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SUGAWARA CONSTRUCTION AND THE Q -DEFORMATION OF VIRASORO (SUPER)ALGEBRA

M. Chaichian¹⁾

Theory Division, CERN,
CH-1211 Geneva 23, Switzerland
and

P. Prešnajder

Department of Theoretical Physics,
Comenius University,
Mlynská dolina F2, CS-84215 Bratislava, Czechoslovakia

Abstract

The q -deformed Virasoro algebra is obtained using the bosonic annihilation and creation operators of the q -deformed infinite Heisenberg algebra $H(\infty)_q$, which has the Hopf structure. The generators of the q -deformed Virasoro algebra are expressed as a Sugawara construction in terms of normal ordered binomials in these annihilation and creation operators, and become double-indexed as the reminder of a degeneracy removal. The obtained q -deformed Virasoro algebra with central extension reduces to the standard one in the non-deformed limit, and in special representations (but not in general) possesses a simple (cocommutative) Hopf structure [not related to the one in $H(\infty)_q$]. The fermionic annihilation and creation operators corresponding to the q -deformed infinite Heisenberg superalgebra $s - H(\infty)_q$, necessary for a similar construction of the q -deformed Virasoro superalgebra, are presented.

Permanent address: Department of High Energy Physics, University of Helsinki,
Siltavuorenpenger 20 C, SF-00170 Helsinki, Finland.

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1 Introduction

For many years great attention has been paid to the Virasoro algebra, the underlying symmetry of the string theory [1]. The Virasoro algebra is the central extension of the Lie algebra of conformal transformations of a complex plane. In the Bargmann representation the generators of the latter are given as

$$L_m = z^{m+1} \partial_z, \quad z \in \mathbb{C}. \quad (1)$$

They satisfy the Lie algebra relations

$$[L_n, L_m] = (m - n)L_{m+n}. \quad (2)$$

The algebra (1) can be re-expressed in terms of annihilation and creation operators satisfying the commutation relation

$$[a, a^+] = 1$$

in the following way:

$$L_m = (a^+)^{m+1} a. \quad (3)$$

During the last few years a growing interest has appeared in the study of quantum groups and algebras [2, 3]. In particular, a great deal of attention has been paid to q -deformations of centreless Virasoro algebra [4–7] and to its central extensions [8–10]. Several of these investigations [8, 9] are based on the use of q -deformed harmonic oscillators satisfying the relations [11]

$$\begin{aligned} [a, a^+]_q &= q^{-N}, \\ [N, a] &= -a, \quad [N, a^+] = a^+, \end{aligned} \quad (4)$$

where $[A, B]_q = AB - qBA$, and q is the deformation parameter.

The q -deformed Virasoro generators are expressed by the following generalization of eq. (3):

$$L_m = q^{-N} (a^+)^{m+1} a, \quad (5)$$

and they satisfy the following algebraic relations

$$[L_n, L_m]_{(q^{m-n}, q^{n-m})} = [m - n] L_{n+m}, \quad (6)$$

where $[A, B]_{(q_1, q_2)} = q_1 AB - q_2 BA$, and $[x] = (q^x - q^{-x})/(q - q^{-1})$. Furthermore, the problem of their central extensions has been investigated by using the q -deformed generalizations of the Jacobi identity [8–10]. However, until now these constructions have not been shown to have a Hopf algebra structure [12–14], even though the q -oscillators (4) are the elements of the quantum Hopf superalgebra $osp_q(1|2)$ [15].

In this letter we shall follow an approach based on the Sugawara construction in the string theory, leading to a Virasoro algebra with central extension (see for example, [1]). The generators L_m are represented by an infinite set of bosonic operators a_n , with n -integer, satisfying the infinite Heisenberg algebra $H(\infty)$, defined by the relations

$$\begin{aligned} [a_m, a_n] &= mH\delta_{m+n,0}, \\ [H, a_m] &= 0, \end{aligned} \quad (7)$$

with H as a central element.

The formula for L_m reads as follows:

$$L_m = \frac{1}{2} \sum_{k,n=-\infty}^{+\infty} : a_k a_n : \delta_{k+n,m} , \quad (8)$$

where the colons denote the normal ordering. Owing to quantization, the operators L_m satisfy the (centrally-extended) Virasoro algebra relations (see ref. [1]):

$$[L_n, L_m] = (m - n)H L_{n+m} + \frac{1}{12} H^2 (n^3 - n) \delta_{n+m,0} . \quad (9)$$

We wish to stress the explicit appearance of the central element H in eqs. (7) and (9), which will be substantiated later (by a suitable choice of units we can put $H = 1$, as is usually assumed).

