative of the \( \pi \)-nucleon scattering amplitude. We should notice however that in order to eliminate unknown quantities we had to go to the limit \( q = q' = 0 \) that is to threshold scattering for unphysical zero mass pions.

In the following we shall assume that physics is not too different from this limit.

PDDAC is enough to determine the \( \pi \)-nucleon amplitude but we need \( f \) in order to have its first derivative. It is at this stage that current algebra enters into the picture.

If (24-c) is satisfied, \( F \) can be readily calculated and one finds:

\[
F^\pi_{\nu}(0) = \langle p | V^3_{\nu}(0) | p \rangle = \frac{1}{2m} p_{\nu}.
\]  (39)

Since we consider the limit \( q' = 0 \) only, gradient terms do not contribute and, even though we considered (24), we actually use only (26). Combining (38) and (39) one writes:

\[
1 - \left( \frac{G_{\nu}}{G_A} \right)^2 - \left( \frac{2m}{g} \right)^2 \frac{\partial}{\partial s} \pi^- \rho (s,0) \bigg|_{s = m^2}.
\]  (40)

which is the Adler-Weisberger relation.

Current algebra complements therefore dispersion relations. It provides us, through its non-linear character, with one particular value of the amplitude, a piece of information which is enough to calculate a subtraction
constant. With the help of (37) and (40) we can therefore calculate two subtraction constants. We know however that the \( \pi \)-nucleon forward amplitude requires only one subtraction. Equation (40) is therefore a constraint which can be checked directly.

The \( \pi \)-nucleon forward scattering amplitude satisfies a one-subtracted dispersion relation. Performing the subtraction at \( s = m^2 \), we write:

\[
m \tilde{\kappa}_{\pi-p}(s,o) = C^s + \frac{s-m^2}{2\pi} \left[ \int_{s-m^2}^{\infty} \frac{\sigma^{-}(s') - \sigma^{+}(s')}{(s'-s)(s'+s-2m^2)} \, ds' \right].
\]

and after derivation with respect to \( s \):

\[
m \frac{\partial}{\partial s} \tilde{\kappa}_{\pi-p}(s,o) \bigg|_{s=m^2} = \frac{1}{2\pi} \int_{s-m^2}^{\infty} \frac{ds}{s-m^2} \left( \sigma^{-}(s) - \sigma^{+}(s) \right).
\]

Combining (40) and (41) we get the Adler-Weisberger relation in terms of a sum rule (30)[8].

The dispersion relation has allowed us to calculate the \( \pi \)-nucleon scattering amplitude at an unphysical point \( s = m^2 \) in terms of physical quantities assuming total cross sections for zero mass pions and actual pions not to be too different.
This illustrates the type of relations which may be obtained from current algebra and PCAC. It is possible to relate a strong amplitude with two external pions to an electromagnetic hadronic amplitude in the limit where these two external pions are soft, that is have a vanishing energy. In other words, even if we do not know in any detail the structure of strong interactions, when they manifest themselves through currents, they have to satisfy specific relations which, in turn, imply strong constraints on the singular (and purely hadronic) terms associated with these currents.

The beautiful agreement reached with experiment shows how fruitful it is to introduce the $\text{SU}(3) \times \text{SU}(3)$ symmetry group, even though it is so badly broken that it is no longer apparent in hadronic spectroscopy. This does not mean of course that weak and electromagnetic interactions have anything to do with the symmetries of strong interactions. Symmetries imply conserved current, as a consequence of the Noether's theorem. It so happens that weak and electromagnetic interactions prefer these conserved, or almost conserved, currents rather than selecting some particle or field to couple to (the bare nucleon as opposed to a pion in its cloud say). They thus provide us with a simple way to measure the matrix elements of these currents between hadronic states, something which we could calculate in exact symmetry but which we cannot yet calculate in broken symmetry.

As a final remark the extension of the Adler-Weisberger relation to strangeness changing currents will involve $K$ nucleon instead of $\pi$-nucleon cross sections. They are not too different and clearly show that the Cabibbo angle is not a renormalization effect due to strong interactions. Symmetry breaking effect could however explain two slightly different vector and axial angles.
REFERENCES AND FOOTNOTES


These notes are an English, and slightly enlarged and modified, version of the first part of a set of notes, written in French, which corresponds to a series of lectures on the application of current algebra given at the Institut National des Sciences et Techniques Nucléaires, Saclay during the first semester of the 66-67 Academic years.

These notes, available as a Rapport CEA, could provide the reader with a broader outlook on the applications of current algebra to weak processes. They also contain a more detailed discussion of some subtle technical points which are left aside in this introduction.


Physics 1, 63 (1964)

S. ADLER - Proceedings of the Argonne Conference on Weak interactions (1965)


The two original papers in which the A-W sum rule is derived are:

S. ADLER - Phys. Rev. Letters 14, 1051, (1965)


More detailed discussions can be found in the excellent and longer articles.
W. WEISBERGER - Phys. Rev. 143, 1302 (1966)

[3] - For a best fit of the parameters involved in semi-leptonic decays

The values given are: \( \theta = 0.245 \pm 0.10, \) \( D = -0.766 \pm 0.037, \)
\( F = -0.415 \pm 0.035. \) This is a 3 parameter fit. The calculation
of the Cabibbo angle from \( K \) and \( \pi \) decay would give \( \theta_A = 0.266 \pm 0.066 \)
and \( \theta_V = 0.222 \pm 0.006. \)

See also N. Brene et al. - Phys. Rev. 149, 1288 (1966).

