Mean, Median and Mode of Hadron Spectrum in QCD Jets*

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Abstract

We present predictions of the three common statistical measures of the 'typical' value of the quantity $\xi = \ln(1/x)$ where x is the momentum fraction of a hadron in a QCD jet. Although the distribution of ξ is asymptotically Gaussian, its mean, median and mode (or peak) should differ by constants plus corrections of order $\sqrt{\alpha_s}$. An experimental study of these quantities would provide a detailed test of the underlying QCD dynamics of jet fragmentation.

1. Introduction

During the past year, the LEP experiments have provided a wealth of new information on the properties of QCD jets. One of the most striking features to emerge so far has been the close similarity between the observed distribution [1] of the single-hadron momentum fraction, x , and the theoretical predictions [2-5] based on perturbative QCD supplemented by the hypothesis of local parton-hadron duality (LPHD) [6].

QCD predicts that the momentum fraction distribution has the form of a 'hump-backed plateau' [7] which is asymptotically Gaussian in the variable $\xi = \ln(1/x)$. Taking proper account of leading and next-to-leading infrared logarithms to all orders, the shape of the distribution has calculable next-to-leading distortions that are substantial at present energies [2-4]. According to the LPHD hypothesis this shape should not be strongly modified by non-perturbative hadronization corrections, since these involve only small momentum transfers. The observed distribution is indeed in very good agreement with the perturbative predictions, including the next-to-leading corrections, over a wide range of energies.

Prompted by this success, we consider in this paper some more refined tests that could be performed making full use of the very precise data available from LEP in the near future. One natural feature to investigate more fully is the 'typical' momentum of a single hadron in a jet, as represented by the typical value of ξ . For this purpose statisticians commonly employ three seemingly unrelated measures of the 'centre' of the probability distribution $P(\xi)$, namely

1. The *mean* value, $\bar{\xi} = \int_0^\infty \xi P(\xi) d\xi$,
2. The *median* value ξ_m , which is such that the probabilities of lying above or below this value are equal, i.e., $\int_0^{\xi_m} P(\xi) d\xi = \frac{1}{2}$,
3. The *modal* or peak value ξ_p , at which $P(\xi)$ reaches its maximum.

We shall show that although the distribution $P(\xi)$ is asymptotically Gaussian, nevertheless these three measures do not coincide. They should differ by prescribed constant amounts asymptotically, and should have next-to-leading corrections proportional to $\sqrt{\alpha_s(Q^2)}$ where Q is of the order of the centre-of-mass energy. These predictions are not inconsistent with the Gaussian asymptotic form, since the width of the distribution grows like $\tau^{\frac{1}{2}}$ where $\tau = \ln(Q/\Lambda)$, and therefore a constant difference represents a vanishing fraction of the total width.

2. Calculation

Each of the quantities ξ ; $= \bar{\xi}, \xi_m$ or ξ_p should have an energy dependence of the form

$$\xi_i = a_i \tau + b_i \sqrt{\tau} + c_i + d_i / \sqrt{\tau} + \mathcal{O}(\tau^{-1}) \quad (1)$$

where the leading and next-to-leading coefficients a_i and b_i have been computed for the mean and the peak and are identical in these cases:

$$a_p = \frac{1}{2}, \quad b_p = \rho / \sqrt{48\beta}, \quad (2)$$

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where $\rho = 11 + 2N_f/N_c$ and $\beta = 11 - 2N_f/N_c$ for N_c colours and N_f flavours. This is in good agreement with the observed energy dependence of the peak position [1]. For the median ξ_m , no previous results are available, so we consider first the calculation of this quantity. In the course of the calculation we shall see that information on the subleading coefficients c_i and d_i can also be obtained for certain combinations of the mean, median and mode.