The deformation we use for a_n was inspired by the work in ref. [16], where, for one pair of annihilation and creation operators, the quantum algebra $H(1)_q$ was obtained from a contraction of the quantum algebra $su(2)_q$. The algebra $H(1)_q$ is generated by the elements $a_{\pm}$, H , satisfying the relations

$$[a_-, a_+] = \frac{\sinh(\varepsilon H/2)}{\varepsilon/2} , \quad (10)$$

$$[H, a_{\pm}] = 0 ,$$

with $q = e^{\varepsilon}$. It was shown in ref. [16] that $H(1)_q$ has a Hopf structure.

By introducing the Hopf algebra $H(\infty)_q$, we perform the q -deformation of the Sugawara construction of the Virasoro generators. It appears that the latter become double-indexed, L_m^{α} , and are formed by normal-ordered binomials in q -deformed annihilation and creation operators, and by factors of $q^{\alpha H/2}$, with α -integer. The introduction of α is inevitable in this approach, but for $q \rightarrow 1$ the α -dependence disappears: $L_m^{\alpha} \rightarrow L_m$. The q -deformed Virasoro algebra is then realized by the usual commutators among L_m^{α} as a Lie algebra with central extension, possessing a simple (cocommutative) Hopf structure, not connected to that in $H(\infty)_q$. It would be interesting to study this problem further and see whether it is possible to construct a q -Virasoro algebra with single-indexed generators and/or with a genuine Hopf structure.

Finally, we generalize the results to the fermionic case and briefly discuss the case of the q -deformed Virasoro superalgebra with central extensions. The infinite Lie (super)algebras associated with our q -deformed Virasoro (super)algebras have trigonometric functions as their structure constants. Infinite Lie (super)algebras, which have been proposed previously [17], have similar trigonometric structure constants.

2 q -Deformation of the Heisenberg algebra

We shall denote by

$$a_n , \quad a_{-n} = a_n^+ , \quad n = 1, 2, \dots ,$$

the infinite set of annihilation and creation operators that satisfy the commutation relations

$$[a_m, a_n] \equiv a_m a_n - a_n a_m = m H \delta_{m+n,0} , \quad (11)$$

where H is a central element commuting with all the operators a_n ,

$$H a_n = a_n H . \quad (12)$$

Equations (11) and (12) define a Lie algebra. As for the Heisenberg algebra $H(\infty)$, we shall denote the corresponding universal enveloping algebra generated by H and a_n . For example, the generators of Virasoro algebra are the elements of $H(\infty)$,

$$L_m = \frac{1}{2} \sum : a_k a_n : \delta_{k+n,m} , \quad m\text{-integer} , \quad (13)$$

where the summation runs over $k, n = \pm 1, \pm 2, \dots$ ¹⁾. In eq. (13) the normal product is defined as

$$: a_k a_n : = a_k a_n - \theta(k) k H \delta_{k+n,0} , \quad (14)$$

where $\theta(k) = 1$ or 0 for k positive or negative, respectively. The operators L_m satisfy the commutation relations [1]

$$[L_m, L_n] = (n - m) H L_{m+n} + \frac{1}{12} H^2 (m^3 - m) \delta_{m+n,0} . \quad (15)$$

We define the deformation of $H(\infty)$ by changing the commutators (11) to

$$[a_m, a_n] \equiv a_m a_n - a_n a_m = \omega_{mn} , \quad (16)$$

where

$$\omega_{mn} = \frac{1}{\varepsilon} \sinh(m\varepsilon H) \delta_{m+n,0} , \quad (17)$$

and the deformation parameter q is taken to be equal to e^ε . The relations (12) remain unchanged.