[4] - For a detailed review of the general properties of the weak cur-
rent see:

R.H. DALITZ - Varenna Lectures Notes (1964)

[5] - One readily verifies that the operator \( \frac{1}{2} \left( I^1 \pm D^1 \right) \)
form two sets of commuting unitary spin operators so that we have twice the algebra
of SU(3).

[6] - ADLER gives \( |G_A/G_V| = 1.24 \pm 0.03 \) while Weisberger without attem-
pting to estimate off mass shell effects gives \( |G_A/G_V| = 1.16. \) The
experimental value is \( 1.18 \pm 0.02. \)

[7] - Such an expression presents some ambiguities. It is associated to
the physical amplitude shown on page 18. It has the same singulari-
ties but may differ from it by a regular function. We shall use only
its singular parts. It is defined as a distribution and the integra-
tion by part which we have performed is justified.

[8] - An other possibility could be to extrapolate bravely up to the physical
threshold \( s = (m+\mu)^2 \), using a linear dependence and thus reaching the
\( \pi \)-nucleon scattering lengths in terms of \( |G_A/G_V| \). The result is satis-
factory even though the method is not, a priori, a reliable one.
Table 1 - General properties of the quarks. I is the isotopic spin, $I_3$ its third component, Y the hypercharge and Q the charge. All quarks have spin 1/2 and baryonic number 1/3.

Table 2 - The Gell-Mann matrices.

Table 3 - Phenomenological test of the Cabibbo theory. The results given are taken from R.E. Marshak "Ten years of the universal (V-A) theory of weak interactions", APS New-York meeting (January 67). They correspond to the same vector and axial Cabibbo angle: $\theta = 0.26$. 

<table>
<thead>
<tr>
<th></th>
<th>( I )</th>
<th>( I_3 )</th>
<th>( \gamma )</th>
<th>( Q )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{2}{3} )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>( n )</td>
<td>( \frac{1}{2} )</td>
<td>( -\frac{1}{2} )</td>
<td>( \frac{1}{3} )</td>
<td>( -\frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0</td>
<td>0</td>
<td>(-\frac{2}{3})</td>
<td>(-\frac{1}{3})</td>
<td>(\frac{1}{3})</td>
</tr>
</tbody>
</table>

**Table 1**
\( \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

\( \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \)

\( \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \)

\( \lambda_8 = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix} \)

**TABLE 2**
<table>
<thead>
<tr>
<th>Decay</th>
<th>Theory</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda \to p + e + \bar{\nu}$</td>
<td>0.87 $10^{-3}$</td>
<td>$(0.88 \pm 0.12) \ 10^{-3}$</td>
</tr>
<tr>
<td>$\Lambda \to \text{all}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Sigma^- \to n + e + \bar{\nu}$</td>
<td>1.48 $10^{-4}$</td>
<td>$(1.5 \pm 1.2) \ 10^{-4}$</td>
</tr>
<tr>
<td>$\Sigma^- \to \text{all}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Pi^- \to p + e + \bar{\nu}$</td>
<td>1.29 $10^{-3}$</td>
<td>$(1.25 \pm 0.17) \ 10^{-3}$</td>
</tr>
<tr>
<td>$\Pi^- \to \text{all}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Sigma^- \to \Lambda + e + \bar{\nu}$</td>
<td>0.62 $10^{-3}$</td>
<td>$(0.62 \pm 0.12) \ 10^{-3}$</td>
</tr>
<tr>
<td>$\Sigma^- \to \text{all}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Sigma^- \to \Lambda + e + \bar{\nu}$</td>
<td>0.43 $10^{-3}$</td>
<td>$(2 \pm 1) \ 10^{-3}$</td>
</tr>
<tr>
<td>$\Sigma^- \to \text{all}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Sigma^+ \to \Lambda + \bar{\nu} + \nu$</td>
<td>0.70 $10^{-4}$</td>
<td>$(0.61 \pm 0.16) \ 10^{-4}$</td>
</tr>
<tr>
<td>$\Sigma^+ \to \text{all}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K^+ \to \pi^0 + \bar{\nu} + \nu$</td>
<td>0.21 $10^{-4}$</td>
<td>0.2 $10^{-4}$</td>
</tr>
<tr>
<td>$K^+ \to \text{all}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K^+ \to \pi^0 + \bar{\nu} + \nu$</td>
<td>4.28 $10^{-2}$</td>
<td>$(4.83 \pm 0.14) \ 10^{-2}$</td>
</tr>
<tr>
<td>$K^+ \to \text{all}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi^+ \to \pi^0 + \bar{\nu} + \nu$</td>
<td>3.25 $10^{-2}$</td>
<td>$(3.44 \pm 0.13) \ 10^{-2}$</td>
</tr>
<tr>
<td>$\pi^+ \to \text{all}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 3**