Since the distribution $P(\xi)$ is asymptotically Gaussian, it may be expanded around the mean in the quantity $\delta = (\xi - \bar{\xi})/\sigma$ where σ is the width:

$$P(\xi = \bar{\xi} + \sigma\delta) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\delta^2\right) \sum p_n \delta^n. \quad (3)$$

To the precision required, one easily finds by Taylor expansion about the mean that the median is at

$$\delta_m = -(p_3 + 7p_5 + 57p_7)/p_0 \quad (4)$$

while the peak is at

$$\delta_p = -3(p_3 + 5p_5 + 35p_7)/(p_0 - 2p_2). \quad (5)$$

Now the coefficients p_n are related to the *cumulants* κ_n that were introduced in [3]:

$$\begin{aligned} \kappa_1 &\equiv \bar{\xi} \\ \kappa_2 &\equiv \sigma^2 = \langle (\xi - \bar{\xi})^2 \rangle \\ \kappa_3 &\equiv \sigma^3 s = \langle (\xi - \bar{\xi})^3 \rangle \\ \kappa_4 &\equiv \sigma^4 k = \langle (\xi - \bar{\xi})^4 \rangle - 3\sigma^4 \\ \kappa_5 &\equiv \sigma^5 k_5 = \langle (\xi - \bar{\xi})^5 \rangle - 10\sigma^2 \langle (\xi - \bar{\xi})^3 \rangle \end{aligned} \quad (6)$$

where s is the *skewness* and k the *kurtosis* of the distribution. The cumulants are the derivatives of the logarithm of the Laplace transform of $P(\xi)$; they vanish for $n > 2$ in the case of a Gaussian distribution. We find that to the accuracy needed in Eqs. (4,5)

$$\begin{aligned} p_0 &= 1 + k/8 \\ p_2 &= -k/4 \\ p_3 &= s/6 - k_5/12 + 35sk/48 \\ p_5 &= k_5/120 - 7sk/48 \\ p_7 &= sk/144 \end{aligned} \quad (7)$$

Substituting these expressions we find

$$\begin{aligned} \xi_m - \bar{\xi} &= -\frac{1}{6}\sigma s \left(1 - \frac{3}{20}\frac{k_5}{s} + \frac{1}{2}k\right), \\ \xi_p - \bar{\xi} &= -\frac{1}{2}\sigma s \left(1 - \frac{1}{4}\frac{k_5}{s} + \frac{5}{6}k\right). \end{aligned} \quad (8)$$

Comparing with the known results [3]

$$\begin{aligned} \sigma &= \sqrt{\frac{\tau}{3}} \left(\frac{\beta\tau}{48}\right)^{\frac{1}{2}} \left(1 - \frac{\beta}{64}\sqrt{\frac{48}{\beta\tau}}\right) + \mathcal{O}(\tau^{-\frac{1}{2}}) \\ s &= -\frac{\rho}{16}\sqrt{\frac{3}{\tau}} \left(\frac{48}{\beta\tau}\right)^{\frac{1}{2}} + \mathcal{O}(\tau^{-\frac{1}{2}}) \\ k &= -\frac{27}{5\tau} \left(\sqrt{\frac{\beta\tau}{48}} - \frac{\beta}{24}\right) + \mathcal{O}(\tau^{-\frac{3}{2}}) \\ k_5 &= \frac{9\rho}{16} \left(\frac{3}{\tau}\right)^{\frac{3}{2}} \left(\frac{\beta\tau}{48}\right)^{\frac{1}{2}} + \mathcal{O}(\tau^{-\frac{1}{2}}), \end{aligned} \quad (9)$$

we see that the differences given in Eq. (8) are asymptotically constant at the values[†]

$$\xi_m - \bar{\xi} \sim \rho/96 = 0.12, \quad \xi_p - \bar{\xi} \sim \rho/32 = 0.35. \quad (10)$$

Thus the median, like the peak, has the same leading and next-to-leading behaviour as the mean, given by Eqs. (1,2), but lies three times closer to the mean than does the peak asymptotically. Notice that both the peak and the median lie above the mean, i.e. at smaller momentum, corresponding to about 90% of the momentum at the mean value of ξ in the case of the median and 70% for the peak.