We shall now demonstrate that the associative algebra $H(\infty)_q$, generated by H and a_n , with the defining relations (12), (16), and (17), is a Hopf algebra with

i) co-product

$$\begin{aligned} \Delta a_n &= q^{-|n|H/2} \otimes a_n + a_n \otimes q^{|n|H/2} , \\ \Delta H &= 1 \otimes H + H \otimes 1 ; \end{aligned} \quad (18)$$

ii) co-unit

$$\varepsilon(a_n) = 0 , \quad \varepsilon(H) = 0 ; \quad (19)$$

iii) co-inverse (antipode)

$$S(a_n) = -a_n , \quad S(H) = -H . \quad (20)$$

Here 1 denotes the unit element in $H(\infty)_q$ for which $\Delta 1 = 1 \otimes 1$, $\varepsilon(1) = 1$, and $S(1) = 1$. Such a deformation for one pair of annihilation and creation operators was proposed in ref. [16].

It can easily be shown that, for H and a_n , eqs. (18)–(20) satisfy the Hopf algebra axioms (see refs. [12–14]):

$$(id \otimes \Delta) \Delta A = (\Delta \otimes id) \Delta A ,$$

1) With respect to the string theory, we exclude the zero mode describing the position of the string, which is unessential for us.

$$m(id \otimes S)\Delta A = m(S \otimes id)\Delta A = \varepsilon(A) ,$$

$$(\varepsilon \otimes id)\Delta A = \mathbf{1} \otimes A, (id \otimes \varepsilon)\Delta A = A \otimes \mathbf{1} ,$$

where id denotes the identity mapping, and $m(A \otimes B) = AB$ denotes the usual product.

Furthermore, it is possible to show by direct calculation that the defining relations are consistent with the Hopf structure, i.e. the operations Δ , ε , and S are homomorphisms of the defining relations. For example:

$$\begin{aligned} \Delta(a_n a_m) &= \Delta a_n \Delta a_m \\ &= q^{-(|n|+|m|)H/2} \otimes a_n a_m + a_n a_m \otimes q^{(|n|+|m|)H/2} \\ &\quad + q^{-|n|H/2} a_m \otimes a_n q^{|m|H/2} + a_n q^{-|m|H/2} \otimes q^{|n|H/2} a_m . \end{aligned}$$

Subtracting from this equation the same expression, but with m and n interchanged, and using eq. (12), we obtain

$$\begin{aligned} \Delta(a_n a_m - a_m a_n) &= q^{-(|n|+|m|)H/2} \otimes [a_n, a_m] + [a_n, a_m] \otimes q^{(|n|+|m|)H/2} \\ &= q^{-|n|H} \otimes (q^{|n|H} - q^{-|n|H}) \frac{1}{2\varepsilon} \operatorname{sgn}(n) \delta_{n+m,0} \\ &\quad + (q^{|n|H} - q^{-|n|H}) \otimes q^{|n|H} \frac{1}{2\varepsilon} \operatorname{sgn}(n) \delta_{n+m,0} \\ &= \{q^{nH} \otimes q^{nH} - q^{-nH} \otimes q^{-nH}\} \frac{1}{2\varepsilon} \delta_{n+m,0} \\ &= \frac{1}{\varepsilon} \Delta \sinh(n\varepsilon H) \delta_{n+m,0} . \end{aligned}$$

For completeness, we present the corresponding R -matrix, which is given by the formula

$$R = q^{-(H \otimes N + N \otimes H)} q^{2K} , \quad (21)$$

where

$$N = \sum_{n=1}^{\infty} \frac{n\varepsilon}{\sinh(n\varepsilon H)} a_{-n} a_n , \quad (22)$$

$$K = \sum_{n=1}^{\infty} q^{nH/2} a_n \otimes q^{-nH/2} a_{-n} . \quad (23)$$

The R -matrix (21) satisfies the Yang-Baxter equation

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} \quad (24)$$

and the universality condition [18]

$$\mathcal{R} \Delta A \mathcal{R}^{-1} = \sigma \Delta A , \quad (25)$$

where σ is defined by $\sigma(A \otimes B) = B \otimes A$.

3 q -deformed Sugawara construction for Virasoro algebra

We define the q -deformed Virasoro generators as

$$L_m^\alpha = \frac{1}{2} \sum \cosh\left(\frac{k-n}{2} \varepsilon \alpha H\right) : a_k a_n : \delta_{k+n,m} , \quad (26)$$

where α is an additional index — we later discuss the need for its presence. The normal product is now given as

$$: a_k a_n : = a_k a_n - \theta(k) \omega_{kn} , \quad (27)$$

with ω_{kn} defined in eq. (17). We note that $L_m^{-\alpha} = L_m^\alpha$ and that, in the non-deformed limit, from the Sugawara construction (26) we obtain

$$L_m^\alpha \rightarrow L_m , \quad \text{for } \varepsilon \rightarrow 0 \text{ (i.e. } q \rightarrow 1) , \quad (28)$$

irrespective of the values of the additional index α . The next formula will be used later on:

$$\begin{aligned} [: a_k a_n : , : a_{k'} a_{n'} :] = & - \omega_{n'k} (: a_{k'} a_n : + \theta(k') \omega_{k'n}) \\ & + \omega_{nk'} (: a_k a_{n'} : + \theta(k) \omega_{kn'}) \\ & + \omega_{kk'} (: a_n a_{n'} : + \theta(n) \omega_{nn'}) \\ & + \omega_{nn'} (: a_{k'} a_k : + \theta(k') \omega_{k'k}) . \end{aligned} \quad (29)$$

Using eqs. (26) and (29), we obtain the q -deformed Virasoro algebra relations,

$$\begin{aligned} [L_m^\alpha, L_{m'}^{\alpha'}] = & \frac{1}{2\varepsilon} \sinh\left(\frac{m - m' - m'\alpha + m\alpha'}{2} \varepsilon H\right) L_{m+m'}^{\alpha+\alpha'+1} \\ & + \frac{1}{2\varepsilon} \sinh\left(\frac{m - m' + m'\alpha - m\alpha'}{2} \varepsilon H\right) L_{m+m'}^{-\alpha-\alpha'+1} \\ & + \frac{1}{2\varepsilon} \sinh\left(\frac{m - m' + m'\alpha + m\alpha'}{2} \varepsilon H\right) L_{m+m'}^{\alpha-\alpha'-1} \\ & + \frac{1}{2\varepsilon} \sinh\left(\frac{m - m' - m'\alpha - m\alpha'}{2} \varepsilon H\right) L_{m+m'}^{\alpha'-\alpha-1} \\ & + \frac{1}{16\varepsilon^2} (C_{m-1}^{\alpha,\alpha'} + C_{m-1}^{\alpha,-\alpha'}) \delta_{m+m',0} , \end{aligned} \quad (30)$$

where the quantities in the central term are given by

$$\begin{aligned} C_m^{\alpha,\alpha'} = & \frac{\sinh((\alpha + \alpha' + 1)/2 \, m\varepsilon H)}{\sinh((\alpha + \alpha' + 1)/2\varepsilon H)} - 2 \cosh(m\varepsilon H + \varepsilon H) \frac{\sinh((\alpha + \alpha')/2 m\varepsilon H)}{\sinh((\alpha + \alpha')/2\varepsilon H)} \\ & + \frac{\sinh((\alpha + \alpha' - 1)/2 m\varepsilon H)}{\sinh((\alpha + \alpha' - 1)/2\varepsilon H)} . \end{aligned} \quad (31)$$

We see that the odd integer values of the parameter α represent a minimal set for which the relations (30) are closed. Of course, the set of α integers is also admissible.

Using eq. (28), we can show directly that in the limit $q \rightarrow 1$ the relations (30) reduce to the non-deformed Virasoro algebra relations (15). We would like to emphasize that owing to the form of the q -deformed Sugawara construction (26), we were unambiguously forced to introduce an additional index α , since the q -deformed Virasoro is realized as a commutator Lie algebra in $H(\infty)_q$ (satisfying the usual Jacobi identity) and spanned by quadratic expressions in a_n , with integer power of the group-like element $Q = q^{H/2}$. In the limit $q \rightarrow 1$ the central subgroup Q^α , with α -integer, degenerates to a unit element. We can say that the q -deformation eliminates this degeneracy.

It is remarkable that the structure constants and the central term on the right-hand side of eq. (30) only depend on the central Heisenberg generator H . However, in a

unitary irreducible representation, H is constant, and we can absorb it into ε , or simply put $H = 1$. In this way we can associate with eq. (30) a usual Lie algebra

$$[\hat{L}_m^\alpha, \hat{L}_{m'}^{\alpha'}] = \hat{S}_{m,m'}^{\alpha,\alpha',\alpha''} \hat{L}_{m+m'}^{\alpha''} + \hat{C}_m^{\alpha,\alpha'} \delta_{m+m',0} \hat{H}, \quad (32)$$