The subasymptotic behaviour of the skewness s cannot be calculated in a next-to-leading approach since it vanishes in leading order. Thus we cannot predict the first non-leading corrections to the results (10) separately, but we can predict the ratio, for which the correction is given by the *leading* behaviour of s , k and k_5 :

$$\begin{aligned} \frac{\xi_p - \bar{\xi}}{\xi_m - \bar{\xi}} &= 3 \left(1 - \frac{1}{10}\frac{k_5}{s} + \frac{1}{3}k\right) \\ &= 3 \left(1 + \frac{9}{10}\sqrt{\frac{\beta}{48\tau}}\right). \end{aligned} \quad (11)$$

3. Discussion

The fact that the ratio (Mode - Mean)/(Median - Mean) is asymptotically equal to 3 for a number of ‘almost Gaussian’ probability distributions was discovered empirically long ago by Pearson [8]. The conditions under which Pearson’s ‘rule of 3’ applies have been investigated by Haldane [9]. A sufficient condition is that the reduced cumulants $k_n = \kappa_n/\sigma^n$ with $n > 2$ should vanish more and more rapidly asymptotically with increasing n . It follows from the results of Ref. [3] that in general

$$\kappa_n \leq \mathcal{O}(\tau^{(2-n)/4}) \quad (12)$$

and therefore Haldane’s conditions are satisfied for the distribution of ξ . The leading subasymptotic correction to Pearson’s rule in this case is proportional to $\tau^{-\frac{1}{2}}$, i.e. of order $\sqrt{\alpha_S(Q^2)}$, and is already numerically small at LEP energies (14% for $\alpha_S = 0.12$).

[†]Numerical values are given for $N_f = 3$ active flavours, but would be almost identical for $N_f = 5$.

The mean, median and modal values of ξ can also be computed from the explicit formulae for the soft parton spectrum in Modified Leading Logarithmic Approximation (MLLA), given in Refs. [2,5]. Here one can also study the dependence of the spectrum on the cutoff parameter of the QCD parton cascade, Q_0 [4]. According to the LPHD hypothesis, one would expect the spectrum of hadrons of mass m_h to be approximated by the parton spectrum with Q_0 related to m_h . The observed ξ distribution for pions is in fact well described by the limiting spectrum with $Q_0 = \Lambda \sim m_\pi$.

Fig. 1 shows the differences $\xi_p - \bar{\xi}$, $\xi_m - \bar{\xi}$ and their ratio for the limiting ($Q_0 = \Lambda$) spectrum, as functions of $\tau = \ln(Q/\Lambda)$. The differences tend to the asymptotic values (10) from below while the ratio approaches the form (11) from above. The energy dependence of all these MLLA predictions is weak for the limiting spectrum. For larger values of Q_0 the individual differences are expected to have stronger energy dependence, but their ratio should be less sensitive to Q_0 [4].

To investigate the behaviour of these quantities in a more specific model that obeys the LPHD hypothesis, we used the Monte Carlo simulation program HERWIG [10], which also describes the observed distribution of ξ quite well at present energies. Fig. 2 shows the HERWIG predictions for the differences and ratio as functions of the e^+e^- c.m. energy W . The 'data' points and errors are based on Monte Carlo runs of approximately 10^6 events. The curves through the difference points, drawn simply to guide the eye, approach the values (10) asymptotically. They show a stronger energy dependence than the limiting MLLA predictions in Fig. 1, indicating the presence of further subleading terms in the Monte Carlo simulation, but remain consistent with an approach towards the expected asymptotic values from below.

In the Monte Carlo results on the ratio (Fig. 2b), the additional subleading terms seen in the individual differences tend to cancel, yielding values close to the asymptotic curve and in good agreement with the limiting MLLA prediction. Thus within the framework of the MLLA-LPHD approach, illustrated specifically by the HERWIG simulation, we find that the prediction (11) is not strongly modified by hadronization or non-leading perturbative corrections and represents a clean test of QCD.

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Figure Captions

Figure 1: (a) Limiting MLLA predictions of the differences between mean ($\bar{\xi}$), median (ξ_m) and modal (ξ_p) values of $\xi = \ln(1/x)$ as functions of $\tau = \ln(Q/\Lambda)$.
(b) Limiting MLLA prediction for Pearson's ratio.

Figure 2: (a) HERWIG Monte Carlo predictions of the differences between mean ($\bar{\xi}$), median (ξ_m) and modal (ξ_p) values of ξ for final-state hadrons in e^+e^- annihilation, as functions of the centre-of-mass energy W .
(b) Analytical and HERWIG predictions for Pearson's ratio.