where $\hat{H}$ is a new central element, and the structure constants $\hat{S}_{m,m'}^{\alpha,\alpha',\alpha''}$, as well as $\hat{C}_m^{\alpha,\alpha'}$, are obtained by putting $H = 1$ in the corresponding expressions in (30) and (31). The algebra (32) is an infinite Lie algebra with structure constants being trigonometric functions (but differs from the algebra proposed in ref. [17] with similar structure constants), and possesses a primitive (cocommutative) Hopf structure²⁾, which is not related to that of $H(\infty)_q$, and is given by:

i) co-product

$$\begin{aligned} \Delta \hat{L}_m^\alpha &= q^{m\hat{H}} \otimes \hat{L}_m^\alpha + \hat{L}_m^\alpha \otimes q^{m\hat{H}}, \\ \Delta \hat{H} &= \mathbf{1} \otimes \hat{H} + \hat{H} \otimes \mathbf{1}; \end{aligned} \quad (33)$$

ii) co-unit

$$\varepsilon(\hat{L}_m^\alpha) = 0, \quad \varepsilon(\hat{H}) = 0;$$

iii) co-inverse

$$S(\hat{L}_m^\alpha) = -\hat{L}_m^\alpha, \quad S(\hat{H}) = -\hat{H}.$$

The proof is straightforward and does not depend on the particular form of $\hat{S}_{m,m'}^{\alpha,\alpha',\alpha''}$ and $\hat{C}_m^{\alpha,\alpha'}$.

4 q -deformed Sugawara construction for the Virasoro superalgebra

We now extend our approach to the case of q -deformed fermionic annihilation and creation operators (in the Neveu-Schwarz sector, see ref. [1]),

$$b_r, b_{-r} = b_r^+, \quad r = 1/2, 3/2, \dots,$$

satisfying the commutation relations

$$[b_r, b_s]_+ \equiv b_r b_s + b_s b_r = \omega_{rs}, \quad (34)$$

where

$$\omega_{rs} = \frac{1}{\varepsilon} \sinh(\varepsilon \tilde{H}) \delta_{r+s,0} \quad (35)$$

and $\tilde{H}$ is the central element:

$$\tilde{H} b_r = b_r \tilde{H}. \quad (36)$$

We show that the associative superalgebra $s - H(\infty)_q$, generated by $\tilde{H}$ and b_r , defined by the relations (34)–(36), is a Hopf superalgebra (for a definition see, for example, [13]) with

2) Notice that this primitive Hopf structure only emerges when for a special representation we put $H = 1$ in eq. (30) and obtain eq. (32). In general, however, the element H is present in the arguments of the sinh functions in eq. (30); thus the Hopf structure, if it exists, would be non-trivial and complicated. However, because of the non-trivial Hopf structure in $H(\infty)_q$, it is conceivable that the q -Virasoro generators could have a related non-trivial Hopf structure through their Sugawara construction (26).

i) co-product

$$\begin{aligned}\Delta b_r &= q^{-\tilde{H}/2} \otimes b_r + b_r \otimes q^{\tilde{H}/2} , \\ \Delta \tilde{H} &= 1 \otimes \tilde{H} + \tilde{H} \otimes 1 ;\end{aligned}\tag{37a}$$

ii) co-unit

$$\varepsilon(b_r) = 0 , \quad \varepsilon(\tilde{H}) = 0 ;\tag{37b}$$

iii) co-inverse

$$S(b_r) = -b_r , \quad S(\tilde{H}) = -\tilde{H} .\tag{37c}$$

It can easily be checked that eqs. (37) satisfy the axioms of Hopf superalgebra and that the defining relations (34)–(36) are consistent with them, i.e. Δ , ε , and S are homomorphisms. For example:

$$\begin{aligned}\Delta(b_r b_s) &= \Delta b_r \Delta b_s \\ &= q^{-\tilde{H}} \otimes b_r b_s + b_r b_s \otimes q^{\tilde{H}} \\ &\quad - q^{-\tilde{H}/2} b_s \otimes b_r q^{\tilde{H}/2} + b_r q^{-\tilde{H}/2} \otimes q^{\tilde{H}/2} b_s ,\end{aligned}$$

where we used the equality

$$(A \otimes B)(C \otimes D) = -AC \otimes BD$$

which holds in the case where B and C are odd. Adding the same expression, but with r and s now interchanged, and using eq. (34), we obtain

$$\begin{aligned}\Delta(b_r b_s + b_s b_r) &= q^{-\tilde{H}} \otimes [b_r, b_s]_+ + [b_r, b_s]_+ \otimes q^{\tilde{H}} \\ &= \{q^{-\tilde{H}} \otimes (q^{\tilde{H}} - q^{-\tilde{H}}) + (q^{\tilde{H}} - q^{-\tilde{H}}) \otimes q^{\tilde{H}}\} \frac{1}{2\varepsilon} \delta_{r+s,0} \\ &= \{q^{\tilde{H}} \otimes q^{\tilde{H}} - q^{-\tilde{H}} \otimes q^{-\tilde{H}}\} \frac{1}{2\varepsilon} \delta_{r+s,0} \\ &= \frac{1}{\varepsilon} \Delta \sinh(\varepsilon \tilde{H}) \delta_{r+s,0} .\end{aligned}$$

In the fermionic realization we define the q -deformed Virasoro generators by

$$\tilde{L}_m^\beta = \frac{1}{2} \sum \frac{\sinh(s-r)/2\varepsilon\beta\tilde{H}}{2\sinh(\varepsilon\tilde{H}/2)} : b_r b_s : \delta_{r+s,m} ,\tag{38}$$

where the summation runs over $r, s = \pm 1/2, \pm 3/2, \dots$. The normal ordered product is defined in the usual way,

$$: b_r b_s : = b_r b_s - \theta(r) \omega_{rs} .\tag{39}$$

with ω_{rs} given in eq. (35).

Evidently, $\tilde{L}_m^{-\beta} = -\tilde{L}_m^\beta$, and in the non-deformed limit

$$\tilde{L}_m^\beta \rightarrow \beta \tilde{L}_m ,\tag{40}$$

where $\tilde{L}_m$ are the standard Virasoro generators in the fermionic realization,

$$\tilde{L}_m = \frac{1}{2} \sum \frac{s-r}{2} : b_r b_s : \delta_{r+s,m} ,\tag{41}$$

with b_r and b_s , the usual fermionic operators, satisfying

$$[b_r, b_s]_+ = \delta_{r+s,0} .$$

The generators $\tilde{L}_m$ satisfy the usual, non-deformed, Virasoro algebra relations,

$$[\tilde{L}_m, \tilde{L}_n] = (n-m)\tilde{H}\tilde{L}_{m+n} + \frac{1}{24}\tilde{H}^2(m^3-m)\delta_{m+n,0} . \quad (42)$$

The q -deformed Virasoro algebra in the fermionic realization is obtained by a direct calculation:

$$\begin{aligned} [\tilde{L}_m^\beta, \tilde{L}_{m'}^{\beta'}] &= \frac{\cosh[(\varepsilon\tilde{H})/2]}{\varepsilon} \sinh\left(\frac{\beta m' - \beta' m}{2}\right) \tilde{L}_{m+m'}^{\beta+\beta'} \\ &\quad - \frac{\cosh[(\varepsilon\tilde{H})/2]}{\varepsilon} \sinh\left(\frac{\beta m' + \beta' m}{2}\right) \varepsilon \tilde{H} \tilde{L}_{m+m'}^{\beta-\beta'} \\ &\quad + \frac{\cosh^2[(\varepsilon\tilde{H})/2]}{\varepsilon^2} \left(S_m^{\beta+\beta'} - S_m^{\beta-\beta'}\right) \delta_{m+m',0} , \end{aligned} \quad (43)$$

where the quantities in the central term are given by

$$S_m^\beta = \frac{\sinh(\beta/2 \, m \varepsilon \tilde{H})}{\sinh(\beta/2 \, \varepsilon \tilde{H})} . \quad (44)$$

By definition we put $\tilde{L}_m^0 = 0$. The algebra (43) is closed if the parameter β is a multiple of some given natural number. For example, the set of β -even is admissible and is complementary to the minimal set of α -odd appearing in the bosonic realization [cf. the note after eq. (31)]. Another example of an admissible set is given by the β -integer.

The q -deformed Virasoro algebra in the fermionic construction is realized as a (commutator) Lie algebra in $s - H(\infty)_q$, formed by expressions quadratic in b_r , and integer powers of $Q = q^{\tilde{H}/2}$. The corresponding (32) Lie algebra in this case has the form:

$$\begin{aligned} [\hat{L}_m^\beta, \hat{L}_{m'}^{\beta'}] &= \frac{\cosh(\frac{\varepsilon}{2})}{\varepsilon} \sinh\left(\frac{\beta m' - \beta' m}{2}\right) \varepsilon \hat{L}_{m+m'}^{\beta+\beta'} \\ &\quad - \frac{\cosh(\frac{\varepsilon}{2})}{\varepsilon} \sinh\left(\frac{\beta m' + \beta' m}{2}\right) \varepsilon \hat{L}_{m+m'}^{\beta-\beta'} \\ &\quad + \frac{\cosh^2(\frac{\varepsilon}{2})}{\varepsilon^2} \left(\hat{S}_m^{\beta+\beta'} - \hat{S}_m^{\beta-\beta'}\right) \delta_{m+m',0} \hat{H} , \end{aligned} \quad (45)$$

where

$$\hat{S}_m^\beta = \frac{\sinh(\beta/2 m \varepsilon)}{\sinh(\beta/2 \varepsilon)}$$

and $\hat{H}$ is a new central element. This is a Lie algebra of the form (32) and it has the simple (primitive) Hopf structure (33) (see footnote 2).

The centreless double-indexed q -deformed Virasoro algebra found in ref. [19] (within the Bargmann realization) is, after rescaling, given by the generators in eq. (45) without the central term. In ref. [19] a simple Hopf structure [similar to eq. (33)] was given.

The algebra (45) is closely related to that proposed in ref. [17],

$$\begin{aligned} [\hat{K}_m^\beta, \hat{K}_{m'}^{\beta'}] &= \frac{1}{2\varepsilon} \sinh\left(\frac{\beta m' - \beta' m}{2}\right) \varepsilon \hat{K}_{m+m'}^{\beta+\beta'} \\ &\quad + (cm + d\beta) \delta_{m+m',0} \delta_{\beta+\beta',0} \hat{H} , \end{aligned} \quad (46)$$

where c and d are constants specifying the general central term. By putting

$$\hat{L}_m^\beta = \cosh\left(\frac{\varepsilon}{2}\right)(\hat{K}_m^\beta - \hat{K}_m^{-\beta}), \quad (47)$$

we obtain the Lie algebra (45), but with a central term linear in m . This indicates that although the algebras (45) and (46) seem to be closely related, they differ considerably [also the relations (47) are not invertible].

To construct the Virasoro superalgebra we have to take bosonic and fermionic annihilation and creation operators simultaneously. We shall assume that they commute,

$$b_r a_n = a_n b_r, \quad \text{i.e. } n = \pm 1, \pm 2, \dots, \quad r = \pm 1/2, \pm 3/2, \dots, \quad (48)$$

and that they satisfy the relations (16) and (34) with the common central element, i.e. $H = \tilde{H}$. Then the even Virasoro generators in the bosonic realization commute with their fermionic partners:

$$[L_m^\alpha, \tilde{L}_n^\beta] = 0. \quad (49)$$

We shall add to them the odd Virasoro generators, which we define as

$$G_r^\gamma = \sum q^{(n-s)\gamma H/2} a_n b_s \delta_{n+s,r}, \quad (50)$$

where the summation runs over $n = \pm 1, \pm 2, \dots$ and $s = \pm 1/2, \pm 3/2, \dots$. The commutation relations containing the odd Virasoro generators are then

$$\begin{aligned} [L_m^\alpha, G_{r'}^{\gamma'}] &= \frac{1}{4\varepsilon} q^{-(r'\alpha+m\gamma')H/2} (q^{(m-r')H/2} G_{k+r'}^{\gamma'-\alpha-1} - q^{(r'-m)H/2} G_{k+r'}^{\gamma'-\alpha+1}) \\ &+ \frac{1}{4\varepsilon} q^{(r'\alpha+m\gamma')H/2} (q^{(m-r')H/2} G_{k+r'}^{\gamma'+\alpha-1} - q^{(r'-m)H/2} G_{k+r'}^{\gamma'+\alpha+1}), \end{aligned} \quad (51)$$

$$\begin{aligned} [\tilde{L}_m^\beta, G_{r'}^{\gamma'}] &= \frac{\cosh(\varepsilon H/2)}{2\varepsilon} q^{(m\gamma'-r'\beta)H/2} G_{k+r'}^{\beta+\gamma'} \\ &- \frac{\cosh(\varepsilon H/2)}{2\varepsilon} q^{(m\gamma'+r'\beta)H/2} G_{k+r'}^{\gamma'-\beta}, \end{aligned} \quad (52)$$

$$\begin{aligned} [G_r^\gamma, G_{r'}^{\gamma'}]_+ &= 2 \frac{\sinh(\varepsilon H)}{\varepsilon} q^{(r\gamma'+r'\gamma)H/2} L_{r+r'}^{\gamma-\gamma'} \\ &+ \frac{\sinh(\varepsilon H/2)}{\varepsilon^2} q^{(r'\gamma+r\gamma')H/2} (q^{(\gamma-\gamma')H/2} \tilde{L}_{r+r'}^{\gamma-\gamma'-1} - q^{(\gamma'-\gamma)H/2} \tilde{L}_{r+r'}^{\gamma-\gamma'+1}) \\ &+ \frac{\sinh(\varepsilon H)}{2\varepsilon^2} q^{(\gamma-\gamma')H/2} (q^{(r+1/2)H/2} S_{r-1/2}^{\gamma-\gamma'+1} - q^{-(r+1/2)H/2} S_{r-1/2}^{\gamma-\gamma'-1}). \end{aligned} \quad (53)$$

The minimal set of indices α, β , and γ , for which the relations (30), (43), (49), and (51)–(53) close among themselves, corresponds to the case where all the additional indices α, β , and γ , take integer values. This then defines a q -deformed Virasoro superalgebra. We can associate with it [in analogy with eq. (32)] a Lie superalgebra possessing a simple Hopf structure.

The q -deformed Virasoro superalgebra (30), (43), (49), and (51)–(53) in the limit $\varepsilon \rightarrow 0$ reduces to the usual Virasoro superalgebra (here the case where $N = 1$) [1]:

$$[L'_m, L'_n] = (n - m)L'_{n+m} + \frac{1}{8}(m^3 - m)\delta_{m+n,0} ,$$

$$[L'_m, G_r] = \left(\frac{1}{2}m - r\right)G_{m+r} ,$$

$$[G_r, G_s]_+ = 2L_{r+s} + \frac{1}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} ,$$

where $L'_m = L_m + \tilde{L}_m$.

5 Conclusions

In this letter we present the q -deformed analogue of the infinite Heisenberg algebra $H(\infty)_q$, formed by the central element H and the infinite set of bosonic and fermionic annihilation and creation operators, which has the structure of the Hopf algebra.

Using the q -analogue of the Sugawara construction, we construct the double-indexed q -deformed Virasoro algebra with central extension, realized by the commutators of the generators L_m^α . These generators are normal-ordered binomials in the introduced q -deformed annihilation and creation operators of $H(\infty)_q$, and contain the factors $q^{\alpha H/2}$, with α -integer, as powers of the group-like element $q^{H/2}$. Thus, the additional index α is inevitably connected with the q -deformation, and for $q \rightarrow 1$ the α -dependence disappears. With this q -deformed Virasoro algebra for special representations, but not in general, we can associate a Lie algebra which has a simple (cocommutative) Hopf structure (see footnote 2). At present it seems to us that the appearance of an extra index α is a necessity, at least within the framework of a Sugawara construction. It is worth considering further investigation of this problem and its relation to the Hopf structure.

We generalize these results to the fermionic realization — the infinite Heisenberg superalgebra $s - H(\infty)_q$ is formed by a central element $\tilde{H}$, and fermionic annihilation and creation operators³⁾. Assuming simultaneously bosonic and fermionic operators, we perform the Sugawara construction of the q -deformed Virasoro superalgebra, the generators of which are again double-indexed and reduce to the usual Virasoro superalgebra in the non-deformed limits. The situation that emerges in connection with the double-indexing of generators reminds us of a kind of degeneracy removal as a result of q -deformation. It is also of interest to study the problem of the corresponding additional symmetry involved and its origin.

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3) After the completion of this work we received a preprint [20] in which the q -deformed Heisenberg algebra $H(1)_q$ and the super one, $s - H(1)_q$, have been obtained from the contraction of the quantum superalgebra $osp_q(1|2)$. Our annihilation and creation operators used for $H(\infty)_q$ and $s - H(\infty)_q$ coincide with those of the special case in ref. [20] with the same Hopf structure.